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ANALYTIC SOLUTION FOR THE CAUCHY-RIEMANN EQUATION WITH NON-LOCAL BOUNDARY CONDITIONS IN THE FIRST QUARTER

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Abstract: The Cauchy-Riemann equation has an important role in the complex theory and boundary value problems. In this paper, we consider this equation with a non-local boundary condition in the first quarter. We give an analytic solution for this boundary value problem by making use of analytic continuation in complex theory.

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Key Words: Cauchy-Riemann equation, analytic solution, non-local boundary condition

1. Introduction

Laplace equation and Cauchy-Riemann equations are the best kind of ellipitic equations. Boundary value problems for the laplace and possion equations are often reduced to the second kind of Fredholm integral equations [1] and [2]. Then the solving process can be continued by numerical methods. But this reduction process to the integral equation about Cauchy-Riemann equation in the first quarter is not possible. Because for this equation in this region, we

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cannot determine and obtain the Greens function [4]. Since the fundamental solution of Cauchy-Riemann equation is not dependent on the distance in the plane, and for this, we cannot determine the Green function. In this paper, we give the analytic solution for this equation in the first quarter.

For this, we will use the analytic solution of Cauchy-Riemann equation in the upper half plane by same process in [4].

2. Mathematical Statement of Problem

We consider the Cauchy-Riemann equation in the first quarter of plane with non-local boundary conditions:

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = 0, \quad x_1 > 0, x_2 > 0\,, \tag{1}$$

$$u(t,o) + u(0,t) = \phi(t), \quad t \geq 0\,, \tag{2}$$

where $i = \sqrt{-1}$, $\phi(t)$ is a arbitrary and continues function.

It is easy to see that general solution of equation (1) in the following form:

$$u(x) = \Phi(x_1 - ix_2)\,, \tag{3}$$

where $\Phi(x)$ is a arbitrary function.

Now, we consider this solution with boundary condition (2), we have

$$\Phi(t) + \Phi(-it) = \phi(t), \quad t \geq 0\,, \tag{4}$$

we suppose the function $\phi(t)$ is defined for complex values on imaginary-axis. By replacing $(-it)$ with (t) in (4), we have:

$$\Phi(-it) + \Phi(-t) = \phi(-it), \quad t \geq 0\,. \tag{5}$$

By subtracting (4) from (5), we obtain:

$$\Phi(-t) - \Phi(t) = \phi(-it) - \phi(t), \quad t \geq 0\,. \tag{6}$$

We conclude from (3):

$$\Phi(-t) = u(-t,0)$$

and

$$\Phi(t) = u(t,0)\,.$$

By considering these values, (6) is written in the following form:

$$u(-t,0) - u(t,0) = \phi(-it) - \phi(t), \quad t \geq 0\,. \tag{7}$$

Hence, the non-local boundary condition on the real-axis is given by (7). By using this condition, we have a new boundary value problem for Cauchy-Riemann equation in the upper half plane. By regarding this situation, we can write the following new boundary condition for problem (1)-(2):

$$u(-t,0)+u(t,0)=\frac{2i}{\pi}\int_{o}^{\infty}\frac{x_1}{x_1^2-t^2}\left[\phi(-ix_1)-\phi(x_1)\right]dx_1,\quad t\geq 0\,. \tag{8}$$

Now by adding and subtracting of relations (7) and (8) together, we have the following relations respectively:

$$\begin{aligned} u(-t,0)&=\frac{i}{\pi}\int_{0}^{\infty}\frac{x_1}{x_1^2-t^2}\left[\phi(-ix_1)-\phi(x_1)\right]dx_1+\frac{\phi(-it)-\phi(t)}{2}\,,\\ u(t,o)&=\frac{i}{\pi}\int_{0}^{\infty}\frac{x_1}{x_1^2-t^2}\left[\phi(-ix_1)-\phi(x_1)\right]dx_1-\frac{\phi(-it)-\phi(t)}{2}\,. \end{aligned} \tag{9}$$

Now, we can replace these values for $u(t,0)$ and $u(-t,0)$ in the expression of analytic solution for the Cauchy-Riemann equation in the half plane. For this, we need to use the same process for obtaining analytic solution for the Cauchy-Riemann equation [4]: At first we consider the fundamental solution of equation (1) in the following form which has been come in[7]:

$$U(X-\xi)=\frac{1}{2\pi}\cdot\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}\,. \tag{10}$$

Now, if we multiply the both sides of equation (1) to (10) and integrate on upper half-plane, then we have:

$$\int_R dx_1\int_0^{\infty}\frac{\partial u(x)}{\partial x_2}U(X-\xi)dx_2+i\int_0^{\infty}dx_2\int_R\frac{\partial u(x)}{\partial x_1}U(X-\xi)dx_1=0\,.$$

If we apply the Astouragradeski formula for the left side of above equation, then by making use the properties of Delta-Dirac function we will have:

$$-\int_R u(x_1,0)U(x_1-\xi_1,-\xi_2)dx_1=\begin{cases} u(\xi) & \text{if}\quad \xi_1\in R\ \ \xi_2>0\,,\\ \frac{1}{2}u(\xi) & \text{if}\quad \xi_1\in R\ \ \xi_2=0\,. \end{cases} \tag{11}$$

We will have from the second case of (11):

$$\frac{1}{2}u(\xi_1,0)=-\int_R u(x_1,0)U(x_1-\xi_1)dx_1$$

By considering of fundamental solution (10), we obtain:

$$\begin{cases} u(-\xi_1,0)=\dfrac{i}{\pi}\displaystyle\int_0^{\infty}\left[\frac{u(x_1,0)}{x_1+\xi_1}-\frac{u(-x_1,0)}{x_1-\xi_1}\right]dx_1 & ;\quad \xi_1>0\,,\\ u(\xi_1,0)=\dfrac{i}{\pi}\displaystyle\int_0^{\infty}\left[\frac{u(x_1,0)}{x_1-\xi_1}-\frac{u(-x_1,0)}{x_1+\xi_1}\right]dx_1 & ;\quad \xi_1>0\,, \end{cases} \tag{12}$$

we consider from the necessary conditions (12) that we can give the following boundary condition on $x_2=0$.

$$u(x_1,0)-u(-x_1,0)=\varphi(x_1)\,,\quad x_1>0\,, \tag{13}$$

where $\varphi(x_1)$ is a continuous and arbitrary function. In this way, we conclude from (12):

$$u(\xi_1,0)+u(-\xi_1,0)=\frac{2i}{\pi}\int_0^\infty \frac{x_1}{x_1^2-\xi_1^2}\varphi(x_1)dx_1\,,\quad \xi_1>0\,. \tag{14}$$

Finally, if we consider (13), (14) together, we have:

$$\begin{cases} u(\xi_1,0)=\frac{\varphi(\xi_1)}{2}+\frac{i}{\pi}\int_0^\infty \frac{x_1}{x_1^2-\xi_1^2}\varphi(x_1)dx_1, & \xi_1>0\,,\\ u(-\xi_1,0)=-\frac{\varphi(\xi_1)}{2}+\frac{i}{\pi}\int_0^\infty \frac{x_1}{x_1^2-\xi_1^2}\varphi(x_1)dx_1, & \xi_1>0\,. \end{cases} \tag{15}$$

If we substitue (15) in the left side of (11), then we will have the following solution for the problem (1)-(2) and (13):

$$\begin{aligned} u(\xi)=\frac{1}{2\pi}\int_0^\infty &\left[\frac{u(\eta_1,0)}{\xi_2-i(\eta_1-\xi_1)}+\frac{u(-\eta_1,0)}{\xi_2+i(\eta_1+\xi_1)}\right]d\eta_1\\ &=\frac{i}{2\pi}\int_0^\infty \frac{\eta_1}{(\xi_2+i\xi_1)^2+\eta_1^2}\varphi(\eta_1)d\eta_1\\ &+\frac{i}{\pi^2}\int_0^\infty \frac{\xi_2+i\xi_1}{(\xi_2+i\xi_1)^2+\eta_1^2}d\eta_1\int_0^\infty \frac{x_1}{x_1^2-\eta_1^2}\varphi(x_1)dx_1\,. \end{aligned} \tag{16}$$

Now, for writing the analytic solution of main problem, we consider the final solution in the following form:

$$\begin{aligned} u(\xi)&=-\frac{1}{\pi}\int_R \frac{u(x_1,0)dx_1}{-\xi_2+i(x_1-\xi_1)}\\ &=\frac{1}{2\pi}\int_{-\infty}^0 \frac{u(t,o)}{\xi_2-i(t-\xi_1)}dt+\frac{1}{2\pi}\int_0^\infty \frac{u(t,0)}{\xi_2-i(t-\xi_1)}\\ &=\frac{1}{2\pi}\int_0^\infty \frac{u(-t,0)}{\xi_2+i(t+\xi_1)}dt+\frac{1}{2\pi}\int_0^\infty \frac{u(t,o)}{\xi_2-i(t-\xi_1)}dt\\ &=\frac{1}{2\pi}\int_0^\infty\left(\frac{i}{\pi}\int_0^\infty \frac{x_1}{x_1^2-t^2}\left[\phi(-ix_1)-\phi(x_1)\right]dx_1+\frac{\phi(-it)-\phi(t)}{2}\right)\\ &\qquad\times\frac{1}{\xi_2+i(t-\xi_1)}dt\\ &+\frac{1}{2\pi}\int_0^\infty\left(\frac{i}{\pi}\int_0^\infty \frac{x_1}{x_1^2-t^2}\left[\phi(-ix_1)-\phi(x_1)\right]dx_1-\frac{\phi(-it)-\phi(t)}{2}\right)\\ &\qquad\times\frac{1}{\xi_2-i(t-\xi_1)}dt\,. \end{aligned} \tag{17}$$

Now, we restrict the values ξ_1 and ξ_2 only for positive values, that is $\xi_2,\xi_1>0$ then the analytic solution of problem (1)-(2) is resulted. At the end, we conclude the following theorem.

Theorem. *Suppose $\phi(t)$ is a continous function which is defined on positive half-real line. If this function is definable on negative half-imaginary axis, then the boundary value problem (1)-(2) has an analytic solution which is defined only for positive values of ξ_1, ξ_2.*

Remark 1. If in the boundary condition (2) instead of addition of boundary values of unknown function, the subtraction of its boundary values is given, then the corresponding relation to (7) will be in the form of addition of its boundary values.

Remark 2. If in the boundary condition (2) points are not symmetric with respect to coordinate origin, then we cannot present analytic solution of problem. In this case, we can reduce the problem to the second kind of Fredholm integral equations.

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