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FAMILY OF ITERATIVE METHODS FOR COMPUTING THE ZEROS OF ANALYTIC FUNCTIONS

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Abstract: We consider a one parameter family of iterative methods for the simultaneous determination of complex zeros of a class of analytic functions which have only simple zeros inside a simple smooth closed contour in the complex plane. The proposed family is based on a cubically convergent family of iterative methods for solving nonlinear equations. The presented convergence analysis shows that the order of convergence of the considered family is four. Numerical examples demonstrate a good convergence properties, fitting very well theoretical results.

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1. Family of Simultaneous Methods

The aim of this paper is the construction and convergence analysis of a one parameter family of simultaneous methods for the computation of simple (real or complex) zeros of a wide class of analytic functions. The derivation of a new family is based on a suitable modification of cubically convergent iterative

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method for solving nonlinear equation $f(x) = 0$, proposed by Gutiérrez and Hernández [2],

$$\hat{x} = x - \frac{f(x)}{f'(x)}\Big(1 + \frac{1}{s(x) - \alpha}\Big), \tag{1}$$

where α is a real parameter and

$$s(x) = \frac{2f'(x)^2}{f(x)f''(x)}. \tag{2}$$

Here x is a current approximation and $\hat{x}$ is a new approximation to a zero of f. The family (1)-(2) converges cubically and includes, for example, Halley's method ($\alpha = 1$) and Chebyshev-Euler's method ($\alpha = 0$). If $|\alpha|$ is very large, then the methods (1) behave as Newton's method

$$\hat{x} = x - f(x)/f'(x)$$

of the second order. For this reason, one should avoid the choice of large parameter α in the presented iterative formulas.

Let $z \mapsto \Phi(z)$ be an analytic function inside and on the simple smooth closed contour Γ, without zeros on Γ and with a known number n of simple zeros inside Γ. Then, following Smirnov [10], Φ can be represented in the form of product

$$\Phi(z) = \Psi(z)\prod_{j=1}^{n}(z - \zeta_j)$$

inside Γ. Here $\zeta_1, \ldots, \zeta_n$ are the zeros of Φ (inside Γ) and Ψ is an analytic function such that $\Psi(z) \neq 0$ for all $z \in \operatorname{int}\Gamma$. According to Iokimidis and Anastasselou [4] the analytic function Ψ can be expressed in the form

$$\Psi(z) = \exp\big(Y(z)\big)$$

inside Γ, where Y is also an analytic function inside Γ given by

$$Y(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\log\big[(w-\eta)^{-n}\Phi(w)\big]}{w - z}\,dw \tag{3}$$

and η is an arbitrary point inside Γ such that $\Phi(\eta) \neq 0$. Therefore,

$$\Phi(z) = \exp\big(Y(z)\big)\prod_{j=1}^{n}(z - \zeta_j).$$

As well known, the number of zeros n of Φ inside Γ may be determined by the *argument principle*

$$n = \frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi'(w)}{\Phi(w)}\,dw = n\big(\Phi(\Gamma), 0\big)\ . \tag{4}$$

$\Phi(\Gamma)$ denotes the image of the curve Γ under the mapping Φ. The integer

$n(\Phi(\Gamma), 0)$ is the so-called *winding number* of $\Phi(\Gamma)$ with respect to the origin and it is equal to the number of times that the curve $\Phi(\Gamma)$ "winds" itself around the origin.

The new family of iterative methods for the simultaneous computation of the zeros of analytic functions is based on the idea presented in Petković et al [9]. Let us note first that the zeros of the analytic function Φ inside Γ coincide to the zeros of the function

$$V_i(z) = \frac{\Phi(z)}{\exp(Y(z)) \prod_{\substack{j=1 \\ j \neq i}}^{n} (z - z_j)}, \tag{5}$$

where $z_1, \dots, z_n$ are some approximations to the zeros $\zeta_1, \dots, \zeta_n$ of Φ. For simplicity, we will write $Y_i' = Y'(z_i)$, $Y_i'' = Y''(z_i)$. Let us introduce the abbreviations

$$\delta_{q,i} = \frac{\Phi^{(q)}(z)}{\Phi(z)}, \quad S_{q,i} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{(z_i - z_j)^q} \quad (q = 1, 2)$$

and

$$F_i = \delta_{1,i} - S_{1,i} - Y_i', \quad H_i = \delta_{1,i}^2 - \delta_{2,i} - S_{2,i} + Y_i''.$$

Starting from (5) and using the logarithmic derivative, we find

$$\left.\frac{(V_i(z))'}{V_i(z)}\right|_{z=z_i} = F_i, \tag{6}$$

$$\left.\frac{(V_i(z))''}{(V_i(z))'}\right|_{z=z_i} = F_i - \frac{H_i}{F_i}. \tag{7}$$

Starting from (1) with the function Φ instead of f and substituting $\Phi'(z)/\Phi(z)$ and $\Phi''(z)/\Phi'(z)$ respectively by $V_i'(z)/V_i(z)$ and $V_i''(z)/V_i'(z)$, evaluated at the point $z = z_i$ and given by (6) and (7), we construct the following one parameter family of iterative methods for finding, simultaneously, simple zeros of the analytic function Φ inside the contour Γ,

$$\hat{z}_i = z_i - \frac{1}{F_i}\left(1 + \frac{F_i^2 - H_i}{2F_i^2 - \alpha(F_i^2 - H_i)}\right) \quad (i \in I_n). \tag{8}$$

In the particular case when $\alpha = 1$, from (8) we get Halley-like method

$$\hat{z}_i = z_i - \frac{2F_i}{F_i^2 + H_i} \quad (i \in I_n), \tag{9}$$

proposed in [8]. If $\alpha \to +\infty$, then (8) reduces to the iterative method

$$\hat{z}_i = z_i - \frac{1}{\dfrac{\Phi'(z_i)}{\Phi(z_i)} - Y'(z_i) - \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{n} (z_i - z_j)^{-1}} \quad (i \in I_n) \tag{10}$$

of the third order, considered by Iokimidis and Anastasselou in [4].

To implement the iterative formula (8), it is necessary to calculate the derivatives $Y'(z)$ and $Y''(z)$ at the point z_i $(i = 1, \ldots, n)$. From (3) we find

$$Y'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\log\left[(w-\eta)^{-n}\Phi(w)\right]}{(w-z)^2}\, dw, \tag{11}$$

wherefrom

$$Y''(z) = \frac{1}{\pi i} \int_\Gamma \frac{\log\left[(w-\eta)^{-n}\Phi(w)\right]}{(w-z)^3}\, dw\ .$$

Without loss of generality, we can take $\eta = 0$ if $\Phi(0) \neq 0$. Applying an integration by parts, from (11) we obtain

$$Y'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi'(w)}{\Phi(w)} \frac{dw}{w-z} \tag{12}$$

(see Smirnov [10]). Then the second derivative is given by

$$Y''(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi'(w)}{\Phi(w)} \frac{dw}{(w-z)^2}. \tag{13}$$

The number of zeros n, given by (4), and $Y'(z)$ and $Y''(z)$, given by (12) and (13), can be calculated by applying a convenient sufficiently accurate quadrature rule for contours of the form

$$\frac{1}{2\pi i} \int_\Gamma g(w)dw \cong \sum_{k=1}^{p} A_{kp}\, g(w_{kp}). \tag{14}$$

A_{kp} are the *weights* and w_{kp} are the corresponding *nodes* of the applied quadrature formula (based on the trapezoidal rule or orthogonal polynomials). Obviously, the same values $\Phi(w_{kp})$ and $\Phi'(w_{kp})$ may be used in the quadrature formulas for the evaluation of n, $Y'(z_i)$ and $Y''(z_i)$ because of the similar structure of the formulas (4), (12) and (13). We omit details about the quadrature procedures and refer the reader interested in this topic to the works [1], [4], [5], [6], [8] and the references cited therein. We will only note that all zero-finding methods applied to the considered class of analytic functions are not effective if the sought zeros are very close to the contour Γ.

Remark 1. In recent time several reliable approaches were developed for computing the number of zeros of an analytic function inside a given positively oriented Jordan curve, see [5]. An efficient algorithm, called *logarithmic residue based quadrature method*, founded on the theory of formal orthogonal polynomials, was proposed in Kravanja [5]. The numerical integration which uses the quadrature rule of the form (14) for computing the number of zeros by (4), produces satisfactory results in practice.

2. Convergence Analysis

The order of convergence of the family (8) is four, which is the subject of the following theorem.

Theorem 1. *If initial approximations $z_1^{(0)},\dots,z_n^{(0)}$ are sufficiently close to the zeros $\zeta_1,\dots,\zeta_n$ of the analytic function Φ, then the family of simultaneous iterative methods* (8) *has the order of convergence equal to four.*

Proof. For simplicity, we omit the iteration indices in the convergence analysis and denote the quantities in the subsequent iteration with the symbol ˆ (*hat*). Let us introduce the errors $u_i = z_i - \zeta_i,\ \hat{u}_i = \hat{z}_i - \zeta_i\ (i \in I_n)$ and the abbreviations

$$B_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j}{(z_i-\zeta_j)(z_i-z_j)},\quad C_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{u_j(2z_i-\zeta_j-z_j)}{(z_i-\zeta_j)^2(z_i-z_j)^2}.$$

Using the identities

$$\begin{aligned}\delta_{1,i} &= \sum_{j=1}^{n} \frac{1}{z_i-\zeta_j} + Y_i',\\ \delta_{1,i}^2 - \delta_{2,i} &= \sum_{j=1}^{n} \frac{1}{(z_i-\zeta_j)^2} - Y_i'',\end{aligned}$$

after some elementary calculations we obtain

$$\begin{aligned}F_i &= \delta_{1,i} - S_{1,i} - Y_i' = \frac{1}{u_i}\big(1 - B_i u_i\big),\\ H_i &= \delta_{1,i}^2 - \delta_{2,i} - S_{2,i} + Y_i'' = \frac{1}{u_i^2}\big(1 - C_i u_i^2\big).\end{aligned}$$

Applying the last two relations, from the iterative formula (8) we find

$$\hat{u}_i = \hat{z}_i - \zeta_i = \frac{u_i^3 P_i}{(1-u_iB_i)Q_i}, \tag{15}$$

where

$$P_i = (3-2\alpha)B_i^2 - C_i + (\alpha-2)u_iB_i^3 + \alpha u_iB_iC_i$$

and

$$Q_i = 2 + 2(\alpha-2)u_iB_i - (\alpha-2)u_i^2B_i^2 - \alpha u_i^2C_i.$$

According to the assumption of the theorem, the approximations $z_1, \dots, z_n$ are good enough, which means that the quantity $u = \max_{1\le j\le n} |u_j|$ is sufficiently small. Since $|B_i| = O(u)$ and $|C_i| = O(u)$, then $|P_i| = O(u)$. According to this and taking into account that the denominator of (15) is bounded and tends to 2 when $u \to 0$, we obtain from (15)

$$|\hat{u}_i| = |u_i|^3 O(u)$$

and whence

$$\hat{u} = O\big(u^4\big),$$

where we adopt $|u_j| = O(u)$, that is, absolute values of all errors u_j ($j = 1, \dots, n$) are of the same order. □

Remark 2. The main advantages of the presented approach are (i) the increase of the convergence order from 3 (the method (1)) to 4 (the method (8)) and (ii) simultaneous computation of all zeros of a given analytic function in the considered region. Furthermore, an extensive numerical experimentation has shown that the domain of convergence of the proposed family (8) is wider in relation to the methods for finding a single zero.

3. Numerical Example

Among a number of tested nonlinear equations, we select the following one to illustrate the convergence behavior of the family (8).

Example 1. To demonstrate the convergence speed of the family of iterative methods (8), we applied it for finding the zeros of the analytic function

$$\Phi(z) = e^{3z} + 2z\cos z - 1$$

inside the contour $\Gamma = \{z \in \mathbb{C} \,:\, |z| = 2\}$ (considered by Kravanja [5]). Figure 1 displays the curve $\Phi(\Gamma)$. The number n of zeros of Φ inside Γ and the values of $Y'(z_i)$ and $Y''(z_i)$ were calculated by the trapezoidal quadrature rule along the circle $\Gamma = \{z \,:\, |z| = 2\}$. We found by this approach that the number of zeros in the disk $\{z \,:\, |z| < 2\}$ is $n = 4$. The number of zeros can be also determined by the winding number $n(\Phi(\Gamma), 0)$; it is obvious from Figure 1 that $n(\Phi(\Gamma), 0) = n = 4$.

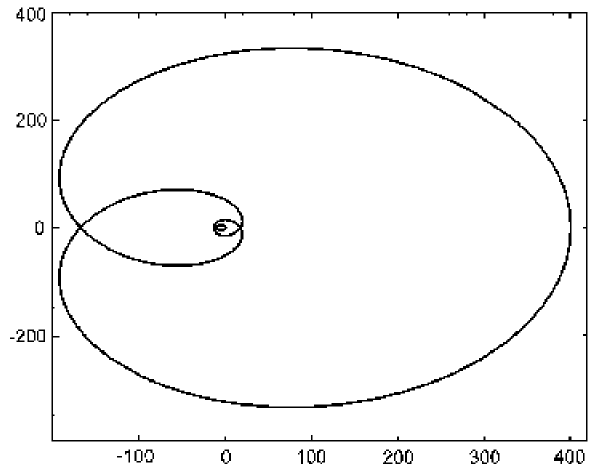

Figure 1: The curve $\Phi(\Gamma)$, $\Phi(z) = e^{3z} + 2z\cos z - 1$ and $\Gamma = \{|z| = 2; z \in \mathbb{C}\}$

	$\alpha = 0$	$\alpha = 1$	$\alpha = -1$	$\alpha = 100$
e_1	3.12(−2)	3.88(−2)	3.50(−2)	0.12
e_2	4.51(−8)	1.18(−7)	3.82(−8)	1.17(−4)
e_3	4.44(−26)	5.82(−25)	1.72(−26)	4.87(−15)

Table 1: The error e_m for the first three iterations

The following initial approximations

$$z_1^{(0)} = 0.3 - 0.3\,i, \quad z_2^{(0)} = 0.7 + 1.\,i,$$
$$z_3^{(0)} = 0.7 - 1.\,i, \quad z_4^{(0)} = -1.3 - 0.4\,i,$$

found by a search algorithm including a proximity test for detection of the presence of a zero, were employed in the realization of the iterative method (8) for $\alpha = 0,\ \alpha = -1,\ \alpha = 1$ (Halley-like method (9)) and $\alpha = 100$. In each iteration we controlled the accuracy of approximations $z_i^{(m)}$ by the Euclidean norm

$$e_m = \Big(\sum_{j=1}^{n} |z_i^{(m)} - \zeta_j|^2\Big)^{1/2} \quad (m = 1, 2, \ldots).$$

For the above initial approximations we have $e_0 = 0.956$. The results of the first three iterations are given in Table 1, where $A(-h)$ means $A \times 10^{-h}$.

For illustration, we give the approximations to the zeros of Φ obtained in the third iterative step by applying Halley-like method (9) ($\alpha = 1$). The underlined digit indicates the first incorrect digit.

$$\begin{aligned} z_1^{(3)} &= 5.18 \times 10^{-26} + 3.43 \times 10^{-25}\, i \\ z_2^{(3)} &= 0.530894930292930532471835948\underline{8}06 + 1.331791876751120929433927437\underline{7}88\, i \\ z_3^{(3)} &= 0.530894930292930532471835948\underline{8}83 - 1.331791876751120929433927437\underline{9}79\, i \\ z_4^{(3)} &= -1.8442339532622133749159244\underline{3}03 + 4.66 \times 10^{-25}\, i. \end{aligned}$$

Let us note that $\zeta_1 = 0$.

From Table 1 we observe that all three methods from the family (8) with small α (in magnitude) possess very fast convergence and almost the same accuracy of the obtained approximations. This example and a number of numerical experiments did not yield a value of α in the family which is optimal for all Φ. The approximations obtained by the method (10) coincide with the theoretical result concerning the cubic convergence of the method (10) which appears when α takes too large values in the iterative formula (8). The tested methods from the family (8) showed good convergence behavior in spite of rough initial approximations. The disadvantage of the iterative methods (8), consisting of high computational costs, is compensated to a good extent by their excellent convergence features: very fast convergence and a wide domain of convergence.

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