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γ^*-REGULAR, γ-LOCALLY COMPACT AND γ-NORMAL SPACES

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Abstract: We continue studying the properties of γ_0-compact, γ^*-regular and γ-normal spaces defined in [5]. We also define and discuss γ-locally compact spaces.

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Key Words: γ-closed(open), γ-closure, γ-regular(open), (γ, β)-continuous (closed, open) function, (γ, β)-homeomorphism, γ-nbd, γ-T_2 spaces, γ_0-compact, γ^*-regular space, γ-normal space and γ-locally compact spaces

1. Introduction

S. Kasahara [8] introduced the concept of α-closed graphs of a function by using the concept of an operation α on topological spaces in 1979. α-closed sets were defined by D.S. Jankovic [7] and he studied the functions with α-closed graphs. H. Ogata [9] introduced the notions of γ-T_i, i = 0,1/2,1,2; (γ, β)-homeomorphism and studied some topological properties.

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B. Ahmad and F.U. Rehamn (see [10], [2]) defined and investigated several properties of γ-interior, γ-exterior, γ-closure and γ-boundary points in topological spaces. They also discussed their properties in product spaces and studied the characterizations of (γ, β)-continuous mappings initiated by H. Ogata [9]. In 2003 and 2005, B. Ahmad and S. Hussain (see [3] and [4]) continued to study the properties of γ-operations on topological spaces introduced by S. Kasahara [8]. They also defined and discuss several properties of γ-nbd, γ-nbd base at x, γ-closed nbd, γ-limit point, γ-isolated point, γ-convergent point and γ^*-regular space. They further defined γ-normal spaces, γ_0-compact, γ^s-regular and γ^s-normal in topological spaces (see [5], [6]) and established many interesting properties.

In this paper we continue to discuss the properties of γ_0-compactness, γ^*-regular, γ-normal spaces defined in [5]. We also define and discuss γ-locally compact spaces.

First, we recall preliminaries used in the sequel. Hereafter we shall write space in place of topological space.

Definition. (see [9]) Let (X,τ) be a space. An operation $\gamma : \tau \to$ P(X) is a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each V $\in \tau$, where V^γ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ.

Definition. (see [10]) Let A $\subseteq$ X. A point a $\in$ A is said to be γ-interior point of A iff there exists an open nbd N of a such that $N^\gamma \subseteq$ A and we denote the set of all such points by int_γ(A). Thus

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\}.$$

Note that A is γ-open [9] iff A =int_γ(A). A set A is called γ- closed [7] iff X-A is γ-open.

Definition. (see [8]) A point x $\in$ X is called a γ-closure point of A $\subseteq$ X, if $U^\gamma \cap A = \phi$, for each open nbd. U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by cl_γ(A). A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq$ A. Note that cl_γ(A) is contained in every γ-closed superset of A.

Definition. (see [6]) An operation $\gamma : \tau \to P(X)$ is said be γ-open, if V^γ is γ open for each $V \in \tau$.

Definition. (see [9]) An operation γ on τ is said be regular, if for any open nbds U, V of x $\in$ X, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition. (see [9]) A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be (γ, β)-

continuous iff for each $x \in X$ and each open set V containing f(x), there exists an open set U such that $x \in U$ and $f(U^\gamma) \subseteq V^\beta$, where $\gamma : \tau_1 \to P(X)$; $\beta : \tau_2 \to P(Y)$ are operations on τ_1 and τ_2 respectively.

Definition. (see [2]) A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be (γ, β)-closed (respt. (γ, β)-open), if for any γ-closed (respt. γ-open) set A of X, f(A) is β-closed (respt. β-open) in Y.

Definition. (see [9]) A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be (γ, β)-homeomorphic, if f is bijective, (γ, β)-continuous and f^{-1} is (β, γ)-continuous.

Definition. (see [9]) A space X is called a γ-T_2 space, if for each distinct points x, y $\in$ X, there exist open sets U, V such that $x \in U$ and $y \in V$ and $U^\gamma \cap V^\gamma = \phi$.

2. γ_0-Compact and γ^*-Regular Spaces

Definition 1. (see [5]) A subset A of a space X is γ_0-compact, if every cover $\{V_i : i \in I\}$ of X by γ-open sets of X, there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{i \in I_0} cl_\gamma(V_i)$. A space X is γ_0-compact [5], if X $= \bigcup_{i \in I_0} cl_\gamma(V_i)$, for some finite subset I_0 of I.

Definition. (see [9]) An operation γ on τ is said to be open if for every nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^\gamma \subseteq B$.

In [5], in the sequel, we shall make use of the following results:

Theorem. (see [5]) *Let X be a γ_0-compact space. Then each γ-closed subset of X is γ_0-compact, where γ is a regular operation.*

Theorem. (see [5]) *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective (γ, β)-continuous function and β be open. Then f is (γ, β)-open (respt. closed), if and only if f^{-1} is (β, γ)-continuous.*

Theorem. (see [5]) *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a (γ, β)-continuous from a γ_0-compact space X onto a space Y. Then Y is γ_0-compact, where β is open.*

We know [5] that G is γ-open in A X iff G $= A \cap O$, where O is γ-open set in X. Using this definition, we prove the following:

Theorem 1. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a (γ, β)-continuous injective function, where β is open. Let C be a γ_0-compact subspace of a space X. Then f(C) is γ_0-compact in Y.*

Proof. Let $\{V_i : i \in I\}$ be any cover by β-open sets of f(C). Then

$$V_i = U_i \cap f(C) \tag{2.1}$$

where each U_i is β-open in Y. Since f is (γ,β)-continuous, therefore $f^{-1}(U_i)$ is γ-open in X. Since f is one-one, therefore, (2.1) gives $f^{-1}(V_i) = f^{-1}(U_i) \cap f^{-1}f(C) = f^{-1}(U_i) \cap C$. Clearly $\{W_i : W_i = f^{-1}(V_i)\}$ is γ-open cover of C. Since C is γ_0-compact, therefore there exists a finite subset I_0 of I such that C $= \cup_{i\in I_0} cl_\gamma(W_i)$. This gives that

$$f(C) = f(\cup_{i\in I_0} cl_\gamma(W_i)) = \cup_{i\in I_0} f cl_\gamma(W_i)$$
$$\subseteq \cup_{i\in I_0} cl_\beta f(W_i) \quad \text{(by [9], Theorem 4.13)}$$
$$= \cup_{i\in I_0} cl_\beta f f^- 1(V_i) = cl_\beta(V_i).$$

Consequently f(C)$\subseteq cl_\beta(V_i)$.

This proves that f(C)is γ_0-compact. This completes the proof. □

Corollary. (see [5]) *Let $f : (X,\tau_1) \to (Y,\tau_2)$ (γ,β)-continuous from a γ_0-compact space X onto a space Y. Then Y is γ_0-compact, where β is open.*

Theorem 2. *Let $f : (X,\tau_1) \to (Y,\tau_2)$ be a (γ,β)-continuous injective function from a γ_0-compact space X into a γ-T_2 space Y and β be open. If γ is regular and γ-open, then f is (γ,β)-closed.*

Proof. Let C be a γ-closed set in X. We show hat f(C) is β-closed in Y. Since γ is regular, therefore by Theorem 4 [5], C is γ_0-compact subset of X. Also by Theorem 1, f(C)is γ_0-compact in Y. Since Y is γ-T_2 and γ is regular and γ-open, therefore by Theorem 2 [6], f(C) is β-closed in Y. This completes the proof. □

The following theorem is immediate:

Theorem 3. *Let $f : (X,\tau_1) \to (Y,\tau_2)$ be a (γ,β)-continuous bijective function from a γ_0-compact space X onto γ-T_2 space Y. Then f is (γ,β)-homeomorphism, where γ is regular and γ-open, and β is open.*

The following is easy to prove:

Theorem 4. *If $A, B \subseteq X$ are γ_0-compact such that $X = cl_\gamma(A) \cup cl_\gamma(B)$, then X is γ_0-compact, where γ is regular and open.*

Definition. (see [5]) A space X is said to be γ^*-regular space, if for any γ-closed set A and $x \in A$, there exist γ-open sets U, V such that $x \in U$, $A \subseteq V$ and $U \cap V = \phi$.

It is known (see [5]) that γ^*-regular space is not regular in general. Next we characterize γ^*-regular spaces as:

Theorem 5. *A space X is γ^*-regular iff for each $x \in X$ and a γ-closed set A such that $x \notin A$, there exist γ-open sets U, V in X such that $x \in U$ and $A \subseteq V$ and $cl_\gamma(U) \cap cl_\gamma(V) = \phi$.*

Proof. For each $x \in X$ and a γ-closed set A such that $x \notin A$, therefore by Theorem 9 [4], there is a γ-open set W such that

$$x \in W,\, cl_\gamma(W) \subseteq X - A.$$

Again by Theorem 9 from [4], there is a γ-open set U containing x such that $cl_\gamma(U) \subseteq W$. Let V = $X - cl_\gamma(W)$. Then $cl_\gamma(U) \subseteq W \subseteq cl_\gamma(W) \subseteq X - A$, implies $A \subseteq X - cl_\gamma(W) = V$. Also

$$\begin{aligned} cl_\gamma(U) \cap cl_\gamma(V) &= cl_\gamma(U) \cap cl_\gamma(X - cl_\gamma(W)) \\ &\subseteq W \cap cl_\gamma(X - cl_\gamma(W)) \subseteq cl_\gamma(W \cap (X - cl_\gamma(W)) \quad ([10], \text{Lemma } 2(3)) \\ &= cl_\gamma(\phi) = \phi. \end{aligned}$$

Thus U, V are the required γ-open sets in X. This proves the necessity. The sufficiency is immediate. This completes the proof. □

3. γ-Locally Compactness

Definition. (see [4]) A γ-nbd of $x \in X$ is a set U of X which contains a γ-open set V containing x. Evidently, U is a γ-nbd of x iff $x \in Int_\gamma(U)$.

Definition 2. A space X is said to be γ-locally compact at $x \in X$ iff x has a γ-nbd which is γ_0-compact in X. If X is γ-locally compact at every point, then X is called a γ-locally compact space.

Example. Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma_(A) = \begin{cases} A & \text{if } b \in A\,, \\ cl(A) & \text{if } b \notin A\,. \end{cases}$$

Clearly, $\{a, b\}, \{a, c\}, \{b\}, X, \phi$ are γ-open sets in X. Then

$$U_a = \{\{a, b\}, \{a, c\}, X\}\,, \quad U_b = \{\{b\}, \{a, b\}, X\}\,, \quad U_c = \{\{a, c\}, X\}$$

are the γ-nbd systems at a, b, c. Routine calculations shows that X is γ-locally compact at each, $x \in X$. Hence X is γ-locally compact.

γ_0-compact spaces are γ-locally compact, since X is γ-nbd of its points which is γ_0-compact. The converse is not true in general. Thus we have the following:

Theorem 6. *Every γ_0-compact space is γ-locally compact.*

The following theorem shows that γ-locally compactness is γ-closed hereditary property:

Theorem 7. *Every γ-closed subspace of γ-locally compact space is γ-locally compact, where γ is regular.*

Proof. Let X be a γ-locally compact space and A γ-closed set in X. Let $x \in A$. Since X is γ-locally compact, therefore for $x \in X$, there is a γ-nbd V of $x \in X$ such that V is γ_0-compact. Let $U = A \cap V$. Then U is a γ-nbd of x in A, since V is γ-nbd of $x \in X$. Clearly U is γ-closed in V, since A is γ-closed in X. Thus by Theorem 4 from [5], U is γ_0-compact in V. This proves that A is γ-locally compact at any point $x \in A$. This completes the proof. □

Theorem 8. *If X is a γ-locally compact and γ-T_2 space, then for all $x \in X$ and for all γ- nbds U of x, there exists a γ-nbd V of x which is γ_0-compact and such that $V \subseteq U$, where γ is regular and open.*

Proof. Let $x \in X$. Since X is γ-locally compact, therefore x has a γ-nbd K which is γ_0-compact. Let U be any γ-nbd of x. Then W = $Int_\gamma(K \cap U)$ is γ-open in X. $W \subseteq K$ implies $cl_\gamma(W) \subseteq cl_\gamma(K)$. Then by Theorem 2 form [6], $cl_\gamma(K)$ = K gives $cl_\gamma(W) \subseteq K$. Also by Theorem 4 from [5], $cl_\gamma(W)$ is γ_0-compact in X. Put $C = cl_\gamma(W) - W$. Clearly, C is γ-closed and hence γ_0-compact in $cl_\gamma(W)$. Moreover $x \notin C$, then by Theorem 1 from [6], there exist open sets G, H in $cl_\gamma(W)$ such that $x \in G^\gamma$, $C \subseteq H^\gamma$ and $G^\gamma \cap H^\gamma = \phi$. Thus G is a γ-nbd of x in $cl_\gamma(W)$ and $cl_\gamma(W)$ is a γ-nbd of x in X imply G is a γ-nbd of x in X. $G^\gamma \cap H^\gamma = \phi$ implies

$$G^\gamma \subseteq cl_\gamma(W) - H^\gamma \subseteq cl_\gamma(W) - C = W \tag{3.1}$$

$cl_\gamma(W) - H^\gamma$ is γ-closed in $cl_\gamma(W)$ and so is in X. By (3.1),

$$G^\gamma \subseteq cl_\gamma(G^\gamma) \subseteq cl_\gamma(cl_\gamma(W) - H^\gamma) = cl_\gamma(W) - H^\gamma \subseteq W \subseteq U.$$

Put $cl_\gamma(G^\gamma) = V$. Then by Theorem 4 form [5], V is γ_0-compact in $cl_\gamma(W)$ and hence in X. Consequently, $x \in V \subseteq U$, where V is γ_0-compact γ-nbd of x. This completes the proof. □

It is known [6], that in γ-T_2 space, every γ_0-compact subset of γ-T_2 space is γ-closed, where γ is regular and γ-open. We use this fact and prove the following:

Theorem 9. *If a space X is γ-locally compact, then each of its points is a γ-interior point of some γ_0-compact subspace of X.*

Proof. Since X is γ-locally compact, then each $x \in X$ has a γ-nbd N such that cl_γ(N)is γ_0-compact, that is, $cl_\gamma(N)$ is a γ_0-compact γ-nbd of x and so x is a γ-interior point. Thus each $x \in X$ is a γ-interior point of some γ_0-compact subspace of X. This completes the proof. □

The partial converse of Theorem 9 is:

Theorem 10. *If in a γ-T_2 space X, each $x \in X$ is a γ-interior point of some γ_0-compact subspace of X, then X is γ-locally compact, where γ is regular and γ-open.*

Proof. Let $x \in X$. Then by hypothesis, there exists a γ_0-compact subspace C of X such that x is the γ-interior point of C. Since X is γ-T_2, by Theorem 2 from [6], C is γ-closed subset of X. $cl_\gamma(C) =$ C gives that every $x \in X$ has a γ-nbd C whose γ-closure is γ_0-compact. This proves that X is γ-locally compact. This completes the proof.

Combining Theorems 9 and 10, we have:

Theorem 11. *A γ-T_2 space X is γ-locally compact iff each $x \in X$ is a γ-interior point of some γ_0-compact subspace of X, where γ is regular and γ-open.*

Definition 3. A property is called (γ, β)- topological property, if it is possessed by a space X, then it is also possessed by all spaces (γ, β)-homeomorphic to X.

In [5], it is proved that if a surjective function $f : (X, \tau_1) \to (Y, \tau_2)$ is (γ, β)-continuous and X is a γ_0-compact, then Y is γ_0-compact, where β is open.

We use this result and prove the following theorem which shows that γ-locally compactness is (γ, β)-topological property.

Theorem 12. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be (γ, β)-open,(γ, β)-continuous surjection. If X is γ-locally compact, then Y is also γ-locally compact, where β is open.*

Proof. Let $y \in Y$. Then there is $x \in X$ such that f(x) = y. Let U_x be a γ_0-compact,γ-nbd of x, since X is γ-locally compact. Also $f(x) \in f(Int_\gamma(U_x)) \subseteq f(U_x)$ gives $f(U_x)$ is γ-nbd of f(x). Since f is (γ, β)-open, then $f(Int_\gamma(U_x))$ is γ-open in Y. Also f is (γ, β)-continuous, therefore by Theorem 3 from [5], $f(U_x)$ is γ_0-compact. Thus each $y \in Y$ has a γ_0-compact, γ-nbd in Y. This proves that Y is γ-locally compact. Hence the proof. □

In [5], it is proved that a γ-closed subset of a γ_0-compact spaces is γ_0-compact, if γ is regular. We use this result and prove that γ-locally compactness is a γ-closed hereditary property.

Theorem 13. *Every γ-closed subspace of a γ-locally compact space is γ-locally compact, if γ is regular.*

Proof. Let X be a γ-locally compact space and A γ-closed set in X. Let $a \in A$. Since X is γ-locally compact, therefore for $a \in X$, there is a γ-nbd V of a such that V is γ_0-compact in X. Let $U = V \cap A$. Then U is a γ-nbd of A.

Clearly U is γ-closed in V and so by Theorem 4 from [5], U is γ_0-compact in A, that is, A is γ-locally compact at a. This proves that A is γ-locally compact. Hence the theorem. □

4. γ-Normal Spaces

In [5], γ-normal spaces have been defined and studies. Here we prove another interesting characterization of such spaces.

Definition. (see [5]) A space X is said to be γ-normal space, if for any disjoint γ-closed sets A, B of X, there exist γ-open sets U, V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

The following examples show that γ-normality and normality are independent notions.

Example 1. Let X= $\{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma_(A) = \begin{cases} A & \text{if } b \in A\,, \\ intcl(A) & \text{if } b \notin A\,. \end{cases}$$

Then X is γ-normal. X is not normal, since for closed sets $\{a, c\}, \{d\}$, there do not exist disjoint open sets containing $\{a, c\}$ and $\{d\}$ respectively.

Example 2. Let X $= \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma_(A) = \begin{cases} A & \text{if } b \in A\,, \\ clint(A) & \text{if } b \notin A\,. \end{cases}$$

Then X is not γ-normal, since for γ-closed sets $\{a\}, \{c\}$, there do not exist disjoint γ-open sets containing $\{a\}, \{c\}$ respectively, where X is normal.

It is known [10], that if A is γ-open, then $A \cap cl_\gamma(B) \subseteq cl_\gamma(A \cap B)$.

In [5], γ-normal spaces have been characterized as:

Theorem A. (see [5]) *A space X is γ-normal iff for any γ-closed set A and γ-open set U containing A, there is a γ-open set V containing A such that $cl_\gamma(V) \subseteq U$.*

We use (3.2) and Theorem A, and prove the following:

Theorem 14. *A space X is γ-normal iff for each pair A, B of disjoint γ-closed sets in X, there exist γ-open sets U, V in X such that $A \subseteq U$, $B \subseteq V$ and $cl_\gamma(U) \cap cl_\gamma(V) = \phi$.*

Proof. The sufficiency is clear. We prove only the necessity. Let A be a γ-closed set and B be a γ-closed set not containing A. Then X - B is γ-open and $A \subseteq X - B$. Then by Theorem A from [5], there is a γ-open set C such that

$$A \subseteq C \subseteq cl_\gamma(C) \subseteq X - B.$$

Since $cl_\gamma(C) \subseteq X - B$, again by Theorem A from [5], there is a γ-open set U containing A such that $cl_\gamma(U) \subseteq C$. Consequently,

$$A \subseteq U \subseteq cl_\gamma(U) \subseteq C \subseteq cl_\gamma(C) \subseteq X - B.$$

$cl_\gamma(C) \subseteq X - B$ implies $B \subseteq X - cl_\gamma(C)$. Put V = X - $cl_\gamma(C)$. Then V is γ-open containing B and moreover

$$\begin{aligned} cl_\gamma(U) \cap cl_\gamma(V) = cl_\gamma(U) \cap cl_\gamma(X - cl_\gamma(C)) \subseteq C \cap cl_\gamma(X - cl_\gamma(C)) \\ \subseteq cl_\gamma(C \cap (X - cl_\gamma(C)) \ \text{(by (3.2))} = cl_\gamma(\phi) = \phi. \end{aligned}$$

Thus U, V are the required γ-open sets in X. This proves the necessity. Hence the theorem. □

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