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COMPOUND NEGATIVE BINOMIAL RISK MODEL FOR DOUBLE TYPE-INSURANCE WITH INVESTMENT AND INTERFERE

Li Suoping[1 §], Liu Qi[2]
[1,2]Department of Applied Mathematics
Lanzhou University of Technology
Lanzhou, 730050, P.R. CHINA
[1]e-mail: lsuop@163.com

Abstract: By considering the effect on company business from the random premium and inflations, and taking the surplus capital as investment to enhance the company payment's capacity for the policy-holder, we proposed a compound negative binomial risk model for the inhomogeneous double type-insurance. For the proposed model, some basic properties of the surplus process were analyzed to obtain its stationary increment properties and statistical character. It also was derived that the formula of the ultimate ruin probability of risk process and its Lundberg inequality.

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1. Introduction

The classical risk model was first considered by Filip Lundberg in 1903. His use of the standard compound Poisson model was later made mathematically rigorous by Harald Cramer in the 1930s (see [3], [6]). The model, also known as the Cramer-Lundberg risk model, has since then been extend in various

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§Correspondence author

ways: general renewal processes and Cox processes replace the Poisson process; a random environment allows for random changes in the intensity of the claims process and of the claim size distribution; dependent claims for the claims process; interest rates are considered in the premium income side; and piecewise deterministic Markov processes provide new insight and models. The quantities of interest in ruin theory are the probabilities associated with the random time of ruin and to obtain the exact formulas or approximations of ruin probabilities in various kinds risk models. Many investigations focused on the continuous time risk models, especially Poisson risk models. But in fact, the research of the discrete time risk models is more practical for insurance business because of dispersing the continuous time. In the discrete time risk models, the compound binomial model, which is a discrete time analogue of the compound Poisson model when the mean of the number of claims exceeds its variance, has been frequently used for the number of insurance claims in a fixed time period when the risks are homogeneous. But when the variance of the number of claims exceeds its mean, the compound binomial model is not appropriate, especially for the inhomogeneous risks. Actually actuaries early recognized that a Poisson distribution seldom fits the real data and estimating parameters in a negative binomial distribution yield satisfactory results.

There are many actuarial literatures on the applications of the negative binomial distributions in insurance (see [1]-[5]). See for example, Carlson (1962), Hewitt (1960), Simon (1960) and Edward (2007). For numerous non-actuarial applications see Johnson et al.(1992). The negative binomial distributions have been frequently proposed as a reasonable model for the number of insurance claims when the risks are not homogeneous. Such suggestions are in general motivated by the following reasons: The negative binomial distributions allow for more flexibility than the traditional Poisson because the former possesses two parameters; the claims generating process may have a contagion; the claims generating process may be a gamma mixture of Poisson distributions because of imperfect classification of risks (see [2]). So in this paper, we proposed a compound negative binomial risk model with the inhomogeneous double type-insurance when the claims occur. We not only consider the random disturbed factors but also involve the investment of the insurance company's surplus to enhance the company payment's capacity for the policy-holder in the proposed model. For the proposed model, some basic properties of the surplus process were discussed, it also was obtained that the formula of its ultimate ruin probability and the Lundberg inequality.

2. Presented Model and Its Basic Properties

First, let us review the negative binomial distribution. Assuming the random event H gets *success* with the probability p at the Bernoulli experiment every times, then call the random variable N, which is the number of H *failing* before n times *success* of the event H, yields the negative binomial distribution with the parameters (n, p) if the probability function given by

$$P(N = k) = \binom{n+k-1}{k} p^n q^k \quad (k = 0, 1, 2, \cdots),$$

where $q = 1 - p$ with $0 < p < 1$ and $n > 0$. For this distribution, it is easy to get $E(N) = nq/p$, $Var(N) = nq/p^2$.

Definition 1. Let $\{N(n)\}_{n=0}^{\infty}$ denote sequence of non-negative integer-valued random variables. For any arbitrary $n_2 > n_1$, $N(n_2) - N(n_1)$ follows the negative binomial distributions with the parameters $(n_2 - n_1, p)$, namely

$$P(N(n_2) - N(n_1) = k) = \binom{n_2 - n_1 + k - 1}{k} p^{n_2 - n_1} q^k,$$

then $\{N(n)\}_{n=0}^{\infty}$ is called as negative binomial random sequence (NBRS) with the parameters (n, p).

From Definition 1, it is easy to prove that the NBRS is stationary and independent increment process. Next we introduce the risk model which will be investigated in this paper and analyze its basic properties.

Definition 2. In a given complete probability space $(\Omega, \mathbb{F}, P)$ endowed with the filtration $\{\mathcal{F}_n, n \geq 0\}$ and a standard Brownian motion $\{W_n, n \geq 0\}$ adapted to $\mathcal{F}_n$, for nonnegative u which is an initial surplus, up to the end of n-th period the surplus process $U(n)$ of the insurance company is defined as

$$U(n) = (u - F) + F(1 + nh) + Z(n) - S_1(n) - S_2(n) + \sigma W_n (n = 0, 1, 2, \cdots)$$

where $R(n) = nhF + Z(n) - S_1(n) - S_2(n) + \sigma W_n$ is payoff process of the insurance company, F is the company fund for investing, h is the income rate of investment, and σ is positive constant. The terms $Z(n)$, $S_1(n)$ and $S_2(n)$ of $U(n)$ follow the different compound negative binomial distributions, that is:

1) $Z(n) = \sum_{k=1}^{M(n)} z_k$ is the cumulated premium corresponding to the number $M(n)$ of insurance policy up to the end of n-th period, and z_k denotes the k-th premium. $\{M(n)\}_{n=1}^{\infty}$ is NBRS with the parameters (n, p) and $q = 1 - p$.

2) $S_1(n) = \sum_{i=1}^{N_1(n)} X_i$ is the aggregate claims corresponding to the number $N_1(n)$ of claims occurring up to the end of n-th period for the 1-th type-insurance, and X_i denotes the i-th claim size. $\{N_1(n)\}_{n=1}^{\infty}$ is NBRS with the parameters (n, a) and $b = 1 - a$.

3) $S_2(n) = \sum_{j=1}^{N_2(n)} Y_j$ is the aggregate claims corresponding to the number $N_2(n)$ of claims occurring up to the end of n-th period for the 2-th type-insurance, and Y_j denotes the j-th claim size. $\{N_2(n)\}_{n=1}^{\infty}$ is NBRS with the parameters (n, c) and $d = 1 - c$.

4) $\{z_k\}_{k=1}^{\infty}$ $\{X_i\}_{i=1}^{\infty}$ and $\{Y_j\}_{j=1}^{\infty}$ are assumed to be positive random variables sequences of independent and identical distribution respectively, and supposing $E(z_k) = \mu_1$, $E(X_i) = \mu_2$, $E(Y_j) = \mu_3$. All the above random sequences also are assumed to be independent each other.

Theorem 3. *The surplus process* $\{R(n) = nhF + Z(n) - S_1(n) - S_2(n) + \sigma W_n, n = 1, 2, \cdots\}$ *is stationary and independent increment process.*

Proof. Letting $0 \le n_0 < n_1 < \cdots < n_{m-1} < n_m < \cdots < n_s$, we have

$$R(n_m) - R(n_{m-1}) = n_m hF + \sum_{k=1}^{M(n_m)} z_k - \sum_{i=1}^{N_1(n_m)} X_i - \sum_{j=1}^{N_2(n_m)} Y_j + \sigma W_{n_m}$$

$$-(n_{m-1}hF + \sum_{k=1}^{M(n_{m-1})} z_k - \sum_{i=1}^{N_1(n_{m-1})} X_i - \sum_{j=1}^{N_2(n_{m-1})} Y_j + \sigma W_{n_{m-1}})$$

$$= (n_m - n_{m-1})hF + (\sum_{k=1}^{M(n_m)} z_k - \sum_{k=1}^{M(n_{m-1})} z_k) - (\sum_{i=1}^{N_1(n_m)} X_i - \sum_{i=1}^{N_1(n_{m-1})} X_i)$$

$$-(\sum_{j=1}^{N_2(n_m)} Y_j - \sum_{j=1}^{N_2(n_{m-1})} Y_j) + \sigma(W_{n_m} - W_{n_{m-1}}).$$

Because the sequences

$$\{\sum_{k=1}^{M(n_m)} z_k - \sum_{k=1}^{M(n_{m-1})} z_k\}_{m=1}^{s}, \ \{\sum_{i=1}^{N_1(n_m)} X_i - \sum_{i=1}^{N_1(n_{m-1})} X_i\}_{m=1}^{s},$$

$$\{\sum_{j=1}^{N_2(n_m)} Y_j - \sum_{j=1}^{N_2(n_{m-1})} Y_j\}_{m=1}^{s},$$

and $\{W_{n_m} - W_{n_{m-1}}\}_{m=1}^{s}$ are independent increment themselves respectively, so the sequence $\{\sum_{k=1}^{M(n_m)} R_k - \sum_{k=1}^{M(n_{m-1})} R_k\}_{m=1}^{s}$ of $\{R(n)\}$ also is independent increment, i.e., $\{R(n), n = 1, 2, \cdots\}$ is an independent increment process.

Further, it holds that

$$R(n+m) - R(n) = mhF + (\sum_{k=1}^{M(n+m)} z_k - \sum_{k=1}^{M(n)} z_k) - (\sum_{i=1}^{N_1(n+m)} X_i - \sum_{i=1}^{N_1(n)} X_i) - (\sum_{j=1}^{N_2(n+m)} Y_j - \sum_{j=1}^{N_2(n)} Y_j) + \sigma(W_{n+m} - W_n)\,.$$

For all $n > 0$, the increments $\sum_{k=1}^{M(n+m)} z_k - \sum_{k=1}^{M(n)} z_k$, $\sum_{i=1}^{N_1(n+m)} X_i - \sum_{i=1}^{N_1(n)} X_i$, $\sum_{j=1}^{N_2(n+m)} Y_j - \sum_{j=1}^{N_2(n)} Y_j$ and $W_{n+m} - W_n$ have the same distributions respectively, so $\{R(n), n = 1, 2, \cdots\}$ also is a stationary increment process. □

Theorem 4. *The risk process* $\{U(n) = u + Fnh + Z(n) - S_1(n) - S_2(n) + \sigma W_n, n = 1, 2, \cdots\}$ *has the following properties:*

(1) $E\left[U\left(n\right)\right] = u + Fnh + n\frac{q}{p}\mu_1 - n\frac{b}{a}\mu_2 - n\frac{d}{c}\mu_3$.

(2) $Var\left(U\left(n\right)\right) = Var\left(R\left(n\right)\right)$

$$= n\left(\frac{q}{p}\gamma_1 + \frac{q^2}{p^2}\mu_1^2 + \frac{b}{a}\gamma_2 + \frac{b^2}{a^2}\mu_2^2 + \frac{d}{c}\gamma_3 + \frac{d^2}{c^2}\mu_3^2 + \sigma^2\right),$$

where $\gamma_1 = E[z_k^2]$, $\gamma_2 = E[X_i^2]$ *and* $\gamma_3 = E[Y_j^2]$.

It is not difficult to complete the proof of Theorem 2, so its proof is omitted here.

3. Ruin Probability of Presented Model

Usually we assume that in unit time the sum of the average premium and average investment income exceeds the average claim size to ensure the insurance company's operation steadily, that means it is necessary for holding $acpFh > cbp\mu_2 + adp\mu_3 - acq\mu_1$. The result leads us to define a *safety load coefficient* by letting

$$\rho = \frac{acpFh}{cbp\mu_2 + adp\mu_3 - acq\mu_1} - 1 > 0\,.$$

In addition, the ruin time of the insurance company is denoted by $T = \{n \geq 1 | U(n) < 0\}$, and the ultimate ruin probability is denoted by $\Psi(u) = P\{T < \infty | U(0) = u\}$.

Theorem 5. *For the risk process $\{U(n), n = 1, 2, \cdots\}$, let*

$$g(r) = \frac{\sigma^2 r^2}{2} - rhF + \ln \frac{p}{1 - qM_Z(-r)} + \ln \frac{a}{1 - bM_X(r)} + \ln \frac{c}{1 - dM_Y(r)},$$

then there exists only one positive solution R in equation $g(r) = 0$, i.e. $g(R) = 0$. The solution R is called as adjust coefficient of risk process $U(n)$.

Proof. First, we have $g(0) = \ln \frac{p}{1-q} + \ln \frac{a}{1-b} + \ln \frac{c}{1-d} = 0$, and

$$g'(r) = \sigma^2 r - hF - \frac{qE[Ze^{-Zr}]}{1 - qM_Z(-r)} + \frac{bE[Xe^{Xr}]}{1 - bM_X(r)} + \frac{dE[Ye^{Yr}]}{1 - dM_Y(r)}$$

So it can be obtained that

$$g'(0) = -hF - \frac{qE[Z]}{1-q} + \frac{bE[X]}{1-b} + \frac{dE[Y]}{1-d} = \frac{-pachF - qac\mu_1 + pcb\mu_2 + pad\mu_3}{pac} < 0$$

$$g''(r) = \sigma^2 + \frac{qE[Z^2 e^{-Zr}][1 - qM_Z(-r)] + (qE[Ze^{-Zr}])^2}{(1 - qM_Z(-r))^2} + \frac{bE[X^2 e^{Xr}][1 - bM_X(r)] + (bE[Xe^{Xr}])^2}{(1 - bM_X(r))^2} + \frac{dE[Y^2 e^{Yr}][1 - dM_Y(r)] + (dE[Ye^{Yr}])^2}{(1 - dM_Y(r))^2}.$$

From the above $g''(r)$, it holds that $g''(r) > 0$ when $r > 0$, so the curve $g(r)$ is convex down. And $g'(0) < 0$ indicates that the tangent slope of $g(r)$ is below zero, where $r = 0$. So as long as the value of r is big enough, the $g(r)$ will become infinite. These properties deduced that there exists only one positive solution R for equation $g(r) = 0$. □

Theorem 6. *For $\forall u \geq 0$, the ruin probability $\Psi(u)$ of the risk process $U(n)$ satisfies with the following equation*

$$\Psi(u) = \frac{e^{-Ru}}{E\left[e^{-RU(T)} | T < \infty\right]},$$

where $U(n) = (u - F) + F(1 + nh) + Z(n) - S_1(n) - S_2(n) + \sigma W_n$ $(n = 1, 2, \cdots)$, and R is the adjust coefficient of $U(n)$.

Proof. For $n > 0$ and $r > 0$, by the full-conditions expectation formula, it holds that

$$E[e^{-rU(n)}] = E[e^{-rU(n)}|T < n]P(T < n) + E[e^{-rU(n)}|T \geq n]P(T \geq n) \quad (1)$$

Applying $U(n) = u + nhF + Z(n) - S_1(n) - S_2(n) + \sigma W_n$, the left side of equation (1) can be transformed to

$$E[e^{-rU(n)}] = e^{-r(u+nhF)} \cdot E[e^{-rZ(n)} \cdot e^{rS_1(n)} \cdot e^{rS_2(n)} \cdot e^{-r\sigma W_n}]$$

$$= e^{-r(u+nhF)} (\frac{p}{1 - qM_Z(-r)})^n (\frac{a}{1 - bM_X(r)})^n (\frac{c}{1 - dM_Y(r)})^n e^{\frac{\sigma^2 r^2}{2} n}$$

$$= e^{-ru} exp\{n[-rhF + \ln \frac{p}{1 - qM_Z(-r)} + \ln \frac{a}{1 - bM_X(r)} + \ln \frac{c}{1 - dM_Y(r)} + \frac{\sigma^2 r^2}{2}]\}$$

$$= e^{-ru} \cdot e^{ng(r)} . \quad (2)$$

For an given $T > 0$, when $T < n$, then $U(n)$ can be written as

$$U(n) = U(T) + U(n) - U(T) = U(T) + Fh(n - T) + (Z(n) - Z(T))$$

$$- (S_1(n) - S_1(T)) - (S_2(n) - S_2(T)) + \sigma (W_n - W_T) .$$

Here all of $Z(n) - Z(T)$, $S_1(n) - S_1(T)$, $S_2(n) - S_2(T)$ and $W_n - W_T$ are independent each other, and follow the compound negative binomial distributions with the parameters $(n - T, p)$, $(n - T, a)$ and $(n - T, c)$ respectively. So the first term in the right side of equation (1) can be transformed into

$$E[e^{-rU(n)}|T < n]P(T < n)$$

$$= E[e^{-rU(T)} \exp\{-rFh(n - T) - r(Z(n) - Z(T)) + r(S_1(n) - S_1(T))$$

$$+ r(S_2(n) - S_2(T)) - r\sigma(W_n - W_T)\}|T < n]P(T < n)$$

$$= E[e^{-rU(T)} e^{-rFh(n-T)} e^{\frac{\sigma^2 r^2}{2}(n-T)}$$

$$\times (\frac{p}{1 - qM_Z(-r)} \frac{a}{1 - bM_X(r)} \frac{c}{1 - dM_Y(r)})^{n-T} |T < n] \cdot P(T < n)$$

$$= E[e^{-rU(T)} e^{(n-T)g(r)}|T < n]P(T < n) . \quad (3)$$

Putting (2)(3) into (1), then equation (1) can be written as

$$e^{-ru} \cdot e^{ng(r)} = E[e^{-rU(T)} e^{(n-T)g(r)}|T < n]P(T < n)$$

$$+ E[e^{-rU(n)}|T \geq n]P(T \geq n) . \quad (3)$$

Letting $r = R$ in (4) and by $g(R) = 0$ of Theorem 5, it can be obtained that

$$e^{-Ru} = E[e^{-RU(T)}|T < n]P(T < n) + E[e^{-RU(n)}|T \geq n]P(T \geq n) \quad (5)$$

In (5), when $n \to \infty$, the first term of the right side will become $E[e^{-RU(T)}| T < \infty]P(T < \infty)$. If it can be proved that $E[e^{-RU(n)}|T \geq n]P(T \geq n) \to 0$

$(n \to \infty)$, the proof can be completed. In order to achieve it, define

$$\alpha = Fh + \frac{q}{p}\mu_1 - \frac{b}{a}\mu_2 - \frac{d}{c}\mu_3\,,$$

$$\beta^2 = \frac{q}{p}\gamma_1 + \frac{q^2}{p^2}\mu_1^2 + \frac{b}{a}\gamma_2 + \frac{b^2}{a^2}\mu_2^2 + \frac{d}{c}\gamma_3 + \frac{d^2}{c^2}\mu_3^2 + \sigma^2\,.$$

Then it holds that $E[U(n)] = u + Fnh + n\frac{q}{p}\mu_1 - n\frac{b}{a}\mu_2 - n\frac{d}{c}\mu_3 = u + n\alpha$,

$$Var[U(n)] = n(\frac{q}{p}\gamma_1 + \frac{q^2}{p^2}\mu_1^2 + \frac{b}{a}\gamma_2 + \frac{b^2}{a^2}\mu_2^2 + \frac{d}{c}\gamma_3 + \frac{d^2}{c^2}\mu_3^2 + \sigma^2) = n\beta^2\,.$$

Further, letting $\delta(n) = u + n\alpha - \beta n^{\frac{3}{4}}$. Due to $\alpha > 0$, so if n becomes big enough, then $\delta(n)$ is positive, and $\delta(n) \to \infty$ as $n \to \infty$. By the full-conditions expectations formula, the second term of the right side of equation (5) can be changed into

$$\begin{aligned} &E[e^{-RU(n)}|T \geq n]P(T \geq n) \\ &= E[e^{-RU(n)}|T \geq n, 0 \leq U(n) \leq \delta(n)]P(T \geq n, 0 \leq U(n) \leq \delta(n)) \\ &\quad + E[e^{-RU(n)}|T \geq n, U(n) > \delta(n)]P(T \geq n, U(n) > \delta(n)) \\ &\leq P\{0 \leq U(n) \leq \delta(n)\} + e^{-R\delta(n)}\,. \end{aligned} \tag{6}$$

Applying the Chebychev inequality, we have

$$\begin{aligned} &P\{0 \leq U(n) \leq \delta(n)\} = P\{0 \leq U(n) \leq E[U(n)] - \beta n^{\frac{3}{4}}\} \\ &\leq P\{|U(n) - E[U(n)]| \geq \beta n^{\frac{3}{4}}\} \leq Var[U(n)]\beta^{-2} n^{-\frac{3}{2}} = n^{-\frac{1}{2}}\,. \end{aligned}$$

Thus, the second term of the right side of equation (5) is

$$E[e^{-RU(n)}|T \geq n]P(T \geq n) \leq n^{-\frac{1}{2}} + e^{-R\delta(n)} \to 0 \quad (n \to \infty)\,.$$

So after letting $n \to \infty$ in equation (5), it can be got that $e^{-Ru} = E[e^{-RU(T)}$ $|T < \infty]\Psi(u)$. The proof was completed. □

Corollary 7. *For the risk process $\{U(n), n = 1, 2, \cdots\}$ in the proposed risk model, the ultimate ruin probability $\Psi(u)$ follows the Lundberg inequality, namely*

$$\Psi(u) \leq e^{-Ru} \quad (\forall u \geq 0)\,.$$

It is because of $-RU(T) \geq 0$ over $\{T < \infty\}$, and $E[e^{-RU(T)}|T < \infty] \geq 1$, so we have $\Psi(u) \leq e^{-Ru} \quad (\forall u \geq 0)$.

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