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A DOUBLE TYPE-INSURANCE RISK MODEL UNDER STOCHASTIC PREMIUM AND REFUND WITH ITS EVENTUAL RUIN PROBABILITY

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Abstract: Based on formulating the arrivals of premiums and claims as the different Poisson flows, a double type-insurance risk model was introduced in insurance business, which the premiums and claim sizes are random variables with general distribution of probability. For the proposed model, some relevant properties of the surplus process were discussed, a kind of formula of eventual ruin probability and the upper bound of the adjustment coefficient were discussed respectively. The upper bound of the eventual ruin probability also was obtained by using martingale.

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1. Introduction

In the classical risk models, the surplus process $U(t)$ at time t for an insurance company is often described as (see [4])

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k \,, \tag{1}$$

where u is the company's initial capital, $c > 0$ is the premium rate, $\{N(t), t \geq 0\}$ is a Poisson process with intensity λ used here to model the number of claims

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in the time interval $(0,t]$ and $\{X_k : k = 1, 2, \cdots, N(t)\}$ is a family independent, identically positive random variables, each independent of $\{N(t), t \geq 0\}$, used to model the size of the k-th claim. The aggregate claim $\sum_{k=1}^{N(t)} X_k$ represents the outflow in the risk process and the income is the premium paid by the policy holders. It is assumed that the premium income accrues linearly through time at some constant rate c, so that at a certain time t, the total premium accrued is ct.

This classical risk model was first considered by Filip Lundberg in 1903. His use of the standard compound Poisson model was later made mathematically rigorous by Harald Cramer in the 1930s (see [3], [7]). This model, also known as the *Cramer-Lundberg* risk model, has since then been extended in various ways: general renewal processes and Cox processes replace the Poisson process; a random environment allows for random changes in the intensity of the claims process and of the claim size distribution; interest rates are considered in the premium income side; and piecewise deterministic Markov processes provide new insight and models (see [6], [2], [1], [5]).

In this paper we consider a following rigorous mathematical formulation defined on a given complete probability space (Ω, F, P) with the filtration $\{F_t, t \geq 0\}$ which represents the information available at time t. Let the premium arrivals of the two type-insurances of an insurance company over $(0,t]$ denoted by the two following Poisson processes $M_1(t)$ with intensity λ_1 and $M_2(t)$ with intensity λ_2 respectively

$$\{M_1(t); t \geq 0\} \sim P(\lambda_1), M_1(0) = 0; \{M_2(t); t > 0\} \sim P(\lambda_2), M_2(0) = 0$$

and the claim arrivals of the two type-insurances over $(0,t]$ denoted by the two following Poisson processes $N_1(t)$ with intensity α_1 and $N_2(t)$ with intensity α_2 respectively

$$\{N_1(t); t \geq 0\} \sim P(\alpha_1), N_1(0) = 0; \{N_2(t); t \geq 0\} \sim P(\alpha_2), N_2(0) = 0\,.$$

Also we assume that the premium incomes of two type-insurances are families of positive i.i.d. variables $\{X_i^{(1)} : i = 1, 2, \cdots, M_1(t)\}$ and $\{X_i^{(2)} : i = 1, 2, \cdots, M_2(t)\}$ respectively independent of $M_1(t)$ and $M_2(t)$. The claim sizes of two type-insurances are denoted by the positive i.i.d. variable sequences $\{Y_i^{(1)} : i = 1, 2, \cdots, N_1(t)\}$ and $\{Y_i^{(2)} : i = 1, 2, \cdots, N_2(t)\}$ respectively independent of $N_1(t)$ and $N_2(t)$. Then, without reinsurance, the surplus of the insurance company at time t can be modeled as

$$U(t) = u + \sum_{i=1}^{M_1(t)} X_i^{(1)} + \sum_{i=1}^{M_2(t)} X_i^{(2)} - \sum_{i=1}^{N_1(t)} Y_i^{(1)} - \sum_{i=1}^{N_2(t)} Y_i^{(2)} \qquad (2)$$

to describe the dynamics of the risk process.

Let $S(t)$ denote the company's payoff at time t, i.e.,

$$S(t) = \sum_{i=1}^{M_1(t)} X_i^{(1)} + \sum_{i=1}^{M_2(t)} X_i^{(2)} - \sum_{i=1}^{N_1(t)} Y_i^{(1)} - \sum_{i=1}^{N_2(t)} Y_i^{(2)},$$

then the stochastic process $\{S(t); t \geq 0\}$ represents the company's payoffs up to time t. Usually the steady operation of insurance company are ensured by assuming $E[S(t)] > 0$ (see [4], [3], [7]).

In the following analysis, assuming that there exist k-order moments with the premium and claim size variables of the risk process $U(t)$ in model (2) ($k = 1, 2, \cdots$). Also let $\lambda_k^{(1)} = E[(X^{(1)})^k]$ and $\lambda_k^{(2)} = E[(X^{(2)})^k]$ denote the k-order moments of the premium variables of $\{X_i^{(1)} : i = 1, 2, \cdots, M_1(t)\}$ and $\{X_i^{(2)} : i = 1, 2, \cdots, M_2(t)\}$ respectively. Also $\mu_k^{(1)} = E[(Y^{(1)})^k]$ and $\mu_k^{(2)} = E[(Y^{(2)})^k]$ denote the k-order moments of the claim sizes of $\{Y_i^{(1)} : i = 1, 2, \cdots, N_1(t)\}$ and $\{Y_i^{(2)} : i = 1, 2, \cdots, N_2(t)\}$ respectively.

2. Main Properties of the Risk Process $U(t)$

The risk process $U(t)$ of model (2) has the following several basic properties.

Theorem 1. *The payoff process $\{S(t); t \geq 0\}$ of risk model (2) satisfies the following propositions:*

(1) $E[S(t)] = \lambda_1 t \cdot E(X^{(1)}) + \lambda_2 t \cdot E(X^{(2)}) - \alpha_1 t \cdot E(Y^{(1)}) - \alpha_2 t \cdot E(Y^{(2)})$.

(2) There exists a function $g(r)$ such that $E[e^{-rS(t)}] = e^{t*g(r)}$, where $g(r) = \lambda_1[E(e^{-rX^{(1)}}) - 1] + \lambda_2[E(e^{-rX^{(2)}}) - 1] + \alpha_1[E(e^{rY^{(1)}}) - 1] + \alpha_2[E(e^{rY^{(2)}}) - 1]$.

(3) The equation $g(r) = 0$ under the condition $r > 0$ has an unique positive solution R. The solution R usually be named as adjustment coefficient of the surplus process $U(t)$.

Proof. (1) By using the property of the compound Poisson process, it follows that

$$\begin{aligned} E[S(t)] &= E[\sum_{i=1}^{M_1(t)} X_i^{(1)} + \sum_{i=1}^{M_2(t)} X_i^{(2)} - \sum_{i=1}^{N_1(t)} Y_i^{(1)} - \sum_{i=1}^{N_2(t)} Y_i^{(2)}] \\ &= \lambda_1 t \cdot E(X^{(1)}) + \lambda_2 t \cdot E(X^{(2)}) - \alpha_1 t \cdot E(Y^{(1)}) - \alpha_2 t \cdot E(Y^{(2)}). \end{aligned}$$

(2) Taking expectation over $e^{-rs(t)}$, we have

$$E[e^{-rs(t)}] = E\{\exp(-r[\sum_{i=1}^{M_1(t)} X_i^{(1)} + \sum_{i=1}^{M_2(t)} X_i^{(2)} - \sum_{i=1}^{N_1(t)} Y_i^{(1)} - \sum_{i=1}^{N_2(t)} Y_i^{(2)}])\}$$

$$= E[\exp(-r\sum_{i=1}^{M_1(t)} X_i^{(1)})] \cdot E[\exp(-r\sum_{i=1}^{M_2(t)} X_i^{(2)})]$$

$$\cdot E[\exp(r\sum_{i=1}^{N_1(t)} Y_i^{(1)})] \cdot E[\exp(r\sum_{i=1}^{N_2(t)} Y_i^{(2)})]\,.$$

By using the conditional full-expectation formula, it follows that

$$E[\exp(-r\sum_{i=1}^{M_1(t)} X_i^{(1)})] = \sum_n E[\exp(\sum_{i=1}^{n} -rX_i^{(1)})|M_1(t)=n] \cdot P[M_1(t)=n]$$

$$= \sum_n E[\exp(\sum_{i=1}^{n} -rX_i^{(1)})] \cdot \frac{(\lambda_1 t)^n}{n!} \cdot e^{-\lambda_1 t} = \exp(\lambda_1 t \cdot E[e^{-rX^{(1)}} - 1]).$$

Similarly, $E[\exp(-r\sum_{i=1}^{M_2(t)} X_i^{(2)})] = \exp(\lambda_2 t \cdot E[e^{-rX^{(2)}} - 1])$, $E[\exp(r\sum_{i=1}^{N_1(t)} Y_i^{(1)})]$ $= \exp(\alpha_1 t \cdot E[e^{rY^{(1)}} - 1])$, $E[\exp(r\sum_{i=1}^{N_2(t)} Y_i^{(2)})] = \exp(\alpha_2 t \cdot E[e^{rY^{(2)}} - 1])$.

Thus $E[e^{-rS(t)}] = \exp\{\lambda_1 t[E(e^{-rX^{(1)}}) - 1] + \lambda_2 t[E(e^{-rX^{(2)}}) - 1]$
$+\alpha_1 t[E(e^{rY^{(1)}}) - 1] + \alpha_2 t[E(e^{rY^{(2)}}) - 1]\}$.

Let $g(r) = \lambda_1[E(e^{-rX^{(1)}}) - 1] + \lambda_2[E(e^{-rX^{(2)}}) - 1] + \alpha_1[E(e^{rY^{(1)}}) - 1] + \alpha_2[E(e^{rY^{(2)}}) - 1]$, then there exists a function $g(r)$ such that

$$E[e^{-rS(t)}] = e^{tg(r)}\,.$$

(3) Note that the function $g(r)$ satisfies the property $g(0) = 0$, and the claim sizes of $Y_i^{(1)}, Y_i^{(2)} (i = 1, 2, \cdots)$ and the premiums of $X_i^{(1)}, X_i^{(2)} (i = 1, 2, \cdots)$ are independent each other. It can be obtained that

$$g'(r) = \lambda_1 E[-X^{(1)} e^{-rX^{(1)}}] + \lambda_2 E[-X^{(2)} e^{-rX^{(2)}}]$$
$$+\alpha_1 E[Y^{(1)} e^{rY^{(1)}}] + \alpha_2 E[Y^{(2)} e^{rY^{(2)}}]$$

$$g'(0) = -\lambda_1 E[X^{(1)}] - \lambda_2 E[X^{(2)}] + \alpha_1 E[Y^{(1)}] + \alpha_2 E[Y^{(2)}] = -\frac{E[S(t)]}{t} < 0.$$

As $r \to \infty$, $g'(r) \to \infty$, and for $\forall r > 0$, it follows that $g''(r) > 0$. Then

$g'(r)$ is monotone increasing at the interval $(0, +\infty)$ and $g(r)$ is down-concave at the interval $(0, +\infty)$. By these properties, there exists $r^* \in (0, +\infty)$ such that $g'(r^*) = 0$. It shows that there exists $R \in (0, +\infty)$ such that $g(R) = 0$, i.e., there is an unique positive solution R with the equation of $g(r) = 0$ under the condition $r > 0$. Usually this solution R is called as the adjustment coefficient of the surplus process of $U(t)$. □

Theorem 2. *For the adjustment coefficient R of the surplus process of $U(t)$ from Theorem 1, it follows that*

$$R < \frac{\lambda_1 + \lambda_2}{\alpha_1 \cdot \mu_1^{(1)} + \alpha_2 \cdot \mu_1^{(2)}},$$

where $\mu_1^{(1)} = E[Y^{(1)}]$, $\mu_1^{(2)} = E[Y^{(2)}]$.

Proof. In the risk model (2), because there exist every order moments with the premiums $X_i^{(1)}$, $X_i^{(2)}$ and the individual claim sizes $Y_i^{(1)}$, $Y_i^{(2)}$ from the double type-insurances, then by

$$E(e^{tX}) = E(1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \cdots)$$
$$= 1 + tEX + \frac{t^2}{2!}EX^2 + \frac{t^3}{3!}EX^3 + \cdots$$

and $\mu_k^{(1)} = E[(Y^{(1)})^k]$, $\mu_k^{(2)} = E[(Y^{(2)})^k]$ $\quad (k = 1, 2, \cdots)$, it follows that

$$E(e^{RY^{(1)}}) = 1 + R\mu_1^{(1)} + \frac{R^2}{2!}\mu_2^{(1)} + \cdots > 1 + R\mu_1^{(1)}, \tag{3}$$

$$E(e^{RY^{(2)}}) = 1 + R\mu_1^{(2)} + \frac{R^2}{2!}\mu_2^{(2)} + \cdots > 1 + R\mu_1^{(2)} \tag{4}$$

and

$$E(e^{RX^{(1)}}) > 0, \qquad E(e^{RX^{(2)}}) > 0. \tag{5}$$

Also it is because that

$$g(R) = \lambda_1[E(e^{-RX^{(1)}}) - 1] + \lambda_2[E(e^{-RX^{(2)}}) - 1]$$
$$+\alpha_1[E(e^{RY^{(1)}}) - 1 + \alpha_2[E(e^{RY^{(2)}}) - 1] = 0$$

or equivalently

$$\lambda_1 E(e^{-RX^{(1)}}) + \lambda_2 E(e^{-RX^{(2)}}) + \alpha_1 E(e^{RY^{(1)}})$$
$$+\alpha_2 E(e^{RY^{(2)}}) - \lambda_1 - \lambda_2 - \alpha_1 - \alpha_2 = 0. \tag{6}$$

Putting (3), (4), (5) into (6), it can be obtained that

$$\alpha_1 + \alpha_1 R\mu_1^{(1)} + \alpha_2 + \alpha_2 R\mu_1^{(2)} - \lambda_1 - \lambda_2 - \alpha_1 - \alpha_2 < 0\,,$$

i.e.

$$R < \frac{\lambda_1 + \lambda_2}{\alpha_1 \cdot \mu_1^{(1)} + \alpha_2 \cdot \mu_1^{(2)}}\,. \qquad \square$$

Remark. Theorem 2 provides an upper bound for the adjustment coefficient R of risk process $U(t)$ of model (2) by the distribution parameters $\lambda_1, \lambda_2, \alpha_1, \alpha_2, \mu_1^{(1)}$ and $\mu_1^{(2)}$ from the premiums and claim sizes of double type-insurance.

3. Eventual Ruin Probability of $U(t)$

Let the ruin time of the company insurance, denoted by T which is the first time the surplus becomes negative, is defined to be $T = \inf\{t : t \geq 0 : U(t) < 0\}$ with $T = \infty$ interpreted as the ruin event never to happen. Apparently, T is a stopping time. Then we can define the eventual ruin probability associated with the initial capital u of as $\varphi(u) = \Pr[T < \infty | U_{(0)} = u]$. In this section, we consider the property of the ruin probability $\varphi(u)$ and the problem of estimating its upper bound by martingale analysis.

Theorem 3. *For the eventual ruin probability $\varphi(u)$ of risk process $U(t)$ in model (2), it follows that*

$$\varphi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)} | T < \infty]}\,. \tag{7}$$

Proof. For arbitrary $t > 0$ and $r > 0$, by the conditional expectation formula (see [7]), it can be obtained that

$$E[e^{-rU(t)}] = E[e^{-rU(t)} | T \leq t] \Pr(T \leq t) + E[e^{-rU(t)} | T > t] \Pr(T > t)\,, \tag{8}$$

where $U(T) = u + \sum_{i=1}^{M_1(T)} X_i^{(1)} + \sum_{i=1}^{M_2(T)} X_i^{(2)} - \sum_{i=1}^{N_1(T)} Y_i^{(1)} - \sum_{i=1}^{N_2(T)} Y_i^{(2)}$.

Applying $g(R) = 0$ when $r = R$ of Theorem 1, the left side of the equation (8) can be written as

$$E[e^{-RU(t)}] = E[e^{-R(u+S(t))}] = e^{-Ru} E[e^{-RS(t)}] = e^{-Ru} e^{-tg(R)} = e^{-Ru}\,.$$

Thus the equation (8) can be transformed to

$$e^{-Ru} = E[e^{-RU(t)} | T \leq t] \Pr(T \leq t) + E[e^{-RU(t)} | T > t] \Pr(T > t)\,. \tag{9}$$

In following we consider the right side of the equation (9). When $T \leq t$,

$U(t)$ can be described as

$$U(t) = U(T) + [U(t) - U(T)]$$
$$= U(T) + \sum_{i=M_1(T)}^{M_1(t)} X_i^{(1)} + \sum_{i=M_2(T)}^{M_2(t)} X_i^{(2)} - \sum_{i=N_1(T)}^{N_1(t)} Y_i^{(1)} - \sum_{i=N_2(T)}^{N_2(t)} Y_i^{(2)}.$$

By applying the above equality, it can be verified that $E[e^{-RU(t)}|T \leq t]\,\Pr(T \leq t) = E[e^{-RU(T)}|T \leq t]\,\Pr(T \leq t)$. So for the first term of the right side in equation (9), we have

$$\lim_{t\to+\infty} E[e^{-RU(t)}|T \leq t]\,\Pr(T \leq t) = E[e^{-RU(T)}|T < \infty]\,\Pr(T < \infty)\,. \qquad (10)$$

For the second term of the right side in equation (9), it can be verified that $\lim\limits_{t\to+\infty} E[e^{-RU(t)}\,|T > t]\,\Pr(T > t) = 0$. This is because that

From $E[U(t)] = u + (\lambda_1\lambda_1^{(1)} + \lambda_2\lambda_1^{(2)} - \mu_1\mu_1^{(1)} - \mu_2\mu_1^{(2)})t$ and

$$Var[U(t)] = (\lambda_1\lambda_2^{(1)} + \lambda_2\lambda_2^{(2)} + \mu_1\mu_2^{(1)} + \mu_2\mu_2^{(2)})t$$

with $\lambda_k^{(1)} = E[(X^{(1)})^k]$, $\lambda_k^{(2)} = E[(X^{(2)})^k]$, $\mu_k^{(1)} = E[(Y^{(1)})^k]$, $\mu_k^{(2)} = E[(Y^{(2)})^k]$ $(k = 1, 2)$, we have the following analysis.

After letting $c = \lambda_1\lambda_1^{(1)} + \lambda_2\lambda_1^{(2)} - \mu_1\mu_1^{(1)} - \mu_2\mu_1^{(2)}$, $d^2 = \lambda_1\lambda_2^{(1)} + \lambda_2\lambda_2^{(2)} + \mu_1\mu_2^{(1)} + \mu_2\mu_2^{(2)}$, define $h(t) = u + ct - d^{2/3}$. Because of the function $h(t) = u + ct - d^{2/3} > 0$ when t is sufficient large, so it can be obtained that

$$E[e^{-RU(t)}|T > t] = E[e^{-RU(t)}|T > t, 0 \leq U(t) \leq h(t)]\,\Pr[0 \leq U(t) \leq h(t)]$$
$$+E[e^{-RU(t)}|T > t, U(t) \geq h(t)]\,\Pr[U(t) > h(t)]$$
$$\leq \Pr[0 \leq U(t) \leq h(t)] + \exp[-R * h(t)]\,.$$

Applying the Chebychev inequality, it follows that

$$0 \leq Pr[0 \leq U(t) \leq h(t)] \leq Pr\{|U(t) - E[U(t)]| \geq d * t^{2/3}\}$$
$$\leq \frac{Var[U(t)]}{d^2 * t^{4/3}} = t^{-1/3}\,.$$

Thus when $t \to \infty$, it follows that $0 \leq E[e^{-RU(t)}|T > t] \leq t^{-1/3} + exp[-R * h(t)] \to 0$. Furthermore, we can get

$$\lim_{t\to+\infty} E[e^{-RU(t)}\,|T > t]\,\Pr(T > t) = 0\,. \qquad (11)$$

Letting $t \to \infty$ in equation (9), and considering the results of equations (10) and (11), it follows that $e^{-Ru} = E[^{-RU(T)}|T < \infty]\,\Pr[T < \infty|U_{(0)} = u]$. That is

$$\varphi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)}|T < \infty]} . \qquad \square$$

Theorem 4. *For the eventual ruin probability $\varphi(u)$ of risk process $U(t)$ in model (2), it follows that*

$$\varphi(u) \le e^{-Ru} .$$

Proof. By the definition of $T_u = \inf\{t : t \ge 0, U(t) < 0\}$, T_u is a F_t^s-stopping time, then for $\forall m_0 < \infty$, $T_u \wedge m_0$ is an bounded stopping time (see [4], [3], [7]). Applying these and the property of martingale, we have

$$\begin{aligned}\Pr(T_u \le m_0) \le E[\exp(-RU(t))] = E[\exp(-RU(t))|\, T_u \le m_0] \cdot \Pr(T_u \le m_0) \\ + E[\exp(-RU(t))|\, T_u > m_0] \cdot \Pr(T_u > m_0) \le e^{-Ru} = M_u(0) .\end{aligned}$$

Letting $m_0 \to \infty$, then it can be obtained that $\varphi(u) \le e^{-Ru}$. $\square$

Remark. Theorem 3 provides a kind of mathematical representation for the eventual ruin probability $\varphi(u)$ of risk process $U(t)$, which is related with the adjustment coefficient R. But Theorem 4 gives an upper bound for the ruin probability $\varphi(u)$. In fact, Theorem 4 also can be obtained from Theorem 3.

4. Conclusion

In collective risk theory, a common viewpoint is that the number and sizes of claims generated from an array of insurance policies are random variables. The double type-insurance model in this paper accords with these characters and also has more general distribution of individual payments. Next we will introduce the other risk model which is more close to reality by considering dividend and random disturbances etc.

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