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POLYNOMIAL APPROXIMATION OF FINITE ORDER ANALYTIC FUNCTION

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Abstract: The aim of this paper is to find seequences $(f_n)_n$ of analytic functions which are the product of a polynomial of degree $\leq n$ and an "easy computable" second factor and such that $(f_n)_n$ converges essentially faster to f on a plane compact set K then the sequence $\{p_n^*\}_n$ of best approximating polynomials of degree $\leq n$. Here K should be thought of as a finite disc or a real interval.

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1. Introduction

Let K be a compact subset of the complex plane C such that K and $C\backslash K$ are connected and K does not consist a single point. According to the Riemann Mapping Theorem there exists a uniquely determined conformal mapping $z = \varphi(w) : \widehat{C}\backslash\Delta \to \widehat{C}\backslash K$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$. Here $\widehat{C} = C\bigcup\{\infty\}$ denote as extended complex plane and we set

$$\Delta_r = \{z \in \mathcal{C} : |z| \leq r\}, \quad \Delta = \Delta_1.$$

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Thus, in a neighborhood of infinity, the function has the representation

$$z = \varphi(w) = d\left[w + C_0 + \frac{C_{-1}}{w} + \cdots\right],$$

where the number $d(> 0)$ is called the conformal radius or transfinite diameter of K. If we define $\eta(w) = \varphi(w/d)$, then η maps $\{w : |w| > d\}$ onto $\widehat{C}\backslash K$ in a one-one conformal manner. If $w = \Omega(z)$ is the inverse function of η, then $\Omega(\infty) = \infty, \lim_{z\to\infty}\left(\frac{\Omega(z)}{z}\right) = 1$ and, in a neighborhood of infinity, the function $\Omega(z)$ has a Laurent expansion of the form

$$\Omega(z) = z + b_0 + \frac{b\_1}{z}\cdots .$$

Thus for each positive integer n and for sufficiently large $|z|$, one has an expansion of the form

$$[\Omega(z)]^n = z^n + b_{n-1,n}z^{n-1} + b_{n-1,n}z^{n-2} + \cdots + b_{1,n}z + b_{0,n} + \frac{b_{-1,n}}{z} + \cdots .$$

The Polynomials

$$P_n(z) = z^n + b_{n-1,n}z^{n-1} + \cdots + b_{1,n}z + b_{0,n}, \quad n = 0, 1, 2, \cdots ,$$

which comprise non-negative power of z in the Laurent series expansion of $[\Omega(z)]^n$ about infinity, are called the Faber polynomials for K. If K is the closed disc $|z - z_0| \le d$, then $w = \Omega(z) = z - z_0$ and so the Faber polynomials for the closed disc are given by $P_n(z) = (z - z_0)^n, n = 0, 1, 2, \cdots$. Thus, the Taylor polynomials $(z - z_0)^n$ are a special case of Faber polynomials. We refer to [6] for more details on Faber polynomials.

Let $L_r = \{z : z = \eta(w), |w| > d\}$. Since $\Omega(z)$ is analytic and univalent, $L_r's$ are analytic, Jordan curves. If K_r denotes the domain bounded by L_r, then $K \subset K_r$ for each $r > d$ and $L_r \subset K_{r_1}$ for $r < r_1$.

Let $H(K; R)$ denote the class of all functions f that are regular in K_R with a singularity on L_R $(d < R < \infty)$. In the sequel we will consider the growth parameters for functions in $H(K; R)$. Thus, growth parameters, analogous to those introduced for functions regular in unit disc, may be defined for functions regular in K_R as follows.

We say that $f \in H(K; R), d < R < \infty$ is of K-order ρ in K_R if

$$\rho = \limsup_{r\to R} \frac{\log^+ \log^+ \overline{M}(r)}{\log(R/(R - r))},$$

where

$$\overline{M}(r) = \max_{z\in L_r} |f(z)|, \quad d < r < R.$$

To compare the growth of functions in $H(K;R)$ that have the same non zero finite K-order, the concept of K-type has been introduced. Then $f \in H(K;R)$ having K-order $\rho, (0 < \rho < \infty)$, is said to be of K-type $T < \infty$ if

$$T = \limsup_{r \to R} \frac{\log \overline{M}(r)}{(Rr/R-r)^{\rho}}.$$

Let Π_n be the set of polynomials of degree $\leq n$ and let

$$E_n(f,K) = \inf_{p \in \Pi_n} \|f-p\|_K,$$

with $\|\varphi\|_K = \sup_{z \in K} |\varphi(z)|$, denote the error of best polynomial approximation of f on K. We can easily obtain the following result by [5].

Theorem 1.1. *Let $f \in H(K;R)$. Then f is the restriction on K of an analytic function of K-order ρ and K-type T if and only if*

$$\limsup_{n \to \infty} n^{1/\rho+1} \log^+ [E_n(f,K)(C(K))^n]^{1/n} = (C(K))^{\rho/\rho+1} \left(\frac{\rho+1}{\rho}\right) (T\rho)^{1/\rho+1}, \quad (1.1)$$

where $C(K) > 0$ is the logarithmic capacity of K.

A sequence $\{p_n\}$ with $p_n \in \Pi_n$ for all $n \in N$ is called maximally convergent on K to f if the asymptotic rate of best polynomial approximation is realized by $\{p_n\}$, that is

$$\limsup_{n \to \infty} n^{1/\rho+1} \log^+ [\|f-p_n\|_K (C(K))^n]^{1/n} = (C(K))^{\rho/\rho+1} \left(\frac{\rho+1}{\rho}\right) (\rho T)^{1/\rho+1}.$$

Besides the polynomials p_n^* of best approximation given by

$$\|f-p_n^*\|_K = \inf_{p \in \Pi_n} \|f-p\|_K$$

the computation of which is rather expansive, here we shall consider two type of maximally convergent sequences $\{p_n\}$: polynomial interpolant in arbitrary system of nodes and Faber expansions. Now first we will give a brief introduction of these two kinds of polynomial approximants.

1. Let $(z_k^{(n)})_{n \in N_0}, k = 0, \cdots, n$ be $(n+1)$ distinct points in the complex plane and let the values $(w_k^{(n)})_{n \in N_0}, k = 0, \cdots, n$ be given. There exist infinitely many polynomials that takes the values $w_k^{(n)}$ at the points $z_k^{(n)} (k = 0, \cdots, n)$. However, if one is interested in a polynomial of degree not exceeding n that assumes the prescribed values w_k at the points z_k then such a polynomial is

unique and is given by

$$H_n(z)=\sum_{k=0}^{n}\frac{w_n(z)}{\left(z-z_k^{(n)}\right)w_n'\left(z_k^{(n)}\right)}w_k\,, \tag{1.2}$$

where

$$w_n(z)=\Pi_{k=0}^{n}\left(z-z_n^{(n)}\right).$$

The polynomial $H_n(z)$ is called Lagrange's interpolation polynomial. Lagrange's interpolation formula takes an elegant form in the case of functions that are regular inside and on a simple closed curve. Thus we have:

— If $L_r=\eta(\partial\Delta_r)$ is a level curve of η^{-1} for some $r>d$, then the (uniquely determined) polynomial interpolant $L_n(f)\in\Pi_n$ to f with respect to the nodes $z_k^{(n)}$ may be expressed y the Hermite interpolation formula

$$L_n(z)=L_n(f)(z)=\frac{1}{2\pi i}\int_{L_r}\frac{w_n(t)-w_n(z)}{t-z}\frac{f(t)}{w_n(t)}dt \quad (z\in K). \tag{1.3}$$

— If $K=\Delta_r$ for some $r>0$ and $z_k^{(n)}=0$ for all k and n, then

$$L_n(f)=S_n(f),$$

where $S_n(f)$ is the n-th partial sum of the Taylor expansion of f around the origin.

Lagrange's interpolation formula and Hermite's interpolation formula can be easily extended to cover the case of multiple interpolation. Thus, if different points $\left(z_k^{(n)}\right)$ are given and each $z_j^{(n)}$ is associated with the quantities $w_j^{(\nu)},\nu=0,1,\cdots,m_j-1$, then the problem of constructing a polynomial $m(z)$ of degree not exceeding $-1+z_{j=1}^{k}m_j$ such that $L_n^{(\nu)}(z_j)=w_j^{(\nu)},\nu=0,1,\cdots,m_j-1,j=1,2,\cdots,k$, where $L_n^{(\nu)}(z)$ denotes the ν-th derivative of $L_n(z)$, is called the problem of multiple interpolation. In the case of Hermite's interpolation formula the values $w_j^{(\nu)}$ are to be replaced by $f^{(\nu)}\left(z_j^{(\nu)}\right)$, where $f^{(\nu)}(z)$ is the ν-th derivative of $f(z)$.

2. The n-th Faber polynomial $F_n=F_{n,K}$ with respect to K may be defined by

$$\frac{\eta'(w)}{\eta(w)-z}=\sum_{n=0}^{\infty}\frac{F_n(z)}{w^{n+1}} \quad (z\in K).$$

It is known [6] that there exist absolute constants $A>0$ and $\alpha<0.5$ such that for every f continuous on K and analytic in the interior of K.

$$\|f - T_n(f)\|_K \leq An^{\alpha} E_n(f, K), \tag{1.4}$$

where $T_n(f)$ denotes the n-th partial sum of the Faber expansion of f, that is

$$T_n(f) = T_{n,K}(f) = \sum_{k=0}^{\infty} a_k(f) F_k \tag{1.5}$$

with

$$a_k(f) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(\eta(w))}{w^{n+1}} dw. \tag{1.6}$$

In view of (1.4) it follows that $(T_n(f))$ converges maximally on K to f. In particular, for $K = K_R (R > d)$ we have by Theorem 1.1 that

$$\limsup_{n\to\infty} n^{1/\rho+1} \log^+ \left[\|f - T_{n,E_R}(f)\|_{K_R} (R/d)^n \right]^{1/n} = (R/d)^{\rho/\rho+1} \left(\frac{\rho+1}{\rho} \right) (\rho T)^{1/\rho+1}$$

(note that $C(K_R) = R/d$). For $K = \Delta_r$ we have $T_{n,\Delta_r}(f) = S_n(f)$, and therefore Theorem 1.1 gives

$$\limsup_{n\to\infty} n^{1/\rho+1} \log^+ \left[\|f - S_n(f)\|_{\Delta_r} r^n \right]^{1/n} = r^{\rho/\rho+1} \left(\frac{\rho+1}{\rho} \right) (\rho T)^{1/\rho+1},$$

$C(\Delta_r) = r$. Now, for $K = [-1, 1], C([-1, 1]) = 1/2$, Theorem 1.1 implies

$$\limsup_{n\to\infty} n^{1/\rho+1} \log^+ \left[\|f - T_n(f)\|_{[-1,1]} \right]^{1/n} = \left(\frac{1}{2} \right)^{\rho/\rho+1} \left(\frac{\rho+1}{\rho} \right) (\rho(T))^{1/\rho+1}. \tag{1.7}$$

Here $T_n(f)$, the n-th partial sum of the Faber expansion of f equals the n-th partial sum of the Tschebyscheff expansion of f.

Generally, the rate of best polynomial approximation on K of an analytic function of finite order is determined by the growth parameters K-order and K-type of f. These information cannot be used to improve the rate of convergence for polynomial approximation of f. The aim of this paper is to modify the function f in such a way that the modified $\widetilde{f}$ is "better" approximable on K by polynomials then f itself, by using the information about the growth of f, and then recover f from an approximation of $\widetilde{f}$.

2. The *RG*-Method

We first describe the idea in the case of $K = K_R$ $(R > d)$ and partial sums T_n of the Faber expansion as approximating polynomials.

Let, for an arbitrary Faber series

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu F_\nu$$

with

$$T_n(f)(z) = \sum_{\nu=0}^{n} a_\nu F_\nu.$$

For f being analytic on K_R $(R > d)$ we obtain from Cauchy integral formula

$$\|f - T_n(f)\|_{K_r} \le \overline{M}(r, f)\left(\frac{r}{r - r'}\right)\left(r'/r\right)^n,$$

where $d < r'' < r' < r < R, n > n_0(r', r''$ and $M(r, f) = \max|f(z)|, z \in L_r$.

Thus, if f is an analytic function of K-order ρ and K-type $T < \infty$, and if (r_n) is an arbitrary sequence with $d < r_n \to R$, we get

$$\begin{aligned} &\limsup_{n\to\infty} n^{1/\rho+1} \log^+\left[\|f - T_n(f)\|_{K_{r_n}} (r_n/d)^n\right]^{1/n} \\ \le\ &\limsup_{n\to\infty} n^{1/\rho+1}\left[\log^+(M(r_n, f))^{1/n} + \log(r'/r_n).(1 + o(1))\right]. \end{aligned}$$

If $T > 0$ and if we take

$$\frac{R.R_n}{R - r_n} = \left(\frac{r_n n}{d\rho T}\right)^{1/\rho+1},$$

then we take

$$\begin{aligned} &\limsup_{n\to\infty} n^{1/\rho+1}\left[\log^+(M(r_n, f))^{1/n} + \log(r'/r_n)\right] \\ &\qquad\qquad \le (r_n/d)^{\rho/\rho+1}\left(\frac{\rho+1}{\rho}\right)(\rho T)^{1/\rho+1}. \end{aligned}$$

Since $C(r_n) = r_n/d$, this implies that $T_n(f)$ converges to f maximally on r_n.

Now consider the case of K being the closed unit disc and the partial sums S_n of the Taylor expansion around the origin as approximating polynomials.

Let, for an arbitrary power series

$$g(z) = \sum_{\nu=0}^{\infty} g_\nu z^\nu$$

around the origin,

$$S_n(g)(z) = \sum_{\nu=0}^{n} g_\nu z^\nu.$$

For g being holomorphic on Δ_r for some $r > 1$, from Cauchy integral formula we get

$$\|g - S_n(g)\|_\Delta \leq \frac{M(r,g)}{r^n(r-1)}.$$

Thus, if f is an analytic function of order ρ and type $T < \infty$ and if (r_n) is an arbitrary sequence with $1 < r_n \to R$, we get

$$\limsup_{n\to\infty} n^{1/\rho+1} \log^+ \|f - S_n(f)\|_\Delta \leq \limsup_{n\to\infty} n^{1/\rho+1} \left[\log(M(r_n,f))^{1/n} + \log(r'/r_n)\right].$$

If $T > 0$ and if we assume

$$\frac{Rr_n}{R - r_n} = \left(\frac{n}{\rho T}\right)^{1/\rho+1},$$

then we obtain

$$\limsup_{n\to\infty} n^{1/\rho+1} \left[\log^+(M(r_n,f))^{1/n} + \log r'/r_n\right] \leq \left(\frac{\rho+1}{\rho}\right)(\rho T)^{1/\rho+1}.$$

Note that $C(\Delta) = 1$, which implies that $S_n(f)$ converges to f maximally on Δ.

Now the main idea is to replace in the above text the function f by $f\varphi_n$, where $(\varphi_n)_n$ is a sequence of functions such that φ_n is continuous on K and analytic in the interior of K. With $\varphi := (r_n, \varphi_n)$ and

$$\mu(\varphi) = \limsup_{n\to\infty} n^{1/\rho+1} \left[\log^+(M(r_n,f))^{1/n} + \log(r'/r_n)\right]$$

it gives

$$\limsup_{n\to\infty} n^{1/\rho+1} \log^+ [\|f\varphi_n - T_n(f\varphi_n)\|_\Delta]^{1/n} \leq \mu(\varphi).$$

If $\mu(\varphi) < \left(\frac{\rho+1}{\rho}\right)(\rho T)^{1/\rho+1}$ and $|\varphi_n|^{1/n}$ converges to 1 uniformly on Δ, then the sequence $\left(\varphi_n^{-1} T_n(f\varphi_n)\right)_n$ converges asymptotically by the factor

$$\left[\exp\left(\mu(\varphi)/\left(\frac{\rho+1}{\rho}\right)(\rho T)^{1/\rho+1}\right)\right]^n$$

faster to f than maximally convergent polynomial sequences.

Using (1.3), one can prove the following more general result.

Theorem 2.1. *Let $f \in H(K;R)$, and let f is the restriction on K of* *an analytic function of* K-order ρ and K-type $T < \infty$. *Suppose further that* $\varphi := (r_n, \varphi_n)_n$ *is a sequence such that* $0 < d < r_n \to R$ *for* $n \to \infty$ *and* φ_n *is a function which is holomorphic on* $\Delta r_n \bigcup K$. *If* $\left(z_k^{(n)}\right)_{n\in N_0, k=0,\cdots,n}$ *is an arbitrary system of nodes on* K, *then*

$$\limsup_{n\to\infty} n^{1/\rho+1} \log^+ [\|f\varphi_n - L_n(f\varphi_n)\|_K (C(K))^n]^{1/n} \le C(K)^{\rho/\rho+1}\mu(\varphi). \quad (2.1)$$

The estimates (1.1) and (2.1) suggest the following idea for an algorithm:

1. Search for a sequence $\varphi = (r_n, \varphi_n)_n$ as in Theorem 1.2 such that

$$\mu(\varphi) < \left(\frac{\rho+1}{\rho}\right)(\rho T)^{1/\rho+1}$$

and $|\varphi_n|^{1/n} \to 1$ locally uniformly on C.

2. Compute an approximating polynomial $P_n = P_n(f\varphi_n, K)$ of $f\varphi_n$.

3. Take $\varphi_n^{-1}.P_n(f\varphi_n)$ as approximation of f.

In view of our proposed algorithm two questions arise:

(i) Let Φ be the set of all sequences $\varphi = (r_n, \varphi_n)_n$ as in Theorem 1.2 with $|\varphi_n|^{1/n} \to 1$ locally uniformly on C. Can we determine

$$\underline{m} = \underline{m}_f = \inf_{\varphi\in\Phi} \mu(\varphi)?$$

(ii) If *so*, how can we find "easy computable" sequences $\varphi \in \Phi$ such that $\mu(\varphi) \approx \underline{m}$?

To answer these questions we have to define the indicator function

$$h = h_f(\theta) = \limsup_{r\to R} \frac{\log\left|f(re^{i\theta})\right|}{(Rr/(R-r))^\rho}, \quad |\theta| \le \alpha.$$

From definition it follows that $h_f(\theta) \le T$ for all θ. The Crucial role in our game is played by

$$I = I_f = \frac{1}{2\pi}\int_0^{2\pi} h_f(\theta) d_\theta.$$

The value of I is intimately related to the number of zeros of f. The fundamental results connecting the modulus of an entire function with the number of its zeros was first given by Jensen [4]. The result states: "Let $f(z)$ be analytic for $|z| < R$"; suppose that $f(0) \neq 0$ and let $r_1, r_2, \cdots, r_n \cdots, r_m$, be the modulii of zeros of $f(z)$ inside the circle $|z| = R$ arranged in non-decreasing order then if $r_n \leq r < r_{n+1}$,

$$\log\left\{\frac{r^n|f(0)|}{r_1r_2\cdots r_n}\right\} = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|d\theta. \tag{2.2}$$

Let $n(r)$ denote the number of zeros of $f(z)$ in Δr. Then $n(r)$ is non-decreasing function of r which is constant in any interval which does not contain the modules of any zero of $f(z)$. Then (2.2) can be written as

$$\int_0^r \frac{n(x)}{x}dx = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|d\theta - \log|f(0)|. \tag{2.3}$$

It is clear from (2.3) that greater the number of zeros of $f(z)$, faster the function must grow. If

$$N(r) = \int_0^r \frac{n(x)}{x}dx,$$

then

$$N(r) \leq \log M(r,f). \tag{2.4}$$

From the above cited results it follows that $I \geq 0$ and $I \in [0,T]$. The following result give an answer to question (i).

Lemma 2.1. *Let f be an analytic function of order $\rho > 0$ and of completely regular growth. Then we have*

$$\underline{m} = \left(\frac{\rho+1}{\rho}\right)(\rho I)^{1/\rho+1}.$$

Proof. Using (2.3) and (2.4) with simple calculation we obtain

$$\underline{m} \geq \left(\frac{\rho+1}{\rho}\right)(\rho I)^{1/\rho+1}.$$ □

Now let us turn to the question (ii) of how to find sequences $\varphi \in \Phi$ such that

$$\mu(\varphi) \approx \left(\frac{\rho+1}{\rho}\right)(\rho I)^{1/\rho+1}$$

and such that the function φ_n are "easy computable". We can consider the sequences (ρ_n) of the form $\varphi_n = e^{-R_n}$, where the R_n are polynomials. Also (φ_n) may be rational functions.

For given h_f we consider a polynomial Q such that $Q(0) = 0$, and we set

$$T(Q) = T_f(Q) = \max_Q(h_f(\theta) - ReQ(e^{i\theta})). \tag{2.5}$$

Since ReQ is sub-harmonic in Δ, we find for $\theta \in [-\pi,\pi]$ (since $h_f(-\pi) =$

$h_f(\pi)$ and $I = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_f(\theta)d\theta$)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[h_f(\theta) - ReQ(e^{i\theta})\right]d\theta \geq I - ReQ(0) = I$$

and we get

$$T(Q) \geq I.$$

Set

$$\frac{Rr_n}{R-r_n} = \left(\frac{n}{T(Q)\rho}\right)^{1/\rho+1} \tag{2.6}$$

and

$$\varphi_n(z) = \varphi_{n,Q}(z) = \exp\left(-(Rr_n/(R-r_n)^{\rho}Q(z/r_n)\right), z \in C. \tag{2.7}$$

Then $\varphi \in (r_n, \varphi_n)_n \in \Phi$ and one can prove the following.

Lemma 2.2. *Let (r_n) and (φ_n) be defined by (2.6) and (2.7). Then we have*

$$\mu(\varphi) \leq \left(\frac{\rho+1}{\rho}\right)(\rho T(Q))^{1/\rho+1}. \tag{2.8}$$

Proof. We have $(Rr_n/(R-r_n))^{\rho}\log|\varphi_n(r_n e^{i\theta})| = ReQ(e^{i\theta})$ for every $n \in N$ and thus

$$\begin{aligned}\frac{\log^+ M(r_n, f\varphi_n)}{(Rr_n/(R-r_n))^{\rho}} &\leq \max_{\theta}\left[\frac{\log^+\left|f(r_n e^{i\theta})\right|}{(Rr_n/(R-r_n))^{\rho}} - h_f(\theta)\right] \\ &+ \max_{\theta}\left[h_f(\theta) - ReQ(e^{i\theta})\right].\end{aligned}$$

From Theorem 28, Chapter I, of [5] one can easily obtain

$$\limsup_{n\to\infty}\max_{\theta}\left[\frac{\log^+\left||f(r_n e^{i\theta})\right|}{(Rr_n/(R-r_n))^{\rho}} - h_f(\theta)\right] \leq 0$$

and thus

$$\limsup_{n\to\infty}\frac{\log^+ M(r_n, f\varphi_n)}{(Rr_n/(R-r_n))^{\rho}} \leq T(Q).$$

In view of (2.6) we get

$$\mu(\varphi) =$$

$$\limsup_{n\to\infty} n^{1/\rho+1}\left[\log^+\left[(M(r_n, f\varphi_n))\left(\begin{array}{c}R-r_n\\ Rr_n\end{array}\right)^{\rho}\right]^{\left(\frac{Rr_n}{R-r_n}\right)^{\rho}/n} + \log(r'/r_n)\right]$$

$$\leq \left(\frac{\rho+1}{\rho}\right)(\rho T(Q))^{1/\rho+1}. \quad \square$$

Now we have

Theorem 2.2. *Let $f \in H(K,R)(R > d)$ and let Q be a polynomial with $Q(0) = 0$. Suppose further that f is an analytic function of K-order $\rho \in (0,\infty)$ and K-type $T < \infty$ and that (φ_n) is given by (2.7).*

1. If $(z_k^{(n)})_{n\in N_0,k=0,\cdots,n}$ is an arbitrary system of nodes on K, then

$$\limsup_{n\to\infty} n^{1/\rho+1} \log^+ \left[\|f-\varphi_n^{-1}L_n(f\varphi_n)\|_k (C(K))^n\right]^{1/n}$$
$$\leq (C(K))^{\rho/\rho+1}\left(\frac{\rho+1}{\rho}\right)(\rho T(Q)^{1/\rho+1}. \tag{2.9}$$

2. Let ε denote the set of all functions $f \in H(K,R)$. If T_n is a sequence of operator $T_n : \varepsilon \to \Pi_n$ such that there exist constants $A, \beta > 0$ with

$$\|g - T_n(g)\|_K \leq An^{\beta}E_n(g,K) \tag{2.10}$$

for all $g \in \varepsilon$, then

$$\limsup_{n\to\infty} n^{1/\rho+1} \log^+ \left[\|f-\varphi_n^{-1}T_n(f\varphi_n)\|_K (C(K))^n\right]^{1/n}$$
$$\leq (C(K))^{\rho/\rho+1}\left(\frac{\rho+1}{\rho}\right)(\rho T(Q))^{\rho/\rho+1}. \tag{2.11}$$

Proof. Since $|\varphi_n|^{-1/n} \to 1$ uniformly on K, then in view of Theorem 2.1 and Lemma 2.2, part 1 follows.

Let $L_n(f\varphi_n)$ denote the n-th polynomial interpolant to $f\varphi_n$ with respect to the system of arbitrary nodes of K. By Theorem 2.1 and Lemma 2.2 we get

$$\begin{aligned}
&\limsup_{n\to\infty} n^{1/\rho+1} \log^+ [\|f\varphi_n - T_n(f\varphi_n)\|_K (C(K))^n]^{1/n}\\
= \ &\lim_{n\to\infty} n^{1/\rho+1} \log^+ [E_n(f\varphi_n, K)]^{1/n}\\
\leq \ &\limsup_{n\to\infty} n^{1/\rho+1} \log^+ [\|f\varphi_n - L_n(f\varphi_n)\|_K (C(K))^n]^{1/n}\\
\leq \ &(C(K))^{\rho/\rho+1}\left(\frac{\rho+1}{\rho}\right)(\rho T(Q))^{1/\rho+1}.
\end{aligned}$$

Since $|\varphi_n|^{-1/n} \to 1$ uniformly on K, we get our result i.e., assertion 2. $\square$

Remark. The condition (2.10) is satisfied by the sequence of best approximation operators, that is, $T_n(g)$ is defined by

$$\|g - T_n(g)\|_K = \inf_{p\in\Pi_n} \|g-p\|_K.$$

Moreover, by a result of Kovari and Pommerenke [6], the same is true for the sequence (T_n) of the n-th partial sums of the Faber expansion with respect to K and therefore, in particular, in the case of $K = [-1, 1]$ for the Tschebyscheff sections.

For a finite set $M \subset N$ we define

$$\Pi_M = \left\{ \sum_{\nu \in M} a_\nu z^\nu : a_\nu \in C \quad \text{for} \quad \nu \in M \right\},$$

i.e., Π_M is the set of polynomials with powers only in M. ($Q(0) = 0$ for $Q \in \Pi_M$). If $Q \in \Pi_M$, then, by definition (2.7), we have $\varphi_n = e^{-R_n}$ when $R_n \in \Pi_M$ for all $n \in N$. Therefore, the approximation of f obtained by the $RG-$ method are of the form

$$\varphi_n^{-1}.P_n = e^{R_n}.Pn \quad \text{with} \quad R_n \in \Pi_M \quad \text{and} \quad p_n \in \Pi_n.$$

Since the effort for the evaluation of the factor e^{R_n} does not increase with n we may regard $\varphi_n^{-1}.P_n$ as a "near polynomial approximation" of f. Theorems 1.1 and 1.3 show that, if $T(Q) < T$, we get a (geometric) acceleration factor $\left[\exp(T(Q)/T)^{1/\rho+1}\right]^n$ if we approximate $f\varphi_n$ instead of f by a polynomial sequence as in Theorem. The "cost" for that is an additional multiplication by $\varphi_n^{-1} = e^{R_n}$.

Now it is the question of ho to choose an appropriate polynomial θ in order to apply RG-method in an efficient way. For let us consider the following problem:

Choose $Q_M \in \Pi_M$ such that

$$\max_\theta (h_f(\theta) - ReQ_M(e^{i\theta})) = \min_{\theta \in \Pi_M} \max_\theta (h_f(\theta) - ReQ(e^{i\theta})).$$

This is a kind of one-sided Tschebyscheff approximation of the (continuous and $2\pi-$ periodic) function h_f by trigonometric polynomials without constant term.

For the important case $M = \{1, \cdots, M\}$ we but $Q_m = Q_{\{1,\cdots,m\}}$ and obtain the following estimates.

Theorem 2.3. *With above assumptions we have*

$$I \leq T(Q_m) \leq I + 2\varepsilon_m(h_f),$$

where $\varepsilon_m(h_f)$ denotes the error of best approximation of h_f by trigonometric polynomials of degree $\leq m$. This implies

$$T(Q_m) \to I \quad (m \to \infty).$$

Proof. Let t_m denote the best approximating trigonometric polynomial of degree $\leq m$ to the function h_f on $[-\pi, \pi]$. If $a_0 m/n$ is the constant term of t_m, that is

$$\frac{|a_0^m|}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} t_m(\theta)d\theta,$$

then

$$\frac{|a_0^m|}{2} = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} h_f(\theta)d\theta + \int_{-\pi}^{\pi}(t_m(\theta) - h_f(\theta))d\theta\right| \leq I + \varepsilon_m(h_f),$$

thus

$$\max_{\theta}\left[h_f(\theta) - \left(t_m(\theta) - \frac{a_0^m}{2}\right)\right] \leq I + 2\varepsilon_m(h_f).$$

Since $t_m - \frac{a_0^m}{2}$ is the real part of a polynomial in $e^{i\theta}$ of degree $\leq m$ without constant term. Hence the proof is completed. □

Assume that we have found a polynomial Q such that $T > T(Q) \approx$. The question now is how to choose the approximating polynomial $p_n = p_n(f\varphi_n, K) \in \Pi_n$ of $f\varphi_n$ on K. Since the polynomial Q does only depend on h_f, we had so far no need to look on our compact set K on which we want to approximate f. This set K now plays an important role in order to choose p_n. Concerning speed of approximation, the best possible choice is given by the sequence $p_n^* = p_n^*(f\varphi_n, K)$ of best approximating polynomials of $f\varphi_n$ with respect to K. However, in the most interesting case of K being a disk or an interval.

Since $L_n(f\varphi_n) = L_n(L_n(f))L_n(\varphi_n)$ for the polynomial interpolant of degree $\leq m$ in an arbitrary system of nodes $(z_k^{(x)})$, the computation of $L_n(f\varphi_n)$ does not requiree more information about f than the computation of $L_n(f)$, namely, the values of f (and, in the case of multiple nodes, derivatives of f) at the nodes $(z_k^{(x)})$.

1. The case $K = \Delta_r$. Let g be holomorphic in Δ_r and let

$$g(z) = \sum_{v=0}^{\infty} g_\nu z^\nu \quad (z \in \Delta_r),$$

be the Taylor expansion of g around the origin. In the case $K = \Delta_r$, the Taylor sections

$$S_n(g)(z) = \sum_{\nu=0}^{n} g_\nu z^\nu \quad (z \in C)$$

represent the interpolation polynomials of degree $\leq n$ to g in the arbitrary system of nodes $z_k^{(n)} = 0$ for $k = 0, \cdots, n$ as well as the n-th partial sum of the

Faber expansion with respect to K of g. Since $C(\Delta_r)=r$, by Theorem 1.3 we have

$$\limsup_{n\to\infty} n^{1/\rho+1}\log^{+}\left[\left\|f-\varphi_n^{-1}S_n(f\varphi_n)\right\|_{\Delta_r} r^n\right]^{1/n} \le r^{\rho/\rho+1}\left(\frac{\rho+1}{\rho}\right)(\rho T(Q)^{\frac{1}{\rho+1}}).$$

If $Q(z)=\sum_{\nu\in M} a_\nu z^\nu$ for some $M\subset N$ and if (r_n) and (φ_n) are given by (2.6) and (2.7), then the Taylor coefficients $\varphi_{k,n}=\varphi_n^{(k)}(0)/k!$ of

$$\begin{aligned}\varphi_n(z) &= \exp\left(-(Rr_n/(R-r_n))^\rho\theta(z/r_n)\right)\\ &= \Pi_{\nu\in M}\exp\left(-a_\nu z^\nu r_n^{-\nu}\left(Rr_n/(R-r_n)\right)^\rho\right),\end{aligned}$$

may be computed by repeated Cauchy product (i.e., by repeated discrete convolution) from the Taylor coefficients of $\exp\left(-a_\nu z^\nu r_n^{-\nu}(Rr_n/(R-r_n))^\rho\right)$. Now, if the Taylor coefficients $f_k=f^{(k)}(0)/k\,!$ of f for $k=0,\cdots,n$ are known, one more Cauchy product gives

$$S_n(f\varphi_n)(z)=\sum_{\nu=0}^{n} z^\nu\left(\sum_{k=0}^{\nu} f_k\varphi_{\nu-k,n}\right).$$

2. The case $K=[a,b]$. It is well known, in the case $K=[-1,1]$ system of arbitrary nodes are for example the zeros of the Tschebyscheff polynomials

$$z_k^{(n)}=\cos\left(\frac{(2k+1)\pi}{2(n+1)}\right),\quad k=0,\cdots,n,$$

or the Fejer nodes given by

$$z_k^{(n)}=\cos\left(\frac{2k\pi}{n+1}\right),\quad k=0,\cdots,n.$$

Since in the second case $z_k^{(n)}=z_{n-k+1}^{(n)}$ for $k=1,\cdots,n$, we have interpolation of f and f' in these nodes.

The Faber polynomials for $K=[-1,1]$ coincide with the (normalized) Tschebyscheff polynomials, more precisely,

$$F_n(x)=\begin{cases} 2\cos(n\arccos x) & \text{if } n=1,2,\dots,\\ 1, & \text{if } n=0.\end{cases}$$

for $x\in[-1,1]$ and the n-th partial sum T_n of the Faber expansion equals the n-th partial sum of the Tschebycheff expansion. Since $C[-1,1]=1/2$, Theorem 2.2 gives

$$\limsup_{n\to\infty} n^{1/\rho+1}\log^{+}\left[\left\|f-\varphi_n^{-1}T_n(f\varphi_n)\right\|_{[-1,1]}\right]^{1/n}\le\left(\frac{1}{2}\right)^{\rho/\rho+1}\left(\frac{\rho+1}{\rho}\right)(\rho T(Q)).$$

Thus, we see that the smaller capacity of $K=[-1,1]$ compared to $K=\Delta$ causes an acceleration factor of $\left[\exp((1/2)^{\rho/\rho+1})\right]^n$ if f is approximated by

$\varphi_n^{-1} T_n(f\varphi_n)$ instead of $\varphi_n^{-1} S_n(f\varphi_n)$ on $[-1,1]$.

The case of an arbitrary interval $K = [a,b]$ with $a, b \in C$ may be reduced to the case $K = [-1,1]$ by a simple linear transformation, so that this case is essentially included above. In particular, for a function g holomorphic on $[a,b]$ the n-th Faber section $T_n(g) = T_{n,[a,b]}(g)$ with respect to $[a,b]$ is given by

$$T_{n,[a,b]}(g)(w) = T_{n,[-1,1]}(\widetilde{g}) = \left(\frac{2}{b-a}w - \frac{b+a}{b-a}\right),$$

where

$$\widetilde{g}(z) = g\left(\frac{b-a}{2}z + \frac{a+b}{2}\right).$$

As in the standard case $[a,b] = [-1,1]$ we denote $T_{n,[a,b]}(g)$ as n-th Tschebyscheff section of g (with respect to $[a,b]$).

More general compact set K (having simply connected complement $\widehat{C}K$) may be handled similar to the above case of $K = [a,b]$ by choosing the n-th partial sum T_n of the Faber expansion instead of the n-th Tschebyscheff section. An efficient method for the numerical evaluation of T_n is described in [2]. Moreover, in [1] and [5] explicit expressions for the Faber polynomials $F_{n,k}$ in the case of K being a circular or an annular sector are given.

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