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A CLASS OF INITIAL VALUE METHODS FOR THE DIRECT SOLUTION OF SECOND ORDER INITIAL VALUE PROBLEMS

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Abstract: A class of Linear Multistep Methods (LMMs) are developed and applied as Initial Value Methods (IVMs) to solve second order Initial Value Problems (IVPs). The main method is derived by interpolating the assumed approximate solution at $x = x_{n+j}$, $j = 1, 2, ..., r-1$ and collocating the differential system at $x = x_{n+j}$, $j = 1, 2, ..., s-1$ respectively, where r and s are the number of interpolation and collocation points. The derivation of the main method leads to a continuous approximation from which IVMs are obtained and simultaneously applied as numerical integrators IVPs. In particular, the Multiple Finite Difference Methods (MFDMs) are implemented without the need for either predictors or other methods to supply the starting values. A numerical example is given to illustrate the efficiency of the methods.

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1. Introduction

The second-order ordinary differential equation of the form

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$$y'' = f(x, y, y'), \tag{1}$$
$$y(a) = y_0,\ y'(a) = \sigma_0\,,$$

where f is a continuous function, is conventionally solved by first reducing it to a system of first-order differential equations and then applying the various methods available for solving systems of first order IVPs. This approach is extensively discussed in the literature and in this paper we cite just a few notable ones such as Lambert [7], Brugnano and Trigiante [3], Onumanyi et al [10], [9]. Although there has been tremendous success with this approach, it has certain draw backs. For instance, computer programs associated with the methods are often complicated especially when incorporating subroutines to supply the starting values for the methods resulting in longer computer time and more computational work.

Considerable attention has been devoted to the development of various methods for solving $y'' = f(x, y)$, $y(a) = y_0$, $y'(a) = \sigma_0$ directly without first reducing it to a system of first order differential equations. For instance, Yusuph and Onumanyi [11], Fatunla [5], and Lambert [8].

Hairer and Wanner [6] proposed Nystrom type methods for (1) and stated order conditions for determining the parameters of the methods. Other methods of the Runge-Kutta type are due to Chawla and Sharma [4]. Methods of the LMM type have been considered by Awoyemi [1] and implemented in a predictor-corrector mode using the Taylor Series algorithm to supply the starting values. The IVMs are cheaper to implement, since they are self-starting and therefore do not share these drawbacks.

In this paper, we discuss IVMs for $k = 2$ to 5 which are applied as simultaneous numerical integrators for IVPs. We also show that the IVMs are zero-stable and consistent, hence the methods are convergent. The main method is derived through interpolation and collocation as in Yusuph and Onumanyi [11].

2. The Derivation of the Method

In this section, we approximate the exact solution $y(x)$ by seeking the continuous method $Y(x)$ of the form

$$Y(x) = \sum_{j=0}^{r+s-1} \lambda_j(x)\Upsilon_j(x)\,, \tag{2}$$

where $x \in [a, b]$, $\lambda_j(x)$'s are unknown coefficients and $\Upsilon_j(x)$'s are polynomial basis functions. The number of interpolation points r and the number of distinct collocation points s are chosen to satisfy $2 \leq r \leq k$, and $0 < s \leq k+1$ respectively, where $k \geq 2$ denotes the step. We then construct a k-step multistep collocation method from (2) by imposing the following conditions.

$$Y\left(x_{n+j}\right) = y_{n+j}, \quad j = 0, 1, 2, \ldots, r-1\,, \tag{3}$$

$$Y''\left(x_{n+j}\right) = f_{n+j}, \quad j = 0, 1, 2, \ldots, s-1\,. \tag{4}$$

Equations (3) and (4) lead to a system of $(r+s)$-equations, which is solved to obtain the continuous coefficients $\lambda_j(x)$'s. The k-step LMM is then, constructed by substituting the values of $\lambda_j(x)$'s into equation (2) and after some manipulation, our method is expressed in the form

$$Y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j} \tag{5}$$

which is used to generate IVMs on the mesh

$$\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \cdots < x_N = b\,,$$

$h = x_{n+1} - x_n$, $n = 0, 1, \ldots, N$, where π_N is a partition of $[a, b]$ and h is the constant step-size.

3. Specification of the Methods

In this section, we use (5) and the formula for the derivative which is expressed as

$$Y'(x) = \frac{1}{h} \sum_{j=0}^{r-1} \alpha'_j(x) y_{n+j} + h \sum_{j=0}^{s-1} \beta'_j(x) f_{n+j} \tag{6}$$

which provides additional equations and derivatives obtained by imposing that

$$Y'(x) = \sigma(x)\,, \tag{7}$$

$$Y'(a) = \sigma_0 \tag{8}$$

to generate IVMs for $k = 2$ to 5

We use (5) to obtain k-step methods with the following specifications: $r = 2$, $s = 3, 4, 5, 6$; $k = 2, 3, 4, 5$; $\Upsilon_i(x) = x^i$, $i = 0, 1, \ldots, s+r-1$. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of t where $t = (x - x_{n+k-1})/h$ as displayed in Table 1. The constant coefficients of the main methods are also given in Table 1.

The coefficients $\alpha'_j(x)$ and $\beta'_j(x)$ are easily obtained by differentiating $\alpha_j(x)$ and $\beta_j(x)$.

k	j	$\alpha_j(t)$	$\beta_j(t)$	$\alpha_j(1)$	$\beta_j(1)$	p
	0	$-t$	$\frac{1}{24}(t^4 - 2t^3 + 3t)$	-1	$\frac{1}{12}$	
2	1	$(1+t)$	$\frac{1}{24}(-2t^4 + 12t^2 + 10t)$	2	$\frac{10}{12}$	4
	2	0	$\frac{1}{24}(t^4 + 2t^3 - t)$	1	$\frac{2}{12}$	
	0	$-(1+t)$	$\frac{1}{360}(-3t^5 + 10t^3 + 23t + 30)$	-2	$\frac{2}{12}$	
3	1	$(2+t)$	$\frac{1}{24}(-2t^4 + 12t^2 + 10t)$	3	$\frac{21}{12}$	4
	2	0	$\frac{1}{120}(3t^5 + 5t^4 - 20t^3 + 122t + 100)$	0	$\frac{12}{12}$	
	3	0	$\frac{1}{360}(3t^5 + 15t^4 + 20t^3 - 8t)$	1	$\frac{1}{12}$	
	0	$-(t+2)$	$\frac{1}{1440}(2t^6 + 6t^5 - 5t^4 - 20t^3 + 119t + 222)$	-3	$\frac{27}{120}$	
	1	$(t+3)$	$\frac{1}{360}(-2t^6 - 9t^5 + 5t^4 + 30t^3 + 324t + 648)$	4	$\frac{332}{120}$	
4	2	0	$\frac{1}{240}(2t^6 + 12t^5 + 5t^4 - 60t^3 + 263t + 222)$	0	$\frac{222}{120}$	5
	3	0	$\frac{1}{360}(-2t^6 - 15t^5 - 25t^4 + 50t^3 + 180t^2 + 160t + 48)$	0	$\frac{132}{120}$	
	4	0	$\frac{1}{1440}(2t^6 + 18t^5 + 55t^4 + 60t^3 - 33t - 18)$	1	$\frac{7}{120}$	
	0	$-(3+t)$	$\frac{1}{10080}(2184 + 674t + 84t^3 + 35t^4 - 21t^5 - 14t^6 - 2t^7)$	-4	$\frac{7}{24}$	
	1	$(4+t)$	$\frac{1}{10080}(28308 + 10021t - 560t^3 - 210t^4 + 147t^5 + 84t^6 + 10t^7)$	5	$\frac{90}{24}$	
	2	0	$\frac{1}{5040}(8904 + 4210t + 840t^3 + 245t^4 - 231t^5 - 98t^6 - 10t^7)$	0	$\frac{66}{24}$	
5	3	0	$\frac{1}{5040}(5964 + 6227t - 1680t^3 - 70t^4 + 357t^5 + 112t^6 + 10t^7)$	0	$\frac{52}{24}$	6
	4	0	$\frac{1}{10080}(168 + 3818t + 5040t^2 + 1820t^3 - 525t^4 - 525t^5 - 126t^6 - 10t^7)$	0	$\frac{23}{24}$	
	5	0	$\frac{1}{10080}(84 - 107t + 336t^3 + 350t^4 + 147t^5 + 28t^6 + 2t^7)$	1	$\frac{2}{24}$	

Table 1: Continuous coefficients $\alpha_j(t)$ and $\beta_j(t)$ as well as discrete coefficients $\alpha_j(1)$ and $\beta_j(1)$ of the main IVMs, for $k = 2, \ldots, 5$, and order $p = 4, 4, 5, 6$

Additional methods are generated by imposing that:

$$Y(x_{n+\tau}) = Y_{n+\tau}, \tau = 2, \ldots, k.$$

It is worth noting that the derivatives are provided by:

$$\sigma(x_{n+\tau}) = \sigma_{n+\tau}, \tau = 0, \ldots, k.$$

Our method is implemented efficiently by combining the IVMs as simultaneous integrators for IVPs without looking for any other methods to provide the starting values. We proceed by explicitly obtaining initial conditions at $x_{n+k}, n = 0, k, \ldots, N-k$ using the computed values $Y(x_{n+k}) = y_{n+k}$ and $\sigma(x_{n+k}) = \sigma_{n+k}$ over sub-intervals $[x_0, x_k], \ldots, [x_{N-k}, x_N]$ which do not overlap (see [11]).

4. Numerical Examples

Example 4.1. Consider the given Cauchy-Euler non-homogeneous linear ODE, which was also solved by Awoyemi [2].

$$x^2y'' - 3xy' + 3y = 2x^3 - x^2, \quad y(1) = 2, \quad y'(1) = 10\,.$$

$$\text{Exact:} \quad y(x) = 3x^3 - 2x + x^2(1 + x\ln x)\,.$$

In this example, the IVMs perform better than those given in Awoyemi [2], despite the fact that we used lager step-sizes. For instance, the maximum absolute error in the solution for $k = 2$, $h = 1/320$ in Awoyemi [2] is 78.5148×10^{-7}, while our corresponding method for $k = 2$, $h = 1/20$ gives 21.3350×10^{-7}, which is roughly 4 times less than the former. Our method for $k = 3$ also performs better than the corresponding method given in Awoyemi [2]. Therefore, for this example, The IVMs are clearly superior. The details of the numerical results at some selected points are given in Table 2.

x	Awoyemi ($k=2$) $h = 1/320$	Our Method ($k=2$) $h = 1/20$	Awoyemi ($k=3$) $h = 1/320$	Our Method ($k=3$) $h = 1/30$
1.1	0.51126×10^{-7}	0.92140×10^{-7}	5.47177×10^{-8}	9.44653×10^{-10}
1.3	5.20749×10^{-7}	3.42082×10^{-7}	53.2739×10^{-8}	8.51594×10^{-9}
1.5	15.9967×10^{-7}	6.94300×10^{-7}	162.226×10^{-8}	2.41639×10^{-8}
1.7	34.2506×10^{-7}	11.6587×10^{-7}	346.036×10^{-8}	4.86983×10^{-8}
1.9	61.2933×10^{-7}	17.7351×10^{-7}	617.937×10^{-8}	8.30551×10^{-8}
2.0	78.5148×10^{-7}	21.3350×10^{-7}	790.965×10^{-8}	10.4229×10^{-8}

Table 2: Absolute errors, $|y(x) - y|$, for Example 4.1, where $y(x) = 3x^3 - 2x + x^2(1 + x\ln x)$

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