International Journal of Pure and Applied Mathematics

Volume 46 No. 2 2008, 165-170

Invited Lecture Delivered at
Forth International Conference of Applied Mathematics
and Computing (Plovdiv, Bulgaria, August 12–18, 2007)

A SIMPLE GRAMMAR FOR GENERATING COCOMPACT FUCHSIAN GROUPS

O. Michael Melko[1 §], Luc Patry[2]

[1]Department of Mathematics
Northern State University
1200 South Jay Str., Aberdeen, South Dakota 57401-7199, USA
e-mail: mike.melko@northern.edu

[2]Department of Mathematics
University of Arkansas
Pine Bluff, AR 71601, USA
e-mail: patryl@uapb.edu

Abstract: In this paper, we present a regular grammar that generates unique representatives of all elements in a cocompact Fuchsian group (CFG) from a given trivalent presentation (to be defined below). This grammar is the simplest possible in the sense that it possesses the fewest productions.

AMS Subject Classification: 53A35, 30F35, 20F05, 68Q42
Key Words: hyperbolic geometry, Fuchsian groups, formal languages

1. Introduction

A *Fuchsian group* G is a discrete subgroup of the isometry group of the hyperbolic plane $\mathbb{H}$ and is said to be *cocompact* if the corresponding quotient space $M = \mathbb{H}/G$ is a compact hyperbolic surface (i.e., a surface of constant negative curvature). Poincaré [4] was the first to systematically study these groups and their geometry. His method is to associate a finite presentation of G with a fundamental domain of the projection $\pi : \mathbb{H} \to M$. In this way, G is seen to

Received: August 17, 2007

§Correspondence author

be isomorphic to the fundamental group of the quotient. Although it is relatively straightforward to produce a presentation of a Fuchsian group given a Poincaré domain, it is a nontrivial matter to uniquely represent each element of the group as a product of the generators in a computationally efficient manner. This problem falls under the purview of the modern theory of automatic groups, for which Epstein, et al [1] is the standard reference.

In this paper, we summarize some results that will be presented in greater detail in [3]. We will begin by reviewing some basic facts about the topology and geometry of Poincaré domains associated with a compact hyperbolic surface. We will then present a regular grammar for generating the elements of a Fuchsian group given a particular kind of presentation, and we will state a a recursion formula for counting the number of words of a given length.

We define an *abstract graph* to be an ordered pair $A = (A_0, A_1)$, where A_0 is a finite set whose elements we may label with positive integers, and the elements of A_1 are undirected incidence relations between pairs of elements in A_0. Formally, $A_1 \subseteq A_0 \times A_0/\sim$, where $(x', y') \sim (x, y)$ if and only if $(x', y') = (x, y)$ or $(x', y') = (y, x)$. We further define an *embedding* of an abstract graph A into a hyperbolic surface M to be a map $\nu : A \to M$. We write v_i for the point $v(i)$ in M and require that each edge $\{i, j\}$ be mapped onto a geodesic segment in M whose endpoints are $\{v_i, v_j\}$. We further require that distinct edges in M are either disjoint or intersect only at vertices.

Let $E := \nu(A)$ denote the image of an abstract graph A into M under an embedding v. Thus $E = (E_0, E_1)$, where E_0 denotes the set of vertices $v(A_0)$, and E_1 denotes the corresponding set of edges. Then E is naturally a one-dimensional CW-complex. Using $\Sigma_k(X)$ to denote the k-skeleton of a CW-complex X, we will say that E is a *Poincaré graph* if $E_2 := M \setminus \Sigma_1(E)$ is a contractible cell, and the valence of each vertex is at least 3. The extended complex $\hat{E} := (E_0, E_1, E_2)$ is a decomposition of M into a CW-complex with a single two-dimensional cell. We further say that A is an *abstract admissible graph* if it admits an embedding $\nu(A)$ into M that is a Poincaré graph. We refer to such an embedding as an *admissible embedding*.

We say that two graphs are *homeomorphic* if there is a one-to-one correspondence between their vertices and edges that preserves their respective incidence relations. Two graphs are *homotopically equivalent* if one can be obtained from the other through a series of edge contractions and expansions. We refer to a graph that has only one vertex as a *one-vertex graph*, and we refer to a graph in which every vertex has valence 3 (the other extreme) as a *trivalent graph*. It is clear that every Poincaré graph is homotopically equivalent to a one-vertex

Poincaré graph through a series of edge contractions. Similarly, by means of edge expansions at vertices with valence greater than 3, every Poincaré graph can be expanded into a trivalent Poincaré graph. Note that the number of cycles in a Poincaré graph must be invariant under homotopy and is equal to $2p$, where p denotes the genus of M. By means of the Euler characteristic, it is easily verified that, for any Poincaré graph E, we must have $1 \leq |E_0| \leq 4p-2$. In particular, $|E_0| = 1 \Rightarrow |E_1| = 2p$ and $|E_0| = 4p-2 \Rightarrow |E_1| = 6p-3$. For the case $p = 2$, we have $|E_0| = 6$ and $|E_1| = 9$.

We refer to the number of vertices $|E_0|$ in an admissible graph E as the *rank* of E, and we say that E has *full rank* if $|E_0| = 4p - 2$. It is easily verified that a Poincaré graph E has full rank if and only if it is trivalent. It is shown in [2] that there are five homeomorphically distinct trivalent abstract admissible graphs with eight homotopically distinct admissible embeddings into M for the case $p = 2$. As the authors point out, the number of homeomorphically distinct abstract admissible graphs grows very rapidly with the genus. The space of Poincaré graphs in a compact hyperbolic surface M may be geometrized by means of the arc-length functional L. It will be shown in [3] that this space is a stratified Riemannian space on which L is Morse.

Now let E be a Poincaré graph in M. Then the open cell $M \setminus E$ may be isometrically lifted to an open set $P_1' \subseteq \mathbb{H}$. This lifting is unique up to an isometry of $\mathbb{H}$. The closure P_1 of P_1' in $\mathbb{H}$ is a geodesic polygon in $\mathbb{H}$. We shall refer to a polygon in $\mathbb{H}$ that arises in this way as a *Poincaré domain* corresponding to E in M. The polygon P_1 necessarily has an even number of edges with a pairing of sides of equal length that arises from projecting the boundary of P_1 onto E. Suppose now that $\{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\}$ is a list of the edges of P_1, where e_k is paired with e_k^* for $1 \leq k \leq n$. For each $1 \leq k \leq n$, we define x_k to be the unique isometry of $\mathbb{H}$ mapping e_k onto e_k^*. The set $X := \{x_1, \ldots, x_n\}$ defines a set of generators for a Fuchsian group G. The set R of relations is found by determining all inequivalent minimal products of generators that fix a vertex of P_1 and one of its edges. We denote the corresponding presentation as $\Gamma = \langle X|R \rangle$. Note that the elements of G are in one-to-one correspondence with tiles in a tessellation of $\mathbb{H}$ that are congruent to the base tile P_1. We will say that Γ is a *one-vertex presentation* if it arises from a one-vertex Poincaré graph, and we will say that it is a *trivalent presentation* if it arises from a trivalent Poincaré graph. One-vertex presentations, in which there is one relation consisting of a product of commutators, are the ones most commonly discussed in the literature. In what follows, we consider only trivalent presentations. For genus p, there are $4p-2$ inequivalent relations of the form $a \cdot b \cdot c = 1$ for some choice of $a, b, c \in X \cup X^{-1}$, one for each "orbit class" of vertices in P_1.

Our objective in the next section is to describe a generative grammar that solves Problem 1. Due to lack of space, we refer the reader to [3] for a detailed derivation of this result.

Problem 1. Given a trivalent presentation $\Gamma = \langle X|R \rangle$ of a cocompact Fuchsian group G, generate a set of words over the alphabet $K = X \cup X^{-1} \cup \{1\}$ that are unique representatives of all group elements in G. Here, X^{-1} denotes the set consisting of the multiplicative inverses of elements in X.

2. A Regular Grammar for Trivalent Presentations

We begin this section by reviewing some preliminaries from the theory of formal languages. We refer to a finite set K as an *alphabet*, and we refer to its elements as *symbols*. It is convenient to augment K with the symbol ϕ, which we call the *null symbol*, and we write $K_\phi = K \cup \{\phi\}$ for the extended alphabet. A *word* $x = x_1 \cdots x_n$ over K is an ordered sequence of symbols $x_i \in K$. A word $x = x_1 \cdots x_{n+1}$ is said to be *null-terminated* if $x_i \in K$ for $1 \leq i \leq n$ and $x_{n+1} = \phi$. We sometimes express the fact that a symbol $y \in K$ appears in a word x by writing $y \in x$. The *concatenation* of two words $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m$ will be written as $x \cdot y := x_1 \cdots x_n y_1 \cdots y_m$. We refer to the number of symbols in a word x as its *length*, and we denote the length of x by $|x|$. This length function is additive in the sense that $|x \cdot y| = |x| + |y|$. We refer to $x[i] := x_i$ as the symbol at position i in the word x. Similarly, for $1 \leq i < j \leq |x|$, we write $x[i,j] := x_i \cdots x_j$, $x[i,j) := x_i \cdots x_{j-1}$, $x(i,j) := x_{i+1} \cdots x_{j-1}$, and $x(i,j] := x_{i+1} \cdots x_j$, for the corresponding parts of x. If $1 \leq j < i \leq |x|$, we understand $x[i,j]$, $x[i,j)$, etc., to be the word obtained by wrapping around the ends of the word x. Thus, if $x = abcdef$, for example, we have $x[5,2] := efab$. We define the *increment operator* $+$ on a word x of length n by the rule $x_i+ = x_{i+1}$ for $1 \leq i < n$, and we set $x_n+ = x_1$. In the same fashion, we define the *decrement operator* $-$ by the rule $x_i- = x_{i-1}$ for $1 < i \leq n$ and $x_1- = x_n$. Note that, if a word x contains no repeated symbols, we may unambiguously identify a symbol with its position in that word. Thus, in the previous example, $x[e,b] = x[5,2]$, $e+ = f$, $e{+}{+} = a$, etc.

The set of all words over K, together with the null word ϕ_K, is a semigroup under concatenation and will be denoted by K^*. A subset $\mathcal{L} \subseteq K^*$ of K^* is a *(formal) language over* K. We may concatenate every word in a language on the right with the null symbol. The language derived in this way consists of null-terminated words and will be denoted by $\mathcal{L}'(K) = \mathcal{L}(K) \cdot \phi$. This artifice is useful for normalizing the language acceptor automaton defined in [3] and associated with the grammar displayed in Table 1. It also corresponds naturally to certain data structures, such as null-terminated strings in the C programming language.

Useful languages are often described in terms of generative grammars, which provide rules for generating words. There is a hierarchy of such grammars, but we only require the most restrictive kind, which we now define.

Definition 2. A (right) regular generative grammar is a pair $\mathfrak{G} = (\Sigma, \Pi)$ consisting of a set of symbols Σ and a collection of rewriting rules or productions Π that satisfy the following conditions:

(i) There are two special subsets Σ_N and Σ_T of Σ such that $\Sigma = \Sigma_N \cup \Sigma_T$ and $\Sigma_N \cap \Sigma_T = \emptyset$. We call Σ_N the set of nonterminal symbols and Σ_T the set of terminal symbols.

(ii) There is a distinguished symbol $\sigma_0 \in \Sigma_N$ called the initial symbol or start symbol. This symbol is used to begin the derivation of words in a language.

(iii) The productions in Π are of the form $X \to wY$ or $X \to w$, where $X, Y \in \Sigma_N$ and $w \in \Sigma_T^*$.

We will write $\mathcal{L}(\Sigma, \Pi)$ or $\mathcal{L}(\mathfrak{G})$ for the language over Σ_T that is generated by the grammar $\mathfrak{G} = (\Sigma, \Pi)$.

We now associate a *cyclic Wick's form* W with a presentation Γ of a Fuchsian group. First, choose a counterclockwise ordering of the edges of a Poincaré domain P_1 and label them accordingly as $\{e_1, e_2, \ldots, e_{2n}\}$. Now set $W = w_1 w_2 w_3 \cdots w_{2n}$, where $w_k \in X \cup X^{-1}$ is the isometry of $\mathbb{H}$ that sends the edge e_k to the other edge in its pairing.

Now imagine that we are building a tessellation of $\mathbb{H}$ by repeated application of the elements of $X \cup X^{-1}$ to P_1 in such a way that none of the resulting tiles overlap. We say that a tile is at *level* l if it takes a minimum of l operations to arrive at its destination, and we further say that it is of *type* k if it has a common edge with exactly k tiles at one lower level. Because the presentation is trivalent, tiles can only be of type 0, 1, or 2. The first tile P_1 is the only tile of type 0, and all tiles at level 1 are of type 1. We use the symbol P_y, resp. $P_{y,y+}$, to denote a type 1, resp. type 2, tile that is arrived at from a previous tile by means of an application of $y \in W$. Our nonterminal symbols are now taken to be $\Sigma_N := \{P_y \,|\, y \in W \cdot 1\} \cup \{P_{y,y+} \,|\, y \in W\}$, and our terminal symbols are taken to be $\Sigma_T := K_\phi = X \cup X^{-1} \cup \{1, \phi\}$. The corresponding productions that generate the group language are given in Table 1. Proposition 3 gives us a way to count tiles at a given level (or words of a given length). For a closed-form formula, see [3].

Proposition 3. *Let P_1 be an m-sided Poincaré domain, and let $p_{k,l}$ denote the number of tiles of type k at level l. Then $p_{1,1} = m$ and $p_{2,1} = 0$, and, for $l \geq 1$,*

$$p_{1,l+1} = (m-5)p_{1,l} + (m-6)p_{2,l}, \quad p_{2,l+1} = p_{1,l} + p_{2,l}.$$

(i)	$\sigma_0 \to yP_y$	$y \in W \cdot 1$
(ii)	$P_1 \to \phi$	
(iii)	$P_y \to zP_z$	$z \in (y{+}{+}, y{-}{-})$
	$P_y \to zP_{z,z+}$	$z = y{+}{+}$
	$P_y \to \phi$	$y \in W$
(iv)	$P_{y,y+} \to zP_z$	$z \in (y{+}{+}{+}, y{-}{-})$
	$P_{z,z+} \to zP_{z,z+}$	$z = y{+}{+}{+}$
	$P_{y,y+} \to \phi$	$y \in W$

Table 1: Productions for generating the elements of a cocompact Fuchsian group

References

[1] David B.A. Epstein, et al *Word Processing in Groups*, Jones and Bartlett Publishers (1992).

[2] Troels Jørgenson, Marjatta Näätänen, Surfaces of genus 2: Generic fundamental domains, *Quart.J. Math. Oxford*, **33** (1982), 451-461.

[3] O. Michael Melko, Luc Patry, The hyperbolic Steiner problem, Teichmüller space, and the efficient computation of cocompact Fuchsian groups, *In Preparation.*

[4] H. Poincaré, Théorie des groupes fuchsiens, *Acta Mathematica*, **1** (1882), 1-62.