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SPECTRAL ANALYSIS OF NATURAL CONVECTION HEAT TRANSFER FROM A RECTANGULAR CYLINDER PARTIALLY HEATED IN AN INFINITE EXTENSION

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Abstract: Two-dimensional steady-state natural convection around a rectangular cylinder placed horizontally is analyzed, using a spectral finite difference scheme by a time marching algorithm. Formulation of doubly-connectedness is introduced, whereas a conformal mapping coordinate system is used in conjunction with Jacobian elliptic functions. Heat and fluid flow characteristics are given for some cases in combinations of configurations and flow parameters.

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1. Introduction

As far as two-dimensional steady-state natural convection around a single body or finitely many bodies in an infinite extension with a far away stationary and uniform thermal field is concerned, surface temperature on the said body or bodies cannot be uniform (different from the far away uniform temperature) in a mathematical point of view for common uniform substances. Thus, far away

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field behaviour (coupled with thermal and dynamical behaviour), independently of infinite extension or finite extension, should be treated with care [1]. In the following, two-dimensional steady-state laminar natural convection around a rectangular cylinder partially heated in an infinite extension is treated.

2. Analysis

2.1. Basic Equations

It is assumed that the two parallel sides of the rectangular cylinder are placed horizontally and that the surounding fluid is Newtonian, substantially incompressible except temperature dependency of density (Boussinesq approximation). Moreover it is assumed that the far away field is substantially stationary with a uniform temperature and that the cylinder surface is kept at a suitable constant uniform temperature (except a heated part at a different temperature). That is, under a conformal mapping system (α, β) with a Cartesian coordinate system (x, y), the dimensionless system of equations consists of

$$\frac{J}{r^2}\frac{\partial}{\partial t}\zeta + \frac{1}{r}\frac{\partial(\zeta,\psi)}{\partial(r,\beta)} = \frac{1}{\sqrt{Gr}}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \beta^2}\right)\zeta + \frac{1}{r}\frac{\partial(T,y)}{\partial(r,\beta)}, \tag{1}$$

$$\frac{J}{r^2}\zeta + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \beta^2}\right)\psi = 0, \tag{2}$$

$$\frac{J}{r^2}\frac{\partial}{\partial t}T + \frac{1}{r}\frac{\partial(T,\psi)}{\partial(r,\beta)} = \frac{1}{Pr\sqrt{Gr}}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \beta^2}\right)T\,, \tag{3}$$

for a vorticity transport equation, the relation between vorticity and a stream function, and an energy equation (neglecting dissipation terms) respectively, where ζ : vorticity, ψ : stream function [such that the complex velocity is given by $((\partial/\partial y) - i(\partial/\partial x))\psi$], T : temperature ($\equiv$ (local temperature $-$ unheated surface temperature)/(heated constant surface temperature $-$ unheated surface temperature)), Pr : Prandtl number, Gr : Grashof number, $J \equiv \partial(x,y)/\partial(\alpha,\beta)$, t : time, and (x, y) is a Cartesian coordinate such that y is verticaliy upward. The coordinate system (α, β) is so chosen that along the cylinder surface α is constant and that the point of infinity corresponds to $\alpha = -\infty$. One choice of such coordinates is given by

$$x + iy = Z(u,k) + \frac{\mathrm{cn}(u,k)}{\mathrm{sn}(u,k)}\,\mathrm{dn}(u,k) + \left\{\frac{E\,(k)}{K\,(k)} - 1 + k^2\right\}u\,, \tag{4}$$

$$i\mathrm{sn}\left\{\omega\left(u + iK\left(k'\right)\right), k^*\right\} = b\,\frac{\xi - 1}{\xi + 1}\,, \tag{5}$$

$$|\Re u| \le K(k),\ |\Im u| \le K(k'),\ \xi \equiv \mathrm{e}^{\alpha + i\beta},\ r \equiv \mathrm{e}^{\alpha},$$
$$-\infty < \alpha \le 0,\ |\beta| \le \pi,$$
$$\omega = \frac{1+k'}{2},\ b = \frac{1+k'}{k},\ k^* = \frac{1-k'}{1+k'},\ k' \equiv \sqrt{1-k^2},$$

where u is a complex; sn, cn, and dn are Jacobian elliptic functions; k and k^* in the argument of elliptic functions are moduli; $Z(,)$ is a Jacobian zeta-function; $K()$ and $E()$ are the complete elliptic integral of the first kind and of the second kind respectively. The surface is given by $\alpha = 0$ and the modulus k is determined so that the aspect ratio of the cylinder section, $\lambda(\equiv$ horizontal width / vertical height), is given by $\lambda = [E(k) - k'^2 K(k)]/[E(k') - k^2 K(k')]$. Reference length L is so defined that (half of the horizontal width) / $L = E(k) - k'^2 K(k)$.

2.2. Boundary Conditions at the Cylinder Surface

For dynamical conditions, so-called no-slip at the interface applies, that is, without loss of generality,

$$\psi(\alpha = 0, \beta, t) = \text{constant} \equiv c,\ \frac{\partial}{\partial \alpha}\psi(\alpha = 0, \beta, t) = 0. \tag{6}$$

For thermal conditions

$$T(\alpha = 0, \beta, t) = \begin{cases} 1 & (\beta_L \le \beta \le \beta_H)\ \text{(heated part)}, \\ 0 & (-\pi < \beta < \beta_L \text{ or } \beta_H < \beta \le \pi)\ \text{(otherwise)}, \end{cases} \tag{7}$$

where β_L, β_H are given a priori (assumption) and c is a constant to be determined as a function of $\lambda, Pr, Gr, \beta_L, \beta_H$, and (possibly) t, including far away conditions.

2.3. Far Away Conditions

For dynamical conditions and a thermal condition, from the assumption that the fluid is substantially stationary at infinity, without loss of generality

$$\psi(\alpha = -\infty, \beta, t) = 0,\ \zeta(\alpha = -\infty, \beta, t) = 0,\ \nabla T(\alpha = -\infty, \beta, t) = \vec{0}. \tag{8}$$

2.4. Auxiliary Condition (Doubly-Connectedness)

For a dimensionless scalar quantity p, pressue, the following identity holds: $\oint (\partial p/\partial \beta)\, d\beta = 0$ (contour integration along the surface). Under a steady-state this gives rise (from the original equations of motion, using boundary conditions) to

$$\frac{1}{\sqrt{Gr}}\oint\frac{\partial\zeta}{\partial\alpha}d\beta+\oint T\frac{\partial y}{\partial\beta}d\beta=0\,, \tag{9}$$

where $\oint$ stands for contour integration along the cylinder surface if $T\geq 0$ (which is the case).

3. Numerical Solution Procedure

3.1. Decomposition of Unknowns

A spectral finite difference scheme using a Fourier series [2] is applied, that is,

$$\begin{bmatrix}\psi\\ \zeta\\ T\end{bmatrix}\equiv\sum_{n=1}^{\infty}\begin{bmatrix}\psi_{sn}(r,t)\\ \zeta_{sn}(r,t)\\ T_{sn}(r,t)\end{bmatrix}\sin n\beta+\sum_{n=0}^{\infty}\begin{bmatrix}\psi_{cn}(r,t)\\ \zeta_{cn}(r,t)\\ T_{cn}(r,t)\end{bmatrix}\cos n\beta\,. \tag{10}$$

3.2. Treatment of Boundary Conditions

Equations (6)−(8) can be decomposed into each Fourier component, details of which are not shown here. Especially a part of equation (6) becomes

$$\begin{aligned}&\frac{9c}{h^3}\int_{-\pi}^{\pi}\frac{1}{J(r=1-h,\beta)}d\beta-\frac{3}{2h^3}\int_{-\pi}^{\pi}\frac{4\psi(1-h,\beta,t)-\psi(1-2h,\beta,t)}{J(r=1-h,\beta)}d\beta\\&-\int_{-\pi}^{\pi}\left\{\frac{r^3}{J}\frac{\partial}{\partial r}\left(\frac{J}{r^2}\right)-1\right\}_{r=1-h}\zeta(1-h,\beta,t)\,d\beta+\sqrt{Gr}\int_{-\pi}^{\pi}\left(T\frac{\partial y}{\partial\beta}\right)_{r=1}d\beta\\&=0\,,\end{aligned} \tag{11}$$

provided that $|\partial\zeta/\partial r|<+\infty$. In equation (11) $r=1-h$ is the grid point adjacent to the surface in the r-coordinate defined later.

3.3. Time Integration Scheme

Substituting equation (10) into equations (1), (2), (3) leads to separation of an independent variable β (Fourier decomposition) to give a system of simultaneous partial differential equations with respect to t (1-st order) and r (2-nd order), which is discretized in time and space using a finite difference scheme [2]. In this case non-uniform grid spacing in r is used, i.e., the n-th grid point

r_n is defined by

$$r_n = 1 - h \left\{ \frac{\sinh \gamma \, (n-1)}{\sinh \gamma} + 1 \right\}, \tag{12}$$

where $n = 0$ corresponds to the surface, and for the infinity point r_m, $r_m = 0$, where number of interior grid points (excluding the infinity and the surface point) is $m - 1$. γ : a suitable real constant (> 0), and $\gamma \to 0$ stands for a uniform mesh in r. Given a stationary initial condition $\psi(\alpha, \beta, t = 0) = \zeta(\alpha, \beta, t = 0) = T(\alpha, \beta, t = 0) = 0$, the system of equations (under finite wave numbers) can be integrated semi-implicitly, independently of the Fourier components, to get a steady-state solution, if diagonal dominant forms are adopted.

4. Numerical Results and Discussions

In the current coordinate system (α, β), along the cylinder surface β increases in the clockwise direction as shown in Figure 1. Steady-state total force per unit depth acting on the cylinder (excluding static buoyancy force) $(\rho U^2 L)$, F, is given by

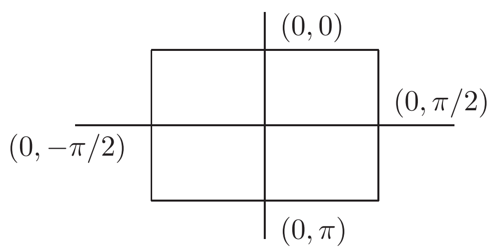

Figure 1 Typical coordinates

$$\boldsymbol{F} = -\frac{1}{\sqrt{Gr}} \int_{-\pi}^{\pi} \zeta \frac{\partial w}{\partial \beta} \, d\beta + \frac{i}{\sqrt{Gr}} \int_{-\pi}^{\pi} \frac{\partial \zeta}{\partial \alpha} w \, d\beta + i \int_{-\pi}^{\pi} wT \frac{\partial y}{\partial \beta} \, d\beta \,, \tag{13}$$

where w is a complex coordinate of a point, $U \equiv (\nu/L)\sqrt{Gr}$, ρ : density of the fluid, ν : kinematic viscosity of the fluid. Integration is to be carried out along $\alpha = 0$. The dimensionless moment of force exerted on the surface, C_M, based on $\rho U^2 L^2$ with respect to the origin is given by

$$2\sqrt{Gr}\; C_M = \int_{-\pi}^{\pi} |w|^2 \frac{\partial \zeta}{\partial \alpha} \, d\beta - \int_{-\pi}^{\pi} \zeta \frac{\partial}{\partial \alpha} |w|^2 \, d\beta + \sqrt{Gr} \int_{-\pi}^{\pi} |w|^2 T \frac{\partial y}{\partial \beta} \, d\beta \,, \tag{14}$$

where contribution from a static buoyancy force is excluded and the sign of C_M is so chosen as $C_M > 0$ or $C_M < 0$ according to a clockwise or anticlockwise

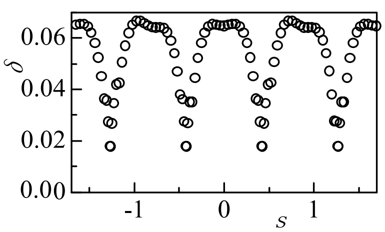

Figure 2: Boundary layer thickness

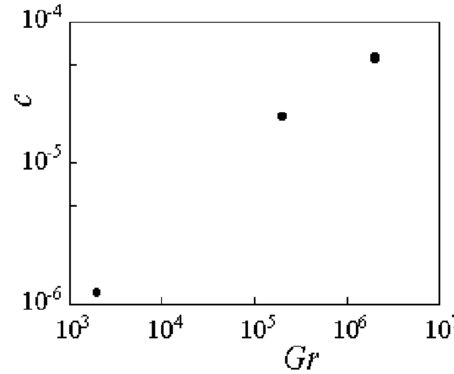

Figure 3: Flow characteristics

moment. Mean Nusselt number, Nu_m, over the heated part is given by

$$Nu_m = \int_{\beta_L}^{\beta_H} \frac{\partial T}{\partial \alpha}\, d\beta \Big/ \int_{\beta_L}^{\beta_H} \sqrt{J}\, d\beta\,, \tag{15}$$

where integration is to be carried out at $\alpha = 0$. Numerical examples are given for cases, where a part of the right vertical surface is heated. Figure 2 shows the flow boundary layer thickness δ (90% flow rate points, i.e., $\psi = 0.1 \times c$) measured normally to the surface along the cylinder surface s (measured in the clockwise direction, $s = 0$ at the top center) at $\lambda = 1 (k = 1/\sqrt{2}), Gr = 2 \times 10^6, Pr = 0.7, w(0, \beta_L) = 0.4236 + 0.35i, w(0, \beta_H) = 0.4236 - 0.35i, (\beta_L = 1.0167, \beta_H = 2.1249), T(s) = 1 (0.4972 < s < 1.1972)$, otherwise 0; used parameters are $\gamma = 0.145$, $m = 32$. For this case $\boldsymbol{F} = 0.03 - 0.369i$, $C_M = -0.077, Nu_m = 2.78$, whereas for the less heated case of $\lambda = 1 (k = 1/\sqrt{2}), Gr = 2 \times 10^6, Pr = 0.7, w(0, \beta_L) = 0.4236, w(0, \beta_H) = 0.4236 - 0.35i, (\beta_L = 1.5708, \beta_H = 2.1249), T(s) = 1 (0.8472 < s < 1.1972)$, otherwise 0, the characteristics are $\boldsymbol{F} = 0.03 - 0.369i$, $C_M = -0.039, Nu_m = 4.39$.

Figure 3 shows the flow characteristics c (the value of the stream function on the surface) against Grashof number Gr at $\lambda = 1, Pr = 0.7, \beta_L = 1.0167, \beta_H = 2.1249$. Possibility of existence of multiple solutions cannot be denyed [3], although multiple solutions will be obtained through different initial condition(s) if any, since time integration schemes are usually unique in algorithm (no branch). Geometrically the cylinder has sharp edges, so that another solution of locally separated flow pattern will be probable for high values of Gr.

5. Conclusions

Formulation on doubly-connectedness for a cylinder with non-uniform surface temperature in a two-dimensional steady-state Newtonian fluid flow under Boussinesq approximation is made. As a result, natural convection heat transfer in an infinite extension from a rectangular cylinder at non-uniform temperature is successfully analyzed under a conformal mapping, using a spectral finite difference scheme.

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