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A COMPARISON BETWEEN THE SELBERG AND THE BRUGGEMAN-KUZNETSOV TRACE FORMULAS

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Abstract: In this paper we further elucidate and make somewhat transparent the clever technique of first introducing and then removing weights (Fourier coefficients of eigenfunctions) when employing the Bruggeman-Kuznetsov trace formula to obtain information on the distribution of the eigenvalues of the hyperbolic Laplacian for the modular group.

Frequently, this technique yields improvement of results obtained by the Selberg trace formula. This gain is realized because the sums on the geometric side of the Bruggeman-Kuznetsov trace formula involve sums and integrals, which apparently package certain cancellations in a more efficient way than do the sums involving class numbers, which appear naturally on the geometric side of the Selberg trace formula.

We do this by obtaining meaningful expansions as T goes to infinity for two functions $E^*(\alpha, T)$ and $F^*(\alpha, T)$ for some $\alpha \in \mathbb{R}$ by means of the Bruggeman-Kuznetsov trace formula. In two previous papers we have shown that one is not able to obtain a meaningful expansion for corresponding functions $E(\alpha, T)$ and $F(\alpha, T)$ by means of the Selberg trace formula because of the limitations in the presently available technique for estimating the hyperbolic classes contribution to said formula.

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1. Introduction

We consider $\Gamma = \mathrm{PSL}(2, Z)$.

Hence we have that

$$\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

are the eigenvalues (i.e. the point spectrum) of the hyperbolic Lacplacian associated with Γ and

$$\lambda_j = s_j(1 - s_j) \qquad \text{with} \qquad s_j = \frac{1}{2} + t_j \, ,$$

so

$$\lambda_j = \frac{1}{4} + t_j^2 \qquad \text{with} \qquad t_j > 0 \quad j = 1, 2, 3, \dots \, .$$

The Weyl-Selberg formula states that for $\Gamma = \mathrm{PSL}(2, Z)$

$$M(T) = \sum_{0 < t_j \le T} m(t_j) = \frac{1}{12} T^2 + c_1 T \log T + O(T), \tag{1.1}$$

where $m(t_j)$ denotes the multiplicity of λ_j, which is equal to the dimension of the eigenspace of λ_j (cf [11]).

In [7] we made the following

Observation 1.1. For fixed $\alpha > 0$ and $T \ge 2$, let

$$E(\alpha, T) = \sum_{0 < t_j} e^{-\frac{t_j^2}{T^2}} \cos(\alpha T t_j).$$

Then

$$E(\alpha, T) = O\left(e^{\left(\frac{\alpha T}{2} + \frac{1}{4T^2} \right)} \right) \quad \text{as} \quad T \to \infty,$$

where the implied constant depends only on α.

To establish the observation we employed the Selberg trace formula with the appropriate test function.

For each α, T let

$$h(t) = h(t, \alpha, T) = e^{-\frac{t^2}{T^2}} e^{i\alpha T t} + e^{-\frac{t^2}{T^2}} e^{-i\alpha T t} = h_+(t, \alpha, T) + h_-(t, \alpha, T).$$

Clearly,

$$h(t) = h(t, \alpha, T) = 2e^{-\frac{t^2}{T^2}} \cos(\alpha T t), \tag{1.2}$$

and the $h(t, \alpha, T)$ satisfy the Selberg conditions for the Selberg trace formula.

In [8] we made the following

Observation 1.2. For fixed $\alpha > 0$ and $T \geq 2$ let

$$F(\alpha, T) = \sum_j W\left(\frac{t_j}{T}\right)\cos(\alpha T t_j),$$

where $W(u) = (1+u^4)^{-1}$. Then

$$F(\alpha, T) = O\left(e^{(1+\epsilon)\frac{\alpha T}{2}}\right),$$

where the implied constant depends only on α.

To establish the observation we employed the Selberg trace formula with the appropriate test function.

For each α, T let

$$h(t) = h(t, \alpha, T) = W\left(\frac{t}{T}\right)e^{i\alpha T t} + W\left(\frac{t}{T}\right)e^{-i\alpha T t} = h_+(t, \alpha, T) + h_-(t, \alpha, T).$$

Clearly,

$$h(t) = h(t, \alpha, T) = 2W\left(\frac{t}{T}\right)\cos(\alpha T t), \tag{1.3}$$

and the $h(t, \alpha, T)$ satisfy the Selberg conditions for the Selberg trace formula.

The weakness of both observations is because of the limitations in the presently available technique for estimating the hyperbolic classes contribution to the Selberg trace formula.

In this paper we establish by means of the Bruggeman-Kutznetsov trace formula the following two theorems:

Theorem 1.1. *For fixed $\alpha \in R$ $|\alpha| \leq (4-\epsilon_0)$ for every $\epsilon_0 > 0$*

$$E^*(\alpha, T) = \frac{T^2}{12}\int_0^\infty xW(x)\cos(\alpha T \log Tx)dx + O(T^{2-\epsilon}),$$

where $W(x) = e^{-x^2}$.

Theorem 1.2. *For fixed $\alpha \in R$ $|\alpha| \leq (4-\epsilon_0)$ for every $\epsilon_0 > 0$*

$$F^*(\alpha, T) = \frac{T^2}{12}\int_0^\infty xW(x)\cos(\alpha T \log Tx)dx + O(T^{2-\epsilon}),$$

where $W(x) = (1+x^4)^{-1}$.

Since the proof of Theorem 1.2 differs from the proof of Theorem 1.1 insignificantly, we only present the proof of Theorem 1.1. Anyone who understands that proof should have no difficulty whatsoever modifying it to establish Theorem 1.2.

In both theorems the $*$ denotes that the corresponding test functions are obtained by changing the T to $\log T$ in the exponential factors in the above-defined corresponding test functions.

2. The Weighted Bruggeman-Kuznetsov Trace Formula

Theorem 2.1. (Bruggeman-Kuznetsov) *Let h satisfy the Selberg trace formula conditions* (cf. [2]). *Then*

$$\sum_{1\leq j} h(t_j)\bar{\mathcal{V}}_j(m)\mathcal{V}_j(n) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\bar{\mathcal{N}}(m,t)\mathcal{N}(n,t)dt$$
$$= \delta_{mn}\frac{1}{\pi}\int_{-\infty}^{\infty} t\tanh(\pi t)h(t)dt \ + \sum_{c=1}^{\infty}\frac{s(m,n,c)}{c}h^{+}\left(\frac{4\pi\sqrt{|mn|}}{c}\right),$$

where $h^{+}(x) = 2i\int_{-\infty}^{\infty} J_{2it}(x)\frac{h(t)t}{\cosh \pi t}dt$ and where $\mathcal{V}_j(n)$, $\mathcal{V}_j(m)$, $\mathcal{N}(n,t)$, $\mathcal{N}(m,t)$ are Fourier coefficients of any arithmetical system of Maass forms, and the eigenpacket of Eisenstein series.

For a nice proof of this theorem, see [2].

Now choose the Hecke basis $\{\phi_j\}$. Then

$$\mathcal{V}_j(n) = \lambda_j(n)\gamma_{\phi_j} \quad \text{and} \quad \mathcal{V}_j(m) = \lambda_j(m)\gamma_{\phi_j},$$

where

$$\gamma_{\phi_j} = \left(\frac{L(1, \operatorname{sym}^2\phi_j)}{2\pi}\right)^{1/2}, \quad \text{and} \quad \lambda_j(1) = 1.$$

Letting $n = m$ we get the so-called weighted Bruggeman-Kuznetsov trace formula.

Theorem 2.2.

$$\sum_{1\leq j} h(t_j)|\lambda_j(n)|^2\gamma_{\phi_j}^2 + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)|\mathcal{N}(n,t)|^2dt$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} t\tanh(\pi t)h(t)dt + \sum_{c=1}^{\infty}\frac{s(n,n,c)}{c}h^{+}\left(\frac{4\pi n}{c}\right).$$

The techniques used in this paper, as described in the abstract, were first used by Iwaniec in [3]. They were refined and elaborated by Sarnak in [10]. See also [5] and [9].

3. Lemmata

Lemma 3.0.

$$h(z,\alpha,T) \quad \textit{is even}\,,$$
$$h(z,\alpha,T) \quad \textit{is entire}\,,$$
$$h(z,\alpha,T) \ll T^{|\alpha|\;|y|} e^{\frac{y^2-x^2}{T^2}}\,.$$

Proof. The proof follows by direct calculation. Note that the lemma is true if $h(z,\alpha,T)$ is replaced with $h_+(z,\alpha,T)$ or $h_-(z,\alpha,T)$. □

Lemma 3.1. $t_j^{-\epsilon} \ll_\epsilon |\gamma_{\phi_j}|^2 \ll \log t_j$.

Proof. This is established in [6]. □

Lemma 3.2. $\sum_{0<t_j\leq T^2} |h(t_j,\alpha,T)| = O(T^2)$.

Proof. The proof follows from (1.1), by partial summation. □

Let

$$S(\phi_j) = \sum_{n=1}^{\infty} \gamma_{\phi_j}^2 |\lambda_j(n)|^2 g\left(\frac{n}{M}\right), \tag{3.1}$$

where $M = T^\Delta$ for $0 < \Delta < \frac{1}{100}$ and $g \in C_c^\infty(1,3)$, $g(x) \geq 0$.

Lemma 3.3. $S(\phi_j) = E_1(t_j)$ *where* $E_1(t_j) = O(M^{\frac{s}{4}}|t_j^\epsilon|)$.

Proof. This is immediate from the estimate due to Kim and Sarnak (cf. [2])

$$|\lambda_j(n)| \ll n^{\frac{7}{64}+\epsilon}$$

and Lemma 3.1. □

In Lemma 1 in [6] let $\sigma = 1-\delta$, where $\delta > 0$ is chosen so that $(\delta b+\epsilon) < \frac{1}{10}$. Consider $0 < t_j \leq T^2$. Define $R = \{\rho = \beta + i\gamma \mid 0 < \delta \leq 1/2, (1-\delta) \leq \beta < 1, |\gamma| \leq \log^3 T^2\}$.
Define $B = \{t_j \mid 0 < t_j \leq T^2$ and $L(\rho, \mathrm{sym}^2\phi_j) = 0$ if $\rho \in R\}$. By Lemma 1 in [6] we have

$$\sum_{t_j \in B} m(t_j) \ll T^{\frac{1}{10}}. \tag{3.2}$$

Define $G = \{t_j | 0 < t_j \leq T^2\} - B$. Clearly, if $t_j \in G$, then $L(s, \mathrm{sym}^2\phi_j)$ has no zero in the domain $(1-\delta) < \sigma < 1$, $|t| \leq \log^3 T^2$. Hence by Lemma 2 in [6] in the domain $(1-\delta/2) < \sigma < 1$, $|t| \le \log^2 T$ we have

$$L(s, \mathrm{sym}^2\phi_j) \ll_\epsilon T^\epsilon \qquad \text{for any } \epsilon > 0. \tag{3.3}$$

Lemma 3.4. *If $t_j \in G$, $\sum_{n=1}^{\infty} |\lambda_j(n)|^2 g\left(\frac{n}{M}\right) = \frac{ML(1,\mathrm{sym}^2\phi_j)}{\zeta(2s)} + O\left(M^{1-\delta}T^\epsilon\right)$.*

Proof.

$$S = \sum_{n=1}^{\infty} |\lambda_j(n)|^2 g\left(\frac{n}{M}\right) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, s=2} \frac{L(s,\mathrm{sym}^2\phi_j)}{\zeta(s)} \tilde{g}(s) M^s ds\,,$$

where $\tilde{g}(s) = \int_0^\infty g(x)x^{s-1}dx$. By moving the line of integration to $(1-\delta)$ we get

$$S = \frac{ML(1,\mathrm{sym}^2\phi_j)}{\zeta(2)} + \frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, s=1-\delta} \frac{L(s,\mathrm{sym}^2\phi_j)}{\zeta(2s)} \tilde{g}(s) M^s ds,$$

and the result follows from (3.3). □

Lemma 3.5. *If $t_j \in G$, then*

$$S(\phi_j) = \frac{12}{\pi} M + E_2(T)\,, \qquad \textit{where} \qquad E_2(T) = O\left(M^{1-\delta}T^\epsilon\right)\,.$$

Proof. This follows immediately from Lemma 3.4 and the fact that $\zeta(2) = \frac{\pi^2}{6}$, $\gamma_{\phi_j}^2 = \frac{2\pi}{L(1,\mathrm{sym}^2\phi_j)}$ and, by Lemma 3.1, $\gamma_{\phi_j}^2 \ll t_j^\epsilon$. □

Lemma 3.6. $\sum_{T^2 < t_j} h(t_j, \alpha, T) = O(1)$.

Proof. The proof follows from (1.1) and partial summation. □

Lemma 3.7. $\sum_{T^2 < t_j} S(\phi_j) h(t_j, \alpha, T) = O(M^{5/4}T^\epsilon)$.

Proof. The proof follows from (1.1), Lemma 3.3 and partial summation. □

Lemma 3.8. $\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) = M + O(1)$.

Proof. Immediate by partial summation, and the trivial observation $\Delta(x) = \sum_{n=1}^{x} 1 = x + O(1)$. □

Lemma 3.9.

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} t \tanh(\pi t) h(t, \alpha, T) dt \right)$$

$$= \frac{4MT^2}{\pi} \int_0^\infty xW(x)\cos(\alpha T \log Tx)dx + O(T^2).$$

Proof. The proof follows in a straightforward manner from the easily established decomposition

$$\frac{2}{\pi}\int_0^\infty t\, h_+(t,\alpha,T)dt + \frac{2}{\pi}\int_0^\infty t\, h_-(t,\alpha,T)dt + O(1)$$

and Lemma 3.8. □

Lemma 3.10. *Assume $0 < x$ and $0 < t$. Let $z = (\frac{x^2}{4} + t^2)^{1/2}$ Then*

$$J_{2it}(x) = \pi^{-1/2} z^{-1/2} e^{\frac{-\pi i}{4}} e^{\pi t} e\left(\frac{z}{\pi} - \frac{t}{\pi}\log\left(\frac{2(z+t)}{x}\right)\right) \cdot \left\{1 + O\left(\frac{1}{t}\right)\right\}$$

Proof. This is established in [1]. □

Clearly, $h_T^+(x) = 2i \int_{-\infty}^{\infty} J_{2it}(x)\left(\frac{h_+(t,\alpha,T)+h_-(t,\alpha,T)}{\cosh \pi t}\right) t\, dt$. Using the fact that $J_{2it}(x) = \overline{J_{-2it}(x)}$, we have

$$h_T^+(x) = 4i^2 \mathrm{Im} \int_{-\infty}^{\infty} J_{2it}(x) \frac{h_+(t,\alpha,T)t}{\cosh \pi t}\, dt.$$

Lemma 3.11. *For every $0 < \epsilon < \frac{1}{100}$ there exists $0 < \Delta(\epsilon) < \frac{1}{100}$ such that if $0 < \Delta < \Delta(\epsilon)$*

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \sum_{c=1}^{T^{1-\epsilon}} \frac{S(n,n,c)}{c} h_T^+(x) \ll T^2 .$$

Proof.

$$h_T^+(x) = 4i^2 \mathrm{Im} \int_0^B \frac{J_{2it}(x) h_+(t,\alpha,T) t\, dt}{\cosh \pi t} - 4i^2 \mathrm{Im} \int_0^A \bar{J}_{2it}(x) h_+(-t,\alpha,T) t\, dt$$

$$\ll \int_0^B |J_{2it}(x)| \frac{\epsilon^{-\frac{t^2}{T^2}} t\, dt}{\cosh \pi t} \ll \int_0^1 \frac{|J_{2it}(x)| t\, dt}{\cosh \pi t} + \int_1^B \frac{|J_{2it}(x)| \epsilon^{-\frac{t^2}{T^2}} t\, dt}{\cosh \pi t} = I_a + I_b .$$

We first consider I_a

$$I_a \ll \int_0^1 \frac{\left(\frac{x^2}{4} + t^2\right)^{-1/4} \epsilon^{\pi t} t\, dt}{\cosh \pi t} + \int_0^1 \frac{\left(\frac{x^2}{4} + t^2\right)^{-1/4} \epsilon^{\pi t} dt}{\cosh \pi t}.$$

But it is easy to see that

$$\left(\frac{x^2}{4}+t^2\right)^{-1/4} \ll T^{1/2-\epsilon/2}.$$

Hence $I_a \ll T^{1/2-\frac{\epsilon}{2}}$. We now consider I_b.

$$I_b \ll \int_1^B \frac{t^{1/2}\epsilon^{\pi t}\epsilon^{-\frac{t^2}{T^2}}}{\left(1+\frac{x^2}{4t^2}\right)^{1/4}\cosh \pi t}dt \ll \int_1^B t^{1/2}\epsilon^{-\frac{t^2}{T^2}}dt.$$

By a simple change of variable we have

$$I_b \ll T^{3/2}\int_{\frac{1}{T}}^{B/T} t^{1/2}\epsilon^{-t^2}dt \ll T^{3/2}.$$

The proof is completed by straightforward application of the Weil bound for Kloosterman sums, namely

$$S(n,n,c) \ll n^{1/2}c^{1/2+\epsilon_2}. \qquad \square \tag{3.4}$$

The Taylor series for $J_{2it}(x)$ is

$$J_{2it}(x) = \left(\frac{x}{2}\right)^{2it}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+1+2it)}\left(\frac{x}{2}\right)^k$$
$$= \frac{\left(\frac{x}{2}\right)^{2it}}{\Gamma(1+2it)} + \left(\frac{x}{2}\right)^{2it}\sum_{k=1}^{\infty}\frac{1}{k!\Gamma(k+1+2it)}\left(\frac{x}{2}\right)^{2k} = J_{2it}^a(x)+J_{2it}^b(x).$$

And if $0<x<1$ we have

$$J_{2it}^b(x) \ll \frac{x^2}{|\Gamma(1+2it)|},$$

since $|\Gamma(1+v)| \le |\Gamma(k+1+v)|$.

Lemma 3.12. *For every $0<\epsilon_0<\frac{1}{100}$ there exists $0<\epsilon_0'<\frac{1}{100}$ and $0<\Delta(\epsilon_0)<\frac{1}{100}$ such that for all $0<\Delta\le\Delta(\epsilon_0)$ for all $0<\epsilon\le\epsilon_0'/2$ (where ϵ_0 is as specified in Theorem 1.1)*

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right)\sum_{c=T^{1-\epsilon}}^{\infty}\frac{S(n,n,c)}{c}\left(4i^2\mathrm{Im}\int_{-\infty}^{\infty}J_{2it}^a(x)\frac{h_+(t,\alpha,T)}{\cosh \pi t}\right) \ll T^2.$$

Proof. Consider

$$I_a = \int_{-\infty}^{\infty}\frac{\left(\frac{x}{2}\right)^{2it}h_+(t,\alpha,T)}{\Gamma(1+2it)}\,dt\,.$$

It is immediate from Stirling's formula that for

$$0 < t_0 \le t \tag{3.5}$$
$$\frac{1}{\Gamma(\sigma+2it)} \ll_\sigma e^{\pi t} t^{1/2-\sigma}.$$

In I_a we move the line of integration to $-i\theta$, where $\theta = \left(\frac{1}{4}+\epsilon_2\right)$, where ϵ_2 is that of (3.4) obtaining

$$I_a = \left(\frac{x}{2}\right)^{2\theta} \int_{-A}^{B} \frac{\left(\frac{x}{2}\right)^{2it} h_+(t-i\theta,\alpha,T)t\,dt}{\Gamma(1+2\theta+2it)\cosh(\pi(t-i\theta))}$$
$$- i\theta\left(\frac{x}{2}\right)^{2\theta} \int_{-A}^{B} \frac{\left(\frac{x}{2}\right)^{2it} h_+(t-i\theta,\alpha,T)dt}{\Gamma(1+2\theta+2it)\cosh(\pi(t-i\theta)}.$$

By a simple change of variable, using the fact that $\frac{1}{\Gamma(z)}$ is entire and $\Gamma(z) = \overline{\Gamma(\bar{z})}$ together with (3.5) we obtain

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{B} |h_+(t-i\theta,\alpha,T)| t^{(1/2-2\theta} dt$$
$$+\left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{A} |h_+(-t-i\theta,\alpha,T)| t^{(1/2-2\theta)} dt + \left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{B} |h_+(t-i\theta,\alpha,T)| t^{(-1/2-2\theta)} dt$$
$$+\left(\frac{x}{2}\right)^{2\theta} \int_{t_0}^{A} |h_+(-t-i\theta,\alpha,T)| t^{(-1/2-2\theta)} dt + O\left(\left(\frac{x}{2}\right)^{2\theta}\right).$$

Then by Lemma 3.0 we have

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{B} t^{(1/2-2\theta)} \epsilon^{-\frac{t^2}{T^2}} dt + \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{A} t^{(1/2-2\theta)} \epsilon^{-\frac{t^2}{T^2}} dt$$
$$+\left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{B} t^{(-1/2-2\theta)} \epsilon^{-\frac{t^2}{T^2}} dt + \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{A} t^{(-1/2-2\theta)} \epsilon^{-\frac{t^2}{T^2}} dt$$
$$+ O\left(\left(\frac{x}{2}\right)^{2\theta}\right).$$

So that

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta} \int_{t_0}^{A} \epsilon^{-\frac{t^2}{T^2}} dt + O\left(\left(\frac{x}{2}\right)^{2\theta}\right),$$

and by a simple change of variable

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta+1} \int_{\frac{t_0}{T}}^{\frac{A}{T}} \epsilon^{-t^2} dt + O\left(\left(\frac{x}{2}\right)^{2\theta}\right).$$

Hence

$$I_a \ll \left(\frac{x}{2}\right)^{2\theta} T^{|\alpha|\theta+1}.$$

The above moving of the line of integration is valid if

$$I_B = \int_0^\theta J_{2iz}^a(x) \frac{h_+(z,\alpha,T) z dz}{\cosh \pi z} \to 0 \quad \text{for each } x \text{ and } T$$

as $B \to \infty$, where $z(t) = B - it$ for $0 \le t \le \theta$, and

$$I_A = \int_0^\theta J_{2iz}^a(x) \frac{h_+(z,\alpha,T) z dz}{\cosh \pi z} \to 0 \quad \text{for each } x \text{ and } T$$

as $A \to \infty$, where $z(t) = -A - it$ for $0 \le t \le \theta$.

By direct substitution and using the fact that

$$\frac{1}{|\Gamma(\sigma + it)|} \le e^{\frac{\pi t}{2}} (ct)^{1/2-\sigma}$$

we have

$$I_B \ll \int_0^\theta \left(\frac{x}{2}\right)^{2t} |h_+(B - it, \alpha, T)| B (cB)^{-1/2-2t} dt$$
$$+ \int_0^\theta \left(\frac{x}{2}\right)^{2t} |h_+(B - it, \alpha, T)| t (cB)^{-1/2-2t} dt .$$

so that by Lemma 3.0 we have

$$I_B \ll \int_0^\theta \left(\frac{x}{2}\right)^{2\theta} \epsilon^{\frac{t^2 - B^2}{T^2}} B dt + \int_0^\theta \left(\frac{x}{2}\right)^{2\theta} \epsilon^{\frac{t^2 - B^2}{T^2}} t\, dt$$

so that $I_B \to 0$ for each x, T as $B \to \infty$. In a similar way we show that $I_A \to 0$ for each x, T as $A \to \infty$. The proof of the lemma is completed by straightforward application of the Weil bound for Kloosterman sums. □

Lemma 3.13. *For every* $0 < \epsilon_0 < \frac{1}{100}$ *there exists* $0 < \epsilon_0' < \frac{1}{100}$ *and* $0 < \Delta(\epsilon_0) < \frac{1}{100}$ *such that for all* $0 < \Delta \leq \Delta(\epsilon_0)$ *for all* $0 < \epsilon < \epsilon_0'/2$ *(where* ϵ_0 *is as specified in Theorem 1.1)*

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \sum_{c=T^{1-\epsilon}}^{\infty} \frac{S(n,n,c)}{c} \left(4i^2 \mathrm{Im} \int_{-\infty}^{\infty} J_{2it}^b(x) \frac{h_+(L(t,\alpha,T))t\,dt}{\cosh \pi t}\right) \ll T^2.$$

Proof. Consider

$$I_b = \int_{-\infty}^{\infty} J_{2it}^b(x) \frac{h_+(L(t,\alpha,T))tdt}{\cosh \pi t}.$$

By a simple change of variable using the fact that $\frac{1}{\Gamma(z)}$ is entire and $\Gamma(z) = \overline{\Gamma(\bar{z})}$ together with (3.5) we obtain

$$I_b \ll x^2 \int_{t_0}^{B} |h_+(L(t,\alpha,T))| t^{1/2} dt + x^2 \int_{t_0}^{A} |h_+(L(-t,\alpha,T))| t^{1/2} dt + O(x^2).$$

Then by Lemma 3.0 we have

$$I_b \ll x^2 \int_{t_0}^{B} t^{1/2} \epsilon^{-\frac{t^2}{T^2}} dt + O(x^2).$$

By a simple change of variable we have

$$I_b \ll x^2 T^{3/2} \int_{\frac{t_0}{T}}^{\frac{B}{T}} t^{1/2} \epsilon^{-t^2} dt + O(x^2).$$

Hence we have

$$I_b \ll x^2 T^{3/2}.$$

The proof of the lemma is completed by straightforward application of the Weil bound for Kloosterman sums. □

Lemma 3.14.

$$\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} H_T(t) |\mathcal{N}(n,t)|^2 dt\right) = O(TM^{1+\epsilon}).$$

Proof. The result follows by straightforward calculation from the wellknown facts and definitions:

$$\mathcal{N}(n,t) \doteq \left(\frac{4\pi|n|}{\cosh \pi t}\right)^{1/2} \varphi(n, 1/2+it),$$

where

$$\varphi(n,s) = \pi^s \Gamma^{-1}(s)\zeta(2s)^{-1}|n|^{-1/2} \sum_{ab=|n|} \left(\frac{a}{b}\right)^{s-1/2}, \qquad \frac{1}{\zeta(s)} = O(\log^{\Delta}(t)),$$

uniformly for $1 \le \sigma$, and Sterling's asymptotic formula in the form

$$\Gamma(\sigma + at) = (2\pi)^{1/2} t^{\sigma-1/2} e^{\frac{-\pi t}{2}} \left(\frac{t}{e}\right)^{it} (1 + O(t^{-1}))$$

if $t > 0$, where the implied constant depends on σ. □

4. Proof

We consider Theorem 1.1.

Let $S = \sum_{0<t_j} S(\phi_j) h(t_j, \alpha, T)$.

$$S = \sum_{0<t_j \le T^2} S(\phi_j) h(t_j,\alpha,T) + \sum_{T^2<t_j} S(\phi_j) h(t_j,\alpha,T).$$

By Lemma 3.5 we have

$$\begin{aligned} S = \frac{12M}{\pi} &\sum_{\substack{0<t_j \le T^2 \\ t_j \in G}} h(t_j,\alpha,T) + \sum_{\substack{0<t_j \le T^2 \\ t_j \in G}} E_2(T) h(t_j,\alpha,T) \\ &+ \sum_{\substack{0<t_j \le T^2 \\ t_j \in B}} S(\phi_j) h(t_j,\alpha,T) \\ &+ \sum_{T^2<t_j} S(\phi_j) h(t_j,\alpha,T) \pm \frac{12M}{\pi} \sum_{\substack{0<t_j \le T^2 \\ t_j \in B}} h(t_j,\alpha,T) \\ &\pm \frac{12M}{\pi} \sum_{T^2 \le t_j} h(t_j,\alpha,T). \end{aligned}$$

Hence

$$S = \frac{12M}{\pi} \sum_{0<t_j} h(t_j,\alpha,T) + \sum_{\substack{0<t_j \le T^2 \\ t_j \in G}} E_2(T) h(t_j,\alpha,T)$$

$$+\sum_{\substack{0<t_j\le T^2\\ t_j\in B}} S(\phi_j)h(t_j,\alpha,T)$$
$$+\sum_{T^2<t_j} S(\phi_j)h(t_j,\alpha,T)-\frac{12M}{\pi}\sum_{\substack{0<t_j\le T^2\\ t_j\in B}} h(t_j,\alpha,T)$$
$$-\frac{12M}{\pi}\sum_{T^2<t_j} h(t_j,\alpha,T).$$

Hence by Lemma 3.2, Lemma 3.3, Lemma 3.5, Lemma 3.6 and Lemma 3.7 we have

$$S=\frac{12M}{\pi}\sum_{0<t_j} h(t_j,\alpha,T)+O(M^{1-\delta}T^{2+\epsilon})+O\left(M^{5/4}T^{\frac{1}{10}+\epsilon}\right)$$
$$+O(M^{5/4}T^{\epsilon})+O(MT^{1/10})+O(M);$$

so that

$$S=\frac{12M}{\pi}\sum_{0<t_j} h(t_j,\alpha,T)+O(M^{1-\delta}T^{2+\epsilon})+O\left(M^{5/4}T^{\frac{1}{10}+\epsilon}\right).$$

But by interchanging the order of summation we have

$$S=\sum_{n=1}^{\infty} g\left(\frac{n}{M}\right)\left(\sum_{0<t_j} h(t_j,\alpha,T)|\lambda_j(n)|^2\gamma_{\phi_j}^2\right).$$

Then Theorem 1.1 follows from Theorem 2.2, Lemma 3.9, Lemma 3.11, Lemma 3.12, Lemma 3.13 and Lemma 3.14.

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