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A NEW SUBCLASS OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS II

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Abstract: We define a new subclass of uniformly starlike and convex functions with negative coefficient, the main object of this paper is to obtain coefficient estimates such as distortion bounds, closure theorems and extreme points for functions belonging to this new class.

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1. Introduction

For $0 < p < 1$, let S_p denote the class of functions f which are meromorphic and univalent in the unit disk $D = \{z : |z| < 1\}$ with the normalization $f(0) = 0, f'(0) = 1$ and $f(p) = \infty$.

Let $A(p)$ denote the set of function analytic in $D\backslash\{p\}$ with the topology given by uniform convergence on compact subsets of $D\backslash\{p\}$. Then $A(p)$ is locally convex linear topological space and S_p is a compact subset of $A(p)$ (cf. [23], p. 55). In the annulus $\{z : p < |z| < 1\}$ every function f in S_p has an expansion of the form

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$$f(z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

where $\alpha = Res(z,p)$, with $0 < \alpha \le 1$ and $z \in D\backslash\{p\}$.

The function f given in (1.1) was studied by Jinxi Ma [12] and Ghanim and Darus [7]. The functions f in S_p is said to be meromorphically starlike functions of order β if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \beta \qquad (z \in D\backslash\{p\}) \tag{1.2}$$

for some β $(0 \le \beta < 1)$. We denote by $S_p^*(\beta)$ the class of all meromorphically starlike functions of order β. Similarly, a function f in S is said to be meromorphically convex of order β if and only if

$$Re\left\{-1-\frac{zf''(z)}{f'(z)}\right\} > \beta \qquad (z \in D\backslash\{p\}) \tag{1.3}$$

for some β $(0 \le \beta < 1)$. We denote by $S_p^c(\beta)$ the class of all meromorphically convex functions of order β. We note that the class $S_0^*(\beta)$ and various other subclasses of $S_0^*(0)$ have been studied rather extensively by Nehari and Netanyahu [17], Clunie [4], Pommerenke [18] [19], Miller [13], Royster [22], and others (cf., e.g., Bajpai [2], Goel and Sohi [9], Mogra et al [14], Uralegaddi and Ganigi [25], Cho et al [3], Aouf [1], and Uralegaddi and Somantha [26], [27]; see also Duren ([5], pp. 29 and 137), and Srivastava and Owa ([24], pp. 86 and 429).

For the function f in the class S_p we define the following:

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = zf'(z) + \frac{\alpha(2z-p)}{(z-p)^2},$$

$$I^2 f(z) = z\left(I^1 f(z)\right)' + \frac{\alpha(2z-p)}{(z-p)^2},$$

and for $k = 1, 2, 3, \ldots$ we can write

$$I^k f(z) = z\left(I^{k-1} f(z)\right)' + \frac{\alpha(2z-p)}{(z-p)^2} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} n^k a_n z^n. \tag{1.4}$$

Ghanim and Darus [7] have studied this operator extensively, and also, for $p = 0$ and $\alpha = 1$, the differential operator I^k reduced to Frasin and Darus [6].

We also let T_p the subclass of S_p consisting of functions of the form

$$f(z) = \frac{\alpha}{z-p} - \sum_{n=1}^{\infty} a_n z^n \qquad (a_n > 0) \tag{1.5}$$

and for $k = 1, 2, 3, \ldots$ we can write

$$I^k f(z) = z\left(I^{k-1}f(z)\right)' + \frac{\alpha(2z-p)}{(z-p)^2} = \frac{\alpha}{z-p} - \sum_{n=1}^{\infty} n^k a_n z^n \quad (a_n > 0). \quad (1.6)$$

With the help of the differential operator I^k, we define the class $ST_p^*(k,\alpha,\xi,\phi)$ as follows:

Definition 1.1. The function $f \in T_p$ is said to be a member of the class $ST_p^*(k,\alpha,\xi,\phi)$ if it satisfies

$$\Re\left\{\frac{z\left(I^k f(z)\right)'}{I^k f(z)} - \xi\right\} > \phi\left|\frac{z\left(I^k f(z)\right)'}{I^k f(z)} - 1\right|, \quad z \in D/\{p\} \quad (1.7)$$

for some ξ $(-1 < \xi \le 1)$ and $\phi \ge 0$ for all z in $D\backslash\{p\}$.

The above criterion with the function of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ was studied by Goodman [10], [11], Rønning [20], [21], and Murugusundaramoorthy and Magesh [15], [16].

The main object of this paper is to obtain necessary and sufficient conditions for the functions $f \in ST_p^*(k,\alpha,\xi,\phi)$. Furthermore we obtain extreme points, distortion bounds and closure properties for the class $ST_p^*(k,\alpha,\xi,\phi)$.

2. Characterization

In this section we obtain necessary and sufficient conditions for functions f in the class $ST_p^*(k,\alpha,\xi,\phi)$.

Theorem 2.1. *A function f of the form (1.5) is in $ST_p^*(k,\alpha,\xi,\phi)$ if*

$$\sum_{n=1}^{\infty} n^k \left[(1+\phi)(n-1) + (1-\xi)\right](1-p)\left|a_n\right| \le \alpha(1-\xi) \quad (2.1)$$

for some $0 < \alpha \le 1$, $-1 < \xi \le 1$, $\phi \ge 0$ and $k \ge 0$.

Proof. It suffices to show that

$$\phi\left|\frac{z\left(I^k f(z)\right)'}{I^k f(z)} - 1\right| - \Re\left\{\frac{z\left(I^k f(z)\right)'}{I^k f(z)} - 1\right\} \le 1 - \xi\,.$$

We have

$$\phi\left|\frac{z\left(I^k f(z)\right)'}{I^k f(z)} - 1\right| - \Re\left\{\frac{z\left(I^k f(z)\right)'}{I^k f(z)} - 1\right\}$$

$$\leq (1+\phi)\left|\frac{z\left(I^k f(z)\right)'}{I^k f(z)}-1\right| \leq \frac{(1+\phi)\sum\limits_{n=1}^{\infty} n^k\,(n-1)\,|a_n|\,|z|^n}{\frac{\alpha}{|z|-p}-\sum\limits_{n=1}^{\infty} n^k\,|a_n|\,|z|^n}\,,$$

then for $|z| \longrightarrow 1$, and $|z-p| \geq |z|-p = 1-p$, we have

$$\leq \frac{(1+\phi)\sum\limits_{n=1}^{\infty} n^k\,(n-1)\,|a_n|}{\frac{\alpha}{1-p}-\sum\limits_{n=1}^{\infty} n^k\,|a_n|} \leq 1-\xi\,,$$

or

$$\sum_{n=1}^{\infty} n^k\left[(1+\phi)\,(n-1)+(1-\xi)\right](1-p)\,|a_n| \leq \alpha\,(1-\xi)\ ,$$

and hence the proof is complete. □

Theorem 2.2. *A necessary and sufficient condition for f of the form (1.5) to be in the class $ST_p^*(k,\alpha,\xi,\phi)$, $0<\alpha\leq 1$, $-1<\xi\leq 1$, $\phi\geq 0$ and $k\geq 0$ is:*

$$\sum_{n=1}^{\infty} n^k\left[(1+\phi)\,(n-1)+(1-\xi)\right](1-p)\,|a_n| \leq \alpha\,(1-\xi)\,. \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f \in ST_p^*(k,\alpha,\xi,\phi)$ and z is real, then

$$\frac{\frac{\alpha}{z-p}-\sum\limits_{n=1}^{\infty} n^{k+1}a_n z^n}{\frac{\alpha}{z-p}-\sum\limits_{n=1}^{\infty} n^k a_n z^n}-\xi \geq \frac{\phi\sum\limits_{n=1}^{\infty} n^k\,(n-1)\,a_n z^n}{\frac{\alpha}{z-p}-\sum\limits_{n=1}^{\infty} n^k a_n z^n}\,.$$

Let $z \to 1$ along the real axis, then we get

$$\frac{\frac{\alpha}{1-p}-\sum\limits_{n=1}^{\infty} n^{k+1}a_n}{\frac{\alpha}{1-p}-\sum\limits_{n=1}^{\infty} n^k a_n}-\frac{\phi\sum\limits_{n=1}^{\infty} n^k\,(n-1)\,a_n}{\frac{\alpha}{1-p}-\sum\limits_{n=1}^{\infty} n^k a_n} \geq \xi\,,$$

or

$$-\frac{\alpha}{1-p}+\sum_{n=1}^{\infty} n^k\left[n+n\phi-\phi\right]a_n \leq -\frac{\alpha\xi}{1-p}+\sum_{n=1}^{\infty} n^k \xi a_n$$

which gives the required result. □

Remark 1. Our assertions in Theorem 2.1 and Theorem 2.2 are sharp, for functions of the form

$$f(z)=\frac{\alpha}{z-p}-\frac{\alpha(1-\xi)}{n^{k}\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)}z^{n},\quad n\geq 1, \tag{2.3}$$

which belong to the class $ST_p^*(k,\alpha,\xi,\phi)$.

Remark 2. The class $ST_p^*(k,\alpha,\xi,1)=ST_p^*(k,\alpha,\xi)$ studied by Ghanim and Darus [8]

Corollary 2.3. *Let the function f defined by (1.5) be in the class $ST_p^*(k,\alpha,\xi,\phi)$. Then*

$$a_n\leq\frac{\alpha(1-\xi)}{n^{k}\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)}, \tag{2.4}$$

$n\geq 1,\quad 0<\alpha\leq 1,\quad -1<\xi\leq 1,$ $\phi\geq 0$ *and* $k\geq 0$.

Equality holds for the functions

$$f(z)=\frac{\alpha}{z-p}-\frac{\alpha(1-\xi)}{n^{k}\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)}z^{n},\quad n\geq 1.$$

Proof. Since $f\in ST_p^*(k,\alpha,\xi,\phi)$, Theorem 2.1 gives

$$\sum_{n=1}^{\infty}n^{k}\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)\left|a_n\right|\leq\alpha(1-\xi).$$

Next, note that

$$n^{k}\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)a_n \leq\sum_{n=1}^{\infty}n^{k}\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)a_n\leq\alpha(1-\xi).$$

Therefore

$$a_n\leq\frac{\alpha(1-\xi)}{n^{k}\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)},\quad n\geq 1.$$

□

Theorem 2.4. *Let f defined by (1.5) and g defined by $g(z)=\frac{\alpha}{z-p}-\sum_{n=1}^{\infty}b_nz^n$ be in the class $ST_p^*(k,\alpha,\xi,\phi)$. Then the function $h(z)$ defined by*

$$h(z)=(1-\lambda)f(z)+\lambda g(z)=\frac{\alpha}{z-p}-\sum_{n=1}^{\infty}q_nz^n,$$

where $q_n=(1-\lambda)a_n+\lambda b_n,\quad 0\leq\lambda<1$ is also in the class $ST_p^*(k,\alpha,\xi,\phi)$.

We prove the following theorem by defining

$$f_j(z)=\frac{1}{z-p}-\sum_{n=1}^{\infty}a_{n,j}z^n\qquad\left(a_{n,j}>0,\quad z\in D/\{p\}\right). \tag{2.5}$$

Theorem 2.5. (Closure Theorem) *Let the functions $f_j(z)$ $(j = 0, 1, 2, \ldots, m)$ defined by (2.5) be in the classes $ST_p^*(k, \alpha, \xi, \phi)$ $(j = 0, 1, 2, \ldots, m)$ respectively. Then the function $h(z)$ defined by*

$$h(z) = \frac{\alpha}{z-p} - \frac{1}{m} \sum_{n=1}^{\infty} \left(\sum_{j=0}^{m} a_{n,j} \right) z^n \tag{2.6}$$

is in the class $ST_p^*(k, \alpha, \xi, \phi)$, where $\xi = \min_{0 \le j \le m} \{\xi_j\}$ where $-1 \le \xi_j < 1$.

Proof. Since $f_j(z) \in ST_p^*(k, \alpha, \xi, \phi)$, $(j = 0, 1, 2, 3, \ldots, m)$ by applying Theorem 2.2, to (2.5) we observe that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^k \left[(1+\phi)(n-1) + (1-\xi)\right](1-p) \left(\frac{1}{m} \sum_{j=0}^{m} a_{n,j} \right) \\ &\qquad = \frac{1}{m} \sum_{j=0}^{m} \left(\sum_{n=1}^{\infty} n^k \left[(1+\phi)(n-1) + (1-\xi)\right](1-p)\, a_{n,j} \right) \\ &\qquad \le \frac{1}{m} \sum_{j=0}^{m} \alpha (1-\xi_j) \le \alpha (1-\xi) \,, \end{aligned}$$

which in view of Theorem 2.2, again implies that $h(z) \in ST_p^*(k, \alpha, \xi, \phi)$ and so the proof is complete. □

3. Distortion and Covering Theorems

Theorem 3.1. *Let the function f defined by (1.5) be in the class $ST_p^*(k, \alpha, \xi, \phi)$. Then*

$$\frac{\alpha}{r-p} - \frac{\alpha}{(1-p)} r \le |f(z)| \le \frac{\alpha}{r-p} + \frac{\alpha}{(1-p)} r \,.$$

Equality holds for the function

$$f(z) = \frac{\alpha}{z-p} - \frac{\alpha}{(1-p)} z.$$

Proof. Since $f \in ST_p^*(k, \alpha, \xi, \phi)$, by Theorem 2.1,

$$\sum_{n=1}^{\infty} n^k \left[(1+\phi)(n-1) + (1-\xi)\right](1-p)\,|a_n| \le \alpha (1-\xi) \,.$$

Now

$$[(1-p)(1-\xi)]\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} [(1-p)(1-\xi)]\, a_n$$
$$\leq \sum_{n=1}^{\infty} n^k \left[(1+\phi)(n-1)+(1-\xi)\right](1-p)\left|a_n\right| \leq \alpha(1-\xi)$$

and therefore

$$\sum_{n=1}^{\infty} a_n \leq \frac{\alpha}{(1-p)}.$$

Since $f(z) = \frac{\alpha}{z-p} - \sum_{n=1}^{\infty} a_n z^n$,

$$|f(z)| = \left|\frac{\alpha}{z-p} - \sum_{n=1}^{\infty} a_n z^n\right| \leq \frac{\alpha}{|z|-p} + \sum_{n=1}^{\infty} a_n |z|^n$$
$$\leq \frac{\alpha}{r-p} + r\sum_{n=1}^{\infty} a_n = \frac{\alpha}{r-p} + \frac{\alpha}{(1-p)} r$$

and

$$|f(z)| = \left|\frac{\alpha}{z-p} - \sum_{n=1}^{\infty} a_n z^n\right| \geq \frac{\alpha}{|z|-p} - \sum_{n=1}^{\infty} a_n |z|^n$$
$$\geq \frac{\alpha}{r-p} - r\sum_{n=1}^{\infty} a_n = \frac{\alpha}{r-p} - \frac{\alpha}{(1-p)} r$$

which yields the theorem. □

Theorem 3.2. *Let the function f defined by (1.5) be in the class $ST_p^*(k,\alpha,\xi,\phi)$. Then*

$$\frac{\alpha}{(r-p)^2} - \frac{\alpha}{(1-p)} \leq \left|f'(z)\right| \leq \frac{\alpha}{(r-p)^2} + \frac{\alpha}{(1-p)}. \tag{3.1}$$

The equality holds for the function

$$f(z) = \frac{\alpha}{z-p} - \frac{\alpha}{(1-p)} z.$$

Proof. We have

$$\left|f'(z)\right| = \left|\frac{-\alpha}{(z-p)^2} - \sum_{n=1}^{\infty} n a_n z^{n-1}\right|$$
$$\leq \frac{\alpha}{(|z|-p)^2} + \sum_{n=1}^{\infty} n a_n |z|^{n-1} \leq \frac{\alpha}{(r-p)^2} + \sum_{n=1}^{\infty} n a_n \tag{3.2}$$

Since, $f \in ST_p^*(k, \alpha, \xi, \phi)$, we have

$$[(1-p)(1-\xi)] \sum_{n=1}^{\infty} n a_n \leq n^{k-1} [(1+\phi)(n-1) + (1-p)(1-\xi)] \sum_{n=1}^{\infty} n a_n$$
$$\leq \sum_{n=1}^{\infty} n^{k-1} [(1+\phi)(n-1) + (1-\xi)] (1-p) n a_n \leq \alpha (1-\xi).$$

Hence,

$$\sum_{n=1}^{\infty} n a_n \leq \frac{\alpha}{(1-p)}. \tag{3.3}$$

Substituting (3.2) in (3.3), we get

$$\left| f'(z) \right| \leq \frac{\alpha}{(r-p)^2} + \frac{\alpha}{(1-p)}.$$

Similarly,

$$\left| f'(z) \right| \geq \frac{\alpha}{(r-p)^2} - \frac{\alpha}{(1-p)}.$$

This completes the proof. □

4. Radii of Starlikeness and Convexity

The radii of Starlikeness and convexity for the class for the class $ST_p^*(k, \alpha, \xi, \phi)$ is given by the following theorems.

Theorem 4.1. *If the function f be defined by (1.5) is in the class $ST_p^*(k, \alpha, \xi, \phi)$ then f is meromorphically starlike of order $\delta (0 \leq \delta < 1)$ in $|z-p| < |z| < r_1$, where*

$$r_1 = r_1(k, \alpha, \xi, \phi)$$
$$= \inf_{n \geq 1} \left\{ \frac{n^k (1-\delta)(1-p)[(1+\phi)(n-1) + (1-\xi)]}{(1-\xi)(n+2-\delta)} \right\}^{\frac{1}{n+1}}. \tag{4.1}$$

The result is sharp for the function f given by (2.3).

Proof. It sufficient to prove that

$$\left| \frac{z f'(z)}{f(z)} + 1 \right| < 1 - \delta \tag{4.2}$$

for $|z-p|<|z|<r_1$. We have

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{\sum\limits_{n=1}^{\infty}(n+1)\,a_n z^n}{\frac{\alpha z}{(z-p)^2}+\sum\limits_{n=1}^{\infty}a_n z^n}\right| \leq \frac{\sum\limits_{n=1}^{\infty}(n+1)\,a_n\,|z|^n}{\frac{\alpha|z|}{|z-p|^2}-\sum\limits_{n=1}^{\infty}a_n\,|z|^n}. \tag{4.3}$$

Hence (4.3) holds true if

$$\sum_{n=1}^{\infty}(n+1)\,a_n\,|z|^n \leq (1-\delta)\left(\frac{\alpha\,|z|}{|z-p|^2}-\sum_{n=1}^{\infty}a_n\,|z|^n\right) \tag{4.4}$$

or

$$\frac{|z-p|^2\sum\limits_{n=1}^{\infty}(n+2-\delta)\,a_n\,|z|^{n-1}}{(1-\delta)\,\alpha} \leq 1 \tag{4.5}$$

with the aid of (2.1), (4.5) is true if

$$\frac{|z-p|^2\sum\limits_{n=1}^{\infty}(n+2-\delta)\,|z|^{n-1}}{\alpha\,(1-\delta)} \leq \frac{n^k\left[(1+\phi)\,(n-1)+(1-\xi)\right](1-p)}{\alpha\,(1-\xi)}, \tag{4.6}$$

$n \geq 1$.

Solving (4.6) for $|z-p|<|z|$, we obtain

$$|z| < \left\{\frac{n^k\,(1-\delta)\,(1-p)\left[(1+\phi)\,(n-1)+(1-\xi)\right]}{(1-\xi)\,(n+2-\delta)}\right\}^{\frac{1}{n+1}}. \tag{4.7}$$

This completes the proof of Theorem 4.1. □

Corollary 4.2. *Let $\alpha=1$ in Theorem 4.1, then we have*

$$|z| < \left\{\frac{n^k\,(1-\delta)\,(1-p)\left[(1+\phi)\,(n-1)+(1-\xi)\right]}{(1-\xi)\,(n+2-\delta)}\right\}^{\frac{1}{n+1}}. \tag{4.8}$$

Theorem 4.3. *If the function f be defined by (1.5) is in the class $ST_p^*(k,\alpha,\xi,\phi)$ then f is meromorphically convex of order δ $(0\leq\delta<1)$ in $|z-p|<|z|<r_2$, where*

$$r_2 = r_2\,(k,\alpha,\xi,\phi) = \inf_{n\geq 1}\left\{\frac{n^{k-1}\,(1-\delta)\,(1-p)\left[(1+\phi)\,(n-1)+(1-\xi)\right]}{(1-\xi)\,(n+2-\delta)}\right\}^{\frac{1}{n+1}}. \tag{4.9}$$

The result is sharp for the function f given by (2.3).

Proof. By using the technique employed in proof of Theorem 4.1, we can show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq (1-\delta) \tag{4.10}$$

for $|z-p|<|z|<r_2$, with the aid of Theorem 2.1. Thus we have the assertion of Theorem 4.3. □

Corollary 4.4. *Let $\alpha = 1$ in Theorem 4.2, then we have*

$$|z| \leq \left\{\frac{n^{k-1}(1-\delta)(1-p)\left[(1+\phi)(n-1)+(1-\xi)\right]}{(1-\xi)(n+2-\delta)}\right\}^{\frac{1}{n+1}}. \tag{4.11}$$

5. Convex Linear Combinations

Our next result involves linear combinations of several functions of the type (2.3).

Theorem 5.1. *Let*

$$f_0(z)=\frac{\alpha}{z-p} \tag{5.1}$$

and

$$f_n(z)=\frac{\alpha}{z-p}-\frac{\alpha(1-\xi)}{n^k\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)}z^n, \tag{5.2}$$

$n \geq 1$, $0<\alpha \leq 1$, $-1<\xi \leq 1$, $\phi \geq 0$ *and* $k \geq 0$.

Then $f \in ST_p^*(k,\alpha,\xi,\phi)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z), \tag{5.3}$$

where $\lambda_n \geq 0$ and $\sum\limits_{n=0}^{\infty}\lambda_n = 1$.

Proof. From (5.1), (5.2) and (5.3), it is easily seen that

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)$$

$$=\frac{\alpha\lambda_n}{z-p}-\frac{\alpha(1-\xi)\lambda_n}{n^k\left[(1+\phi)(n-1)+(1-p)(1-\xi)\right]}z^n. \tag{5.4}$$

Since

$$\sum_{n=1}^{\infty}\frac{n^k\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)}{\alpha(1-\xi)}\lambda_n\frac{\alpha(1-\xi)}{n^k\left[(1+\phi)(n-1)+(1-\xi)\right](1-p)}$$

$$= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1$$

it follows from Theorem 2.1 and Theorem 2.2 that the function $f \in ST_p^*(k, \alpha, \xi, \phi)$.

Conversely, let us suppose that $f \in ST_p^*[k, \alpha, \xi, \phi]$. Since

$$a_n \leq \frac{\alpha (1-\xi)}{n^k [(1+\phi)(n-1) + (1-\xi)] (1-p)},$$

$n \geq 1$, $0 < \alpha \leq 1$, $-1 < \xi \leq 1$, $\phi \geq 0$ and $k \geq 0$. Setting

$$\lambda_n = \frac{n^k [(1+\phi)(n-1) + (1-\xi)] (1-p)}{\alpha (1-\xi)}$$

$n \geq 1$, $k \geq 0$ and $\lambda_0 = 1 - \sum\limits_{n=1}^{\infty} \lambda_n$ it follows that $f(z) = \sum\limits_{n=0}^{\infty} \lambda_n f_n(z)$. This completes the proof of the theorem. □

Finally, we prove the following:

Theorem 5.2. *The class $ST_p^*(k, \alpha, \xi, \phi)$ is closed under convex linear combinations.*

Proof. Suppose that the function $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_{n,j} z^n \qquad (j = 1, 2; z \in D \backslash \{p\}) \tag{5.5}$$

are in the class $ST_p^*(k, \alpha, \xi, \phi)$. Setting

$$f(z) = \mu f_1(z) + (1-\mu) f_2(z) \qquad (0 \leq \mu < 1), \tag{5.6}$$

we find from (5.5) that

$$f(z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} \{\mu a_{n,1} + (1-\mu) a_{n,2}\} z^n \tag{5.7}$$

$((0 \leq \mu < 1),\ z \in D \backslash \{p\})$.

In view of Theorem 2.2, we have

$$\sum_{n=1}^{\infty} n^k (1-p) [(1+\phi)(n-1) + (1-\xi)] (\mu a_{n,1} + (1-\mu) a_{n,2})$$

$$= \mu \sum_{n=1}^{\infty} n^k [(1+\phi)(n-1) + (1-\xi)] (1-p) a_{n,1} +$$

$$(1-\mu) \sum_{n=1}^{\infty} n^k [(1+\phi)(n-1) + (1-\xi)] (1-p) a_{n,2}$$

$$\leq \mu\alpha\left(1-\xi\right)+\left(1-\mu\right)\alpha\left(1-\xi\right)=\alpha\left(1-\xi\right).$$

which shows that $f \in ST_p^*(k, \alpha, \xi, \phi)$. Hence the theorem.

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