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ON INEQUALITIES OF THE HILBERT'S TYPE

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Abstract: By introducing the function $\frac{|\ln x - \ln y|}{x+y+\max\{x,y\}}$, we establish a new inequalities similar to Hilbert's type inequality with the best constants factor.

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1. Introduction

If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (cf. Hardy et al [4])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\}^{\frac{1}{2}}, \qquad (1.1)$$

where the constant factor π is the best possible. Inequality (1.1) is the well known Hilbert's inequality. Inequality (1.1) had been generalized by Hardy-Riesz (see [3]) in 1925 as:

If f, g are real functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

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$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}\{\int_0^\infty f^p(x)dx\}^{1/p}\{\int_0^\infty g^q(x)dx\}^{1/q}, \quad (1.2)$$

where the constant factor $c = \frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible. When $p = q = 2$, (1.2) reduces to (1.1). Inequality (1.2) is named of Hardy-Hilbert's integral inequality, which is important in analysis and its applications (see [9]). It has been studied and generalized in many directions by a number of mathematicians (see [1], [2], [6], [8], [10], [11], [12]).

Recently, Li, Wu and He [7] obtained the following theorem.

Theorem 1.1. *If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Then we have*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}\mathrm{d}x\mathrm{d}y < c\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(x)\mathrm{d}x\right)^{\frac{1}{2}},$$

where the constant factor $c = \sqrt{2}(\pi - 2 - \arctan\sqrt{2}) = 1.7408....$

The main purpose of the present article is to establish a new inequalities similar to Hilbert's type inequalities.

2. Main Results and Applications

Theorem 2.1. *If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Then we have*

$$\int_0^\infty\int_0^\infty \frac{|\ln x - \ln y|}{x+y+\max\{x,y\}}f(x)g(y)\mathrm{d}x\mathrm{d}y < A\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(x)\mathrm{d}x\right)^{\frac{1}{2}}, \quad (2.1)$$

where the constant factor $A = 3.8099...$ is the best possible.

Proof. By Hölder inequality, we have

$$\int_0^\infty\int_0^\infty \frac{|\ln x - \ln y|}{x+y+\max\{x,y\}}f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$= \int_0^\infty\int_0^\infty \left[\left(\frac{|\ln x - \ln y|}{x+y+\max\{x,y\}}\right)^{\frac{1}{2}} f(x)\left(\frac{x}{y}\right)^{\frac{1}{4}}\right]$$
$$\left[\left(\frac{|\ln x - \ln y|}{x+y+\max\{x,y\}}\right)^{\frac{1}{2}} g(y))\left(\frac{y}{x}\right)^{\frac{1}{4}}\right]\mathrm{d}x\mathrm{d}y$$

$$\leq \left\{ \int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} \left(\frac{x}{y}\right)^{\frac{1}{2}} \mathrm{d}y \right) f^2(x) \mathrm{d}x \right\}$$
$$\left\{ \int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} \left(\frac{y}{x}\right)^{\frac{1}{2}} \mathrm{d}x \right) g^2(y) \mathrm{d}y \right\}.$$

Define the weight function $\varpi(u)$ as

$$\omega(u) := \int_0^\infty \frac{|\ln u - \ln v|}{u + v + \max\{u, v\}} \left(\frac{u}{v}\right)^{\frac{1}{2}} \mathrm{d}v.$$

For fixed u, let $v = ut$, we have

$$\omega(u) = \int_0^\infty \frac{|\ln u - \ln tu|}{u + tu + \max\{u, tu\}} \left(\frac{1}{t}\right)^{\frac{1}{2}} u\mathrm{d}t = \int_0^\infty \frac{|\ln t|}{1 + t + \max\{1, t\}} \left(\frac{1}{t}\right)^{\frac{1}{2}} \mathrm{d}t$$
$$= -\int_0^1 \frac{\ln t}{2 + t} \left(\frac{1}{t}\right)^{\frac{1}{2}} \mathrm{d}t + \int_1^\infty \frac{\ln t}{1 + 2t} \left(\frac{1}{t}\right)^{\frac{1}{2}} \mathrm{d}t$$
$$= -2 \int_0^1 \frac{\ln t}{2 + t} \left(\frac{1}{t}\right)^{\frac{1}{2}} \mathrm{d}t = -8 \int_0^1 \frac{\ln s}{2 + s^2} \mathrm{d}s \left(t^{\frac{1}{2}} = s\right) = A.$$

Thus

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$\leq A \left(\int_0^\infty f^2(x)\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_0^\infty g^2(x)\mathrm{d}x \right)^{\frac{1}{2}}. \quad (2.2)$$

If (2.2) takes the form of the equality, then there exist constants c and d, such that they are not all zero and (see [5])

$$c\frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} f^2(x) \left(\frac{x}{y}\right)^{\frac{1}{2}} = d\frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} g^2(y) \left(\frac{y}{x}\right)^{\frac{1}{2}},$$
$$\text{a.e. on } (0, \infty) \times (0, \infty).$$

Then we have

$$cxf^2(x) = dyg^2(y), \quad \text{a.e. on } (0, \infty) \times (0, \infty).$$

Hence we have

$$cxf^2(x) = dyg^2(y) = \text{constant}, \quad \text{a.e. on } (0, \infty) \times (0, \infty).$$

Without losing the generality, suppose $c \neq 0$, then

$$\int_0^\infty f^2(x)\mathrm{d}x = \int_0^\infty \frac{1}{x} \frac{\text{const}}{c} \mathrm{d}x = \frac{\text{const}}{c} \int_0^\infty \frac{1}{x} \mathrm{d}x,$$

which contradicts the facts that $0 < \int_0^\infty f^2(x)dx < \infty$. Hence (2.2) takes the form of strict inequality. So we have (2.1).

Assume that the constant factor $A = -8\int_0^1 \frac{\ln s}{2+s^2}\mathrm{d}s = -4\log\log\frac{3}{2}$ in (2.1) is not the best possible, then there exists a positive number K with $K < A$ and $a > 0$, We have

$$\int_a^\infty\int_0^\infty \frac{|\ln x - \ln y|}{x+y+\max\{x,y\}} f(x)g(y)\mathrm{d}x\mathrm{d}y < K\left(\int_a^\infty f^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^\infty g^2(x)\mathrm{d}x\right)^{\frac{1}{2}}, \quad (2.3)$$

For $0 < \varepsilon < 1$, setting $b > 0 (b < a)$, $f_\varepsilon(x) = x^{\frac{-\varepsilon-1}{2}}$, for $x \in [b,\infty)$; $f_\varepsilon(x) = 0$, for $x \in (0,b)$. $g_\varepsilon(y) = y^{\frac{-\varepsilon-1}{2}}$, for $y \in [b,\infty)$; $g_\varepsilon(y) = 0$, for $y \in (0,b)$.

Since

$$K\left(\int_a^\infty f^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^\infty g^2(x)\mathrm{d}x\right)^{\frac{1}{2}} = K\int_a^\infty x^{-(1+\varepsilon)}\mathrm{d}x = K\cdot\frac{1}{\varepsilon a^\varepsilon},$$

setting $y = ux$, we find

$$\begin{aligned}
&\int_a^\infty\int_0^\infty \frac{|\ln x - \ln y|}{x+y+\max\{x,y\}} f_\varepsilon(x)g_\varepsilon(y)\mathrm{d}x\mathrm{d}y \\
=\ &\int_a^\infty\int_b^\infty \frac{|\ln x - \ln y|}{x+y+\max\{x,y\}} x^{-\frac{1+\varepsilon}{2}}y^{-\frac{1+\varepsilon}{2}}\mathrm{d}x\mathrm{d}y \\
=\ &\int_a^\infty\int_{b/x}^\infty \frac{|\ln x - \ln ux|}{x+ux+\max\{x,ux\}} x\cdot x^{-(1+\varepsilon)}u^{-\frac{1+\varepsilon}{2}}\mathrm{d}x\mathrm{d}u \\
=\ &\int_a^\infty\int_{b/x}^\infty \frac{x^{-(1+\varepsilon)}u^{-\frac{1+\varepsilon}{2}}|\ln u|}{1+u+\max\{1,u\}}\mathrm{d}x\mathrm{d}u.
\end{aligned}$$

By (2.3) and for $b \to 0^+$, we have

$$\int_a^\infty\int_0^\infty \frac{x^{-(1+\varepsilon)}u^{-\frac{1+\varepsilon}{2}}|\ln u|}{1+u+\max\{1,u\}}\mathrm{d}x\mathrm{d}u \le K\cdot\frac{1}{\varepsilon a^\varepsilon},$$

or

$$\frac{1}{\varepsilon a^\varepsilon}\int_0^\infty \frac{u^{-\frac{1+\varepsilon}{2}}|\ln u|}{1+u+\max\{1,u\}}\mathrm{d}u \le K\cdot\frac{1}{\varepsilon a^\varepsilon},$$

that is $\int_0^\infty \frac{u^{-\frac{1+\varepsilon}{2}}|\ln u|}{1+u+\max\{1,u\}}\mathrm{d}u \le K$.

When $\varepsilon \to 0^+$, we have

$$\int_0^\infty \frac{u^{-\frac{1+\varepsilon}{2}}|\ln u|}{1+u+\max\{1,u\}}\mathrm{d}u = \int_0^\infty \frac{u^{-\frac{1}{2}}|\ln u|}{1+u+\max\{1,u\}}\mathrm{d}u + o(1) = A + o(1).$$

This contradicts the hypothesis. Hence the constant factor A in (2.1) is the best possible. □

Theorem 2.2. *Suppose $f \geq 0$ and $0 < \int_0^\infty f^2(x)dx < \infty$. Then*

$$\int_0^\infty [\int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} f(x)dx]^2 dy < A^2 \int_0^\infty f^2(x)dx, \tag{2.3}$$

where the constant factor $A^2 = 14.515338...$ is the best possible. Inequality (2.3) is equivalent to (2.1).

Proof. Let $g(y) = \int_0^\infty \frac{|\ln x - \ln y|}{x+y+\max\{x,y\}} f(x)dx$, then by (2.1), we get

$$\begin{aligned} 0 < \int_0^\infty g^2(y)dy &= \int_0^\infty [\int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} f(x)dx]^2 dy \\ &= \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} f(x)g(y)dxdy \\ &\leq A\{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}} \{\int_0^\infty g^2(y)dy\}^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

Hence we obtain

$$0 < \int_0^\infty g^2(y)dy = A^2 \int_0^\infty f^2(x)dx < \infty. \tag{2.5}$$

By (2.1), both (2.4) and (2.5) take the form of strict inequality, so we have (2.6).

On the other hand, suppose that (2.3) is valid. By Hölder's inequality, we find

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} f(x)g(y)dxdy \\ &= \int_0^\infty [\int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} f(x)dx] g(y)dy \\ &\leq \{\int_0^\infty [\int_0^\infty \frac{|\ln x - \ln y|}{x + y + \max\{x, y\}} f(x)dx]^2 dy\}^{1/2} \{\int_0^\infty g^2(x)dx\}^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

By (2.3), we have (2.1). Thus (2.1) and (2.3) are equivalent.

If the constant A^2 in (2.3) is not the best possible, by (2.6), we may get a contradiction that the constant factor in (2.1) is not the best possible. This completes the proof. □

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