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ON THE EXISTENCE AND THE ASYMPTOTIC BEHAVIOUR OF REGULARLY (SLOWLY) VARYING SOLUTIONS OF SOME CLASSES OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

Kusano Takasi[1 §], Vojislav Marić[2]

[1]Department of Applied Mathematics
Faculty of Science
Fukuoka University
Fukuoka, 841-0180, JAPAN
e-mail: tkusano@cis.fukuoka-u.ac.jp

[2]Serbian Academy of Sciences and Arts
Kneza Mihaila 35, Beograd, 11000, SERBIA
e-mail: vojam@uns.ns.ac.yu

Abstract: Functional differential equations of the second order with deviating arguments are studied in the framework of Karamata regularly varying functions. Sharp conditions are established for the existence of slowly and regularly varying solutions. Some results on the asymptotics of these solutions are also given.

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1. Introduction

The theory of regular variation, which was initiated by Karamata in 1930, has

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§Correspondence author

provided a major tool for various branches of mathematical analysis including Abelian and Tauberian Theorems, analytic number theory and complex analysis, and it is equally important for probability theory.

We recall that a measurable function $f : [0,\infty) \to (0,\infty)$ is said to be regularly varying of index $\rho \in \mathbf{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for any} \quad \lambda > 0.$$

The totality of regularly varying functions of index ρ is denoted by RV (ρ). The symbol SV is used to denote RV(0) and a member of SV $=$ RV(0) is referred to as a slowly varying function. If $f(t) \in$ RV (ρ), then $f(t) = t^\rho L(t)$ for some $L(t) \in$ SV, and so the class of slowly varying functions is of fundamental imporatnce in regular variation. For the most comprehensive exposition of regular variation and its applications, the reader is referred to the book of Bingham, Goldie and Teugels [2].

The study of solutions of differential equations in the framework of Karamata theory began in 1947 by the seminal paper of Avakumović on the Thomas-Fermi equation, [1]. A complete survey of the results on differentail equations, both linear and nonlinear, developed by means of regular variation up to 2000 is given in the monograph of Marić [7]. It is shown therein that the class of Karamata regularly varying functions is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear and nonlinear differential equations.

On the other hand the study of functional differential equations in this spirit started only in 2006 in [4], [5], [6].

We give here a concise presentation of some of the results.

These concern the functional differential equation with retarted argument

$$x''(t) = q(t)x(g(t))\,, \tag{A}$$

i.e. the function $g(t)$ is continuous and increases to infinity as $t \to \infty$ and $g(t) < t$. The function $q(t)$ is continuous, positive and integrable on some positive half-axis $[t_0, \infty)$.

2. Results

2.1. Existence

We prove the following theorem.

Theorem 2.1. *Let $c > 0$ and denote by λ_i, $i = 1, 2, \lambda_1 < \lambda_2$ the two roots of the quadratic equation*

$$\lambda^2 - \lambda - c = 0. \tag{2.1}$$

Suppose that

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1; \tag{2.2}$$

then equation (A) has two regularly varying solutions $x_i(t)$ of indices λ_i, i.e. possessing the form

$$x_i(t) = t^{\lambda_i} L_i(t)\,,$$

where $L_i(t)$ are some normalized slowly varying functions, if and only if

$$Q(t) := t \int_t^\infty q(s)ds - c \to 0, \quad \text{as} \quad t \to \infty. \tag{2.3}$$

Theorem 2.2. *Let $c = 0$, so that $\lambda_1 = 0$, $\lambda_2 = 1$. Instead of (2.2) suppose that*

$$\limsup_{t\to\infty} \frac{t}{g(t)} < \infty; \tag{2.4}$$

then equation (A) has

a) a slowly varying solution $x_1(t) = L_1(t)$,

b) a regularly varying solution $x(t) = tL_2(t)$ of index 1, where $L_i(t)$ are normalized SV-functions, if and only if condition (2.3) holds.

2.2. Asymptotics

The following two results serve as examples of a possible use of the fact that equation (A) has RV-solutions. These are easy consequences of similar results for equation (B) ([7, Corollary 2.1, Theorem 2.7]).

Theorem 2.3. *Let $c = 0$. If*

$$\int_a^\infty s^2 q(s)ds \int_{g(s)}^\infty q(u)du < \infty,$$

then for the slowly varying solution $x_1(t)$ of equation (A) there holds for $t \to \infty$

$$x_1(t) \sim A \exp\left\{ - \int_a^t sq(s)ds \right\}, \quad A > 0.$$

Obviously if the above integral converges for $t \to \infty$, then $x_1(t) \to B > 0$.

Theorem 2.4. *Let conditions of Theorem 2.1 hold with*

$$\frac{g(t)}{t} = 1 + O\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty$$

instead of (2.2).

If

$$\int_a^\infty t \left| q(t) - \frac{c}{t^2} \right| dt < \infty ,$$

then there exist two solutions $x_i(t)$, $i = 1, 2$ of equation (A) such that

$$x_i(t) \sim t^{\lambda_i} \quad \text{as} \quad t \to \infty.$$

3. Method of Proofs for the Existence Theorems

The necessity parts are rather simple and mimic the one for the companion ordinary differential equation (i.e. without deviating argument)

$$x''(t) = q(t)x(t) \tag{B}$$

as given in [7].

For the sufficiency parts the following result is crucial [7], [3].

Lemma 3.1. *Let $c \geq 0$ and λ_i be as in Theorem 2.1. Then the condition (2.3) is necessary and sufficient for equation (B) to possess two regularly varying solutions $x_i(t)$ of indices λ_i for $t \geq T$, $T > 0$, of the form*

$$x_i(t) = A \exp\left\{ \int_T^t \frac{\lambda_i - Q(s) - v_i(s)}{t} ds \right\}, \quad A > 0,$$

where $v_i(t)$ are solutions of the integral equations

$$v_1(t) = t^{1-2\lambda_1} \int_t^\infty s^{2(\lambda_1 - 1)} \{ v_1(s) - Q(s))^2 - 2\lambda_1 Q(s) \} ds ,$$

$$v_2(t) = t^{1-2\lambda_2} \int_T^t s^{2(\lambda_2 - 1)} \{ 2\lambda_2 Q(s) - (v_2(s) - Q(s))^2 \} ds ,$$

respectively.

Next major device is the use of a Fixed Point Principle:

One first constructs for sufficiently large t ($t \geq t_0$ say) a set Ξ of positive continuous functions $\xi(t)$ such that Ξ is a closed, convex subset of the locally convex space $C[t_0, \infty)$ equipped with the metric topology of uniform convergence on compact subintervals of $[t_0, \infty)$.

Secondly, for each $\xi(t) \in \Xi$ one consider the infinite family of equations of the type (B)

$$x''(t) = q(t)\frac{\xi(g(t))}{\xi(t)}x(t). \tag{3.1}$$

Lemma 3.1 ensures for each $\xi(t) \in \Xi$ the existence of RV-solutions, $x_\xi(t)$ for $t \geq t_1 \geq t_0$.

Thirdly, one constructs a mapping Φ which associates to each $\xi \in \Xi$ the function $\Phi\xi$ defined (for $t \geq t_1$) by $\Phi(\xi(t)) = x_\xi(t)$.

Then one shows that Φ is a self-map on Ξ, the relative compactness of the set $\Phi(\Xi)$ in $C[t_0, \infty)$ and the continuity of the mapping Φ, meaning that all the hypotheses of the Schauder-Tychonoff Fixed-Point Theorem are fulfilled. Hence, there exists an element $\xi_0 \in \Xi$ such that $\xi_0 = \Phi\xi_0$ or, due to the definition of Φ, such that $\xi_0''(t) = q(t)\xi_0(g(t))$, i.e. $\xi_0(t)$ is an SV-solution of (A).

We emphasize that the construction of the function $\xi(t)$ is essential for the proof and it is different in each of the three considered cases (Theorem 2.1 and Theorem 2.2).

4. Remarks

Remark 4.1. Theorems 2.2 and 2.3 hold also for the advanced case i.e. when $g(t) > t$, with (2.4) replaced by $\limsup\limits_{t\to\infty} \frac{g(t)}{t} < \infty$.

Remark 4.2. Theorem 2.2 holds for the "mixed type" equations

$$x''(t) \pm p(t)x(g(t)) \pm q(t)x(h(t)) = 0\,, \tag{C}$$

where $p(t), q(t) > 0$, $g(t) < t$, $h(t) > t$ and $\limsup\limits_{t\to\infty} \frac{t}{g(t)} < \infty$, $\limsup\limits_{t\to\infty} \frac{h(t)}{t} < \infty$. The same is true for Theorem 2.3 but with "-" sign in (C) only.

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