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EFFICIENCY OF MULTIQUADRIC COLLOCATION METHOD FOR SOLVING PDE

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Abstract: In this paper, we demonstrate the efficiency of the multiquadric radial basis functions (MQ-RBFs) collocation method for solving partial differential equations (PDEs), as theoretically compared to the finite element method (FEM). The MQ-RBF has the property of exponential convergence with respect to the shape parameter. Although the optimal choice of shape parameter is still an unsettled issue, there exist a wide range c of values in which the RBF solution has high accuracy. Error estimation of the approximate solution is also given and the numerical results indicate that the method provides accurate approximations.

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1. Introduction

The commonly numerical methods for solving PDEs include finite difference method (FDMs), spectral methods, finite element methods (FEMs), boundary element methods (BEMs), and finite volume methods (FVMs). We know that FDM uses a rectilinear grid, which is difficult to form-fit irregular geometry, and the solution only achieves in the mesh grid points. FEM is flexible in dealing with complex geometry, but the mesh generation in high dimensions, their data structure, and the computer coding are complicated [**?**]. Although the FEM can accommodate a more flexible grid work and has been used as an alternative solution scheme for these equations, the finite element solution is not as stable as the finite difference solution and usually requires the use of nonphysical dissipation [**?**]. Furthermore the generation of a finite element grid with several thousand nodes and with element of various sizes, shape and orientation is a nontrivial task.

In the last two decades, RBFs have been used as a basis functions for interpolation of scattered data in higher dimensional spaces, and interesting results are obtained with them. The MQ-RBFs were first developed by Hardy in 1971 as a multidimensional scattered interpolation method in modeling earth's gravitational field. Its theoretical basis and convergence properties have been well established [14], [2]. Since RBFs have excellent performance for function approximation, many researchers turn to explore their ability for solving PDEs. The direct application of RBFs to solve PDEs was first attempted by Kansa in 1992 [4]. In this method, MQ-RBF was used as an interpolant to approximate the solution and point collocation was applied to enforce the governing equation and boundary conditions. Another method that can fall into this category is the method of fundamental solutions (MFS) [10], as MFS can be considered as a form of RBF.

One key feature shared by the RBF collocation method and the MFS is that they do not require a structured grid, and are truly meshless. They can be used to solve complex geometry problems based on a set of scattered points. The computer coding is simple as compared to the above-mentioned methods. Particularly, since no element structure is involved, there is little difference between the two-dimensional and the three-dimensional programs. In addition, RBFs are not restricted to spatial dimensions and they involve a single independent variable regardless of the dimension of the problem, i.e., the norms can be defined in any number of dimensions. Lastly, and most importantly, it has been observed that certain classes of RBFs, such as the MQs and Gaussians, exhibits superior error convergence properties; hence they are very efficient for solving PDEs.

RBFs can be globally supported, infinitely differentiable, and contain a free parameter, a, called the shape parameter. Franke in 1982 [10], [**?**] tested a large number of interpolation methods for scattered data, and ranked MQs one of the best. Micchelli [1] has proven the positive definiteness of the MQ collocation matrices. Madych and Nelson [2] have proven the exponential convergence property of MQ approximation. In short, they have shown that, under certain conditions, the interpolation error is proportional to $\lambda^{\frac{c}{h}}$; where c is the shape parameter of MQ, h is the mesh size, and $0 < \lambda < 1$ is a constant. This implies that there are two ways for the solution to converge either by reducing the size of h, or by increasing the magnitude of c. While reducing h means the increase in computational cost, increasing c can be performed without cost. However, according to the “uncertainty principle” of Schaback [13], as the error becomes smaller, the matrix becomes more ill-conditioned; hence the solution will break down as c becomes too large. Nevertheless, there exists a wide range c of values that highly accurate results can be produced, which are unmatched by any other method.

Because of the above-mentioned advantages, RBF based methods have since been applied to many application problems in the past decades, such as the biphasic mixture model for tissue engineering problems [6], 1D and 2D nonlinear Burgers’ equation [7], shallow water equation for tide and currents simulation [5], and free boundary problems arising from the American option pricing [8]. To the best of our knowledge, there is no direct theoretical comparison between RBF methods and FEMs yet, which is the goal of this paper. Such comparison is necessary because we must carefully read the above statements about the accuracy of using MQs for interpolation. The above mentioned convergence refers to the interpolation of a function by RBF and collocation for known function values at a set of discrete points. There is no proof yet that the convergence property of interpolation will be preserved under such operations. Hence numerical experiments are necessary to demonstrate the superiority of the MQ collocation method.

The paper is organized as follows. In Section 2 we give a brief introduction of finite element method. RBF collocation method and numerical example are presented in Section 3. The example shows that the RBF collocation methods have good accuracy although the condition number is large. At least, Section 4 concludes the paper.

2. Finite Element Method

In many natural, biological and mechanical problems, we impacted by algebra problem, differential equation, integral equation, or problems related them. Most solution of them is a summation of infinitely series or there is no theoretical solution. Therefore it can be used approximation methods. In variational method the approximation solution is the linear combination of trial functions $\{\varphi_i\}_{i=1}^N$

$$U_N = \sum_{i=1}^{N} c_i \varphi_i \, . \tag{1}$$

The coefficients of combination can be obtained by variational method. This method has complicated structure. In variational method integration is always used for obtaining approximation solution of partial differential equation. For integration in one-dimensional space usually part to part method is used and in 2D, divergence and gradient theorem are used. Ritz method, the method of weighted residuals, the Petrov-Galerkin method, Galerkin method, the least square method and collocation method are classical variational methods [12], [9]. All of this methods are a linear combination of suitable functions. The fundamental difference of this methods, is the choice of the basis functions, $\{\varphi_i\}_{i=1}^N$. When domain is complicated the choice of basis function is difficult and the matrix of linear system is full and non symmetric. To obtain this system we must calculate a lot of integration without collocation method, and numerical method is used for integration. We know numerical integration has a lot of errors. FEM reduces the difficulty level of this problem, and it is a regular and systematic method for approximation. In this method to answer best, we consider two cases:

1. Discretization of the irregular geometry to simple geometry so called "eleman".

2. The approximation solution is the linear combination of polynomial. The Concept of this method is obtained from interpolation [12].

So FEM is application of a piecewise variational method. The number of the parameters depends on the choice of nodes in eleman. Moreover this method contains various steps, such as domain discretization, obtaining equation for each eleman, assembling of eleman equation for obtaining equations all over the domain, applying boundary conditions and solving the assembled equation. But in general, domain discretization in higher dimension, selecting geometric shapes, and integral calculation is difficult. With attention to this, the approximation idea and basis function choice in FEM is taken from interpolation to polynomial and there is no classical method for choosing polynomial basis function.

As we know, in Error! Objects cannot be created from editing field codes. there is no Haar subspaces of continuous function (except one-dimensional ones) [1] and we cannot use every n-dimensional subspace of functions for interpolation of arbitrary multivariate function in n distinctive points. Multivariate interpolation with polynomial depends on the position of nods and for approximate solution of PDE the choice of polynomials depends on the position of inter nods of eleman and the number of unknown coefficients. But the choice of polynomials is independent than structure of PDE.

3. RBF Collocation Method

Consider the following boundary value problem (BVM):

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ Bu = g, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^d$, L is a linear differential operator, B is an operator imposed as boundary conditions, such as Dirichlet, Neumann, or other types and d is the dimension of the problem. Let $\{x^{(i)}\}_{i=1}^{N}$ be N collocation points in Ω, of which,$\{x^{(i)}\}_{i=1}^{N_l}$ are interior points and $\{x^{(i)}\}_{i=N_l+1}^{N}$ are boundary points. It is assumed that the approximate solution for the problems (2) can be expressed as

$$u \approx \overline{u} = \sum_{i=1}^{N} w^{(i)} \varphi^{(i)}(x)\,, \tag{3}$$

where $\{w^{(i)}\}_{i=1}^{N}$ are the unknown coefficients to be determined and $\varphi^{(i)}$ are the RBFs. Some commonly used parametric RBFs are as follows.

$$\begin{aligned} \varphi_k &= e^{\frac{-r_k^2}{a^2}} && - \quad \text{Gaussian}\,, \\ \varphi_k &= (r_k^2 + a^2)^{\frac{n}{2}} && - \quad \text{Hardy multiquadric (MQ)}\,, \\ \varphi_k &= \frac{1}{\sqrt{r_k^2 + a^2}} && - \quad \text{Inverse multiquadric (IMQ)}\,, \\ \varphi_k &= \frac{1}{r_k^2 + a^2} && - \quad \text{Inverse quadric (IQ)}\,, \end{aligned}$$

where n is a positive integer, a is the shape parameter of RBF, $r_k = \|x - c^{(k)}\|$ is the Euclidean norm between points x and $c^{(k)}$, and $c^{(k)}$ is called the center of RBF. For more details, we refer the readers to the excellent review papers on the theory of RBF interpolation by Powell [2], [**?**].

By substituting equation (3) in equation (2), and collocating at the N interior and boundary points, we have:

$$\sum_{i=1}^{N_l} w^{(i)} L\varphi^{(i)}(x^{(j)}) = f(x^{(j)}), \qquad j = 1, \dots, N_l\,, \tag{4}$$

$$\sum_{i=N_l+1}^{N} w^{(i)} B\varphi^{(i)}(x^{(j)}) = g(x^{(j)}), \qquad j = N_l + 1, \dots, N\,. \tag{5}$$

The above system can be solved for the unknown coefficients $\{w^{(i)}\}_{i=1}^{N}$. This shows that the implementation of RBF collocation method is simple and straightforward. This is one of the reasons that this technique is getting popular.

For numerical verification purposes, we focus on the solving of the following 2D elliptic problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = f(x_1, x_2), & (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \\ u|_{\partial\Omega} = g(x_1, x_2), & (x_1, x_2) \in \partial\Omega. \end{cases} \tag{6}$$

In this paper, many kinds of RBFs can be used to construct the solution. In order to express our algorithm clearly and concisely, we will only use the MQ-RBFs. The chosen MQ is given by:

$$\varphi^{(i)}(x) = \sqrt{r_i^2 + a^2}\,.$$

Given in equation (3) and MQ-RBFs, as a basis functions, the derivatives of the function $u(x)$ are calculated by

$$\frac{\partial^k u}{\partial x_j \dots \partial x_l} = \sum_{i=1}^{N} w^{(i)} \frac{\partial^k \varphi^{(i)}}{\partial x_j \dots \partial x_l}\,. \tag{7}$$

The derivatives (e.g. up to second order with respect to x_j) are calculated by

$$u_j(x) = \sum_{i=1}^{N} w^{(i)} h^{(i)}(x), \tag{8}$$

$$u_{jj}(x) = \sum_{i=1}^{N} w^{(i)} \overline{h}^{(i)}(x), \tag{9}$$

a	Condition number	Max error
0.8	6.21100×10^{13}	6.2×10^{-5}
1.0	1.22047×10^{15}	3.5×10^{-5}
1.2	1.48000×10^{15}	1.5×10^{-5}
1.5	2.47900×10^{18}	0.8×10^{-5}

Table 1: Numerical results for Example 1

where

$$h^{(i)}(x) = \frac{\partial \varphi^{(i)}}{\partial x_j} = \frac{x_j - c_j^{(i)}}{(r^2 + a^2)^{.5}}, \qquad \overline{h}^{(i)}(x) = \frac{\partial h^{(i)}}{\partial x_j} = \frac{r^2 + a - (x_j - c_j^{(i)})^2}{(r^2 + a^2)^{1.5}},$$

Substituting the above derivatives into Eqs (6), we have

$$\sum_{i=1}^{N} w^{(i)} \Big(\frac{\partial^2 \varphi^{(i)}}{\partial x_1^2} + \frac{\partial^2 \varphi^{(i)}}{\partial x_2^2} \Big)(x_1^{(j)}, x_2^{(j)}) = f(x_1^{(j)}, x_2^{(j)}), \qquad j = 1, \ldots, N_l, \quad (10)$$

$$\sum_{i=1}^{N} w^{(i)} \varphi(x_1^{(j)}, x_2^{(j)}) = g(x_1^{(j)}, x_2^{(j)}), \qquad j = N_l + 1, \ldots, N\,, \quad (11)$$

which is a $N \times N$ linear system for the unknowns $\{w^{(i)}\}_{i=1}^{N}$. For solving this linear system the LU factorization method is used. Notice that the representation (3) gives a global approximation solution at any point in the domain.

Example 1. Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-5\pi^2}{4} \sin(\pi x)\cos(\frac{\pi y}{2}), \qquad (x, y) \in (0,1) \times (0,1),$$

$$u(x, y) = \sin(\pi x)\cos(\frac{\pi y}{2}), \qquad (x, y) \in \partial\Omega,$$

with exact solution

$$u(x, y) = \sin(\pi x)\cos(\frac{\pi y}{2}).$$

Using MQ collocation method with $a = 0.8$ and few points in domain the absolute errors are shown in Figure 1. Also, by using different value of a the results, the condition number of collocation matrix and maximum error of this method, are given in Table 1.

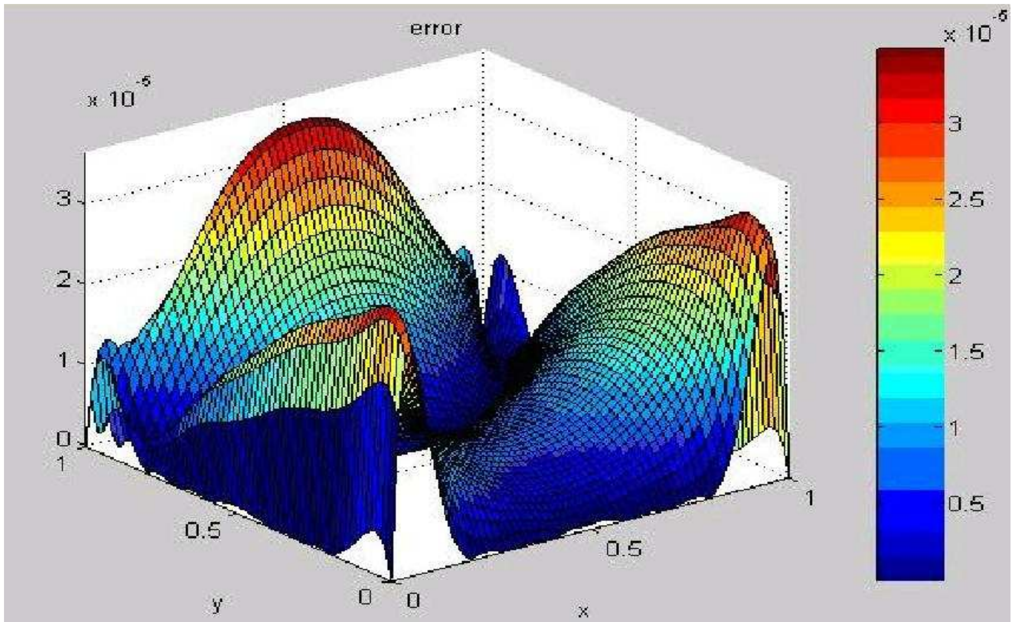

Figure 1: Error of MQ-RBF method

4. Conclusions

In this paper we present the MQ-RBF collocation method capable of solving PDEs of the Dirichlet type. We observe that the method has a lot of advantages such as simplicity, accuracy and efficiency respect to the FEM. RBF methods are truly meshless methods. Using RBF, no domain discretization and domain integration are required. We can use easily the RBFs in complicated geometry and higher dimensional. But using FEM in complicated geometry and higher dimensional is very difficult. For RBF collocation method, there is a classic way for choosing basis function, and basis function is easily obtained with collocation point. They are infinitely derivable and derivatives are classical. But in FEM there is no way to choose polynomials. Results obtained from RBF approach show that the obtained approximation has good accuracy although the condition number is large.

References

[1] W. Cheney, W. Light, *A Course in Approximation Theory*, Brooks-Colen Publishing Company (2000).

[2] A.H.-D. Cheng, M.A. Golberg, *Multiquadric Collocation Method for Partial Differential Equations* (2002), 572-94.

[3] A.I. Fedoseytev, M.J. Friedman, *Improved Multiquadric Method for Elliptic Partial Differential Equations via PDE Collocation on the Boundary*, Computers and Mathematics with Applications, **43** (2002).

[4] C. Franke, R. Schaback, Solving partial differential equations by collocation using radial basis functions, *Appl. Math. Comput.*, **93** (1998), 73-82.

[5] Y.C. Hon, K.F. Cheung, X.Z. Mao, E.J. Kansa, Multiquadric solution for shallow water equations, *J. Hydraulic Engng. ASCE*, **125** (1999), 524-33.

[6] Y.C. Hon, M.W. Lu, W.M. Xue, Y.M. Zhu, Multiquadric method for the numerical solution of a biphasic mixture model, *Appl. Math. Comput.*, **88** (1997), 153-75.

[7] Y.C. Hon, X.Z. Mao, An efficient numerical scheme for Burger's equation, *Appl. Math. Comput.*, **95** (1998), 37-50.

[8] Y.C. Hon, X.Z. Mao, A Radial basis function method for solving options pricing model, *Financial Engng.*, **8** (1999), 31-49.

[9] C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press (1992).

[10] J. Li, Y.C. Hon, C.S. Chen, Numerical comparisons of two meshless methods using radial basis functions, *Engng. Anal. Bound. Elem.*, **26** (2002), 205-225.

[11] J.M. Melenk, I. Babuska, The partition of unity finite element method: basic theory and applications, *Comput. Meth. Appl. Mech. Engng.*, **139** (1996), 289-314.

[12] J.N. Reddy, *An Introduction to the Finite Element Method*, McGrawill Publishing Company, New York (1984).

[13] R. Schaback, Error estimates and condition numbers for radial basis function interpolation, *Adv. Comput Math.*, **3** (1995), 251-64.

[14] R. Schaback, Improved error bounds for scattered data interpolation by radial basis functions, *Math. Comput.*, **68** (1999), 201-216.