UNBALANCED RANK TWO VECTOR BUNDLES
ON SCROLLS WITHOUT INTERMEDIATE COHOMOLOGY

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Abstract: Let $C$ be a smooth curve of genus $g \geq 2$, and $E$ a rank $n$ ample and
spanned vector bundle on $C$ such that $h^1(C, E) = 0$. Set $X := \mathbb{P}(E)$. Here we
study the “unbalanced” rank 2 vector bundles $F$ on $X$ such that $h^i(X, F(t)) = 0$
for all $1 \leq i \leq n-1$ and all $t \in \mathbb{Z}$. We show that they are an extension of two
line bundles.

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* Let $Y$ be an integral projective variety and $\mathcal{O}_Y(1)$ an ample line bundle on $Y$. A vector
bundle on $Y$ is said to be arithmetically Cohen-Macaulay with respect to $\mathcal{O}_Y(1)$ if $h^i(Y, E \otimes \mathcal{O}_Y(t)) = 0$ for all $t \in \mathbb{Z}$ and all $1 \leq i < \dim(Y)$. If $Y$ is a
smooth hypersurface and $\dim(Y) \geq 3$, every arithmetically Cohen-Macaulay a
vector bundle is a direct sum of line bundles (see [2], [3], [10]). Several papers
are devoted to the study of arithmetically Cohen-Macaulay vector bundles on
other varieties (see [1], [4], [5], [6], [7], and the list is growing). Here we explore
a similar notion in a case with $h^1(Y, \mathcal{O}_Y) > 0$. Fix an integer $g > 0$, a smooth
curve with genus $g$, an integer $n \geq 0$ and a rank $n$ vector bundle $E$ on $C$
such that $h^1(C, S^t(E)) = 0$ for all $t \geq 1$. We will call $\diamond$ this condition on $E$.
Condition $\diamond$ is satisfied if either $\mu_-(E) > 2g - 2$ or $\mu(E) > 2g - 2$ and $E$ is
semistable or $\mu(E) \geq 2g - 2$ and $E$ is stable or if $g \geq 2$, $E$ is a general stable
bundle on $C$ with its degree and rank and $\mu(E) \geq g - 1$. By Riemann-Roch
Condition \( \diamond \) implies \( \mu_-(E) \geq g - 1 \). Set \( X := \mathbf{P}(E) \) and let \( \pi : X \to C \) denote the ruling. Let \( \mathcal{O}_X(1) \) denote the tautological line bundle on \( X \). Hence \( \pi_*\mathcal{O}_X(t) \cong S^t(E) \) for all integers \( t \geq 1 \). Hence Condition \( \diamond \) is equivalent to the condition \( h^i(X, \mathcal{O}_X(t)) = 0 \) for all \( t \geq 1 \). We have \( h^i(X, \mathcal{O}_X) = g > 0 \). We have \( h^i(X, \mathcal{O}_X(t)) = 0 \) for all \( 1 \leq i \leq n - 1 \) and all \( t < 0 \) (Remark 1). For any coherent sheaf \( F \) on \( X \) and any \( t \in \mathbb{Z} \) set \( F(t) := F \otimes \mathcal{O}_X \mathcal{O}_X(t) \). For any integer \( r \geq 1 \) let \( A(r) \) denote the set of the isomorphism classes of all rank \( r \) vector bundles on \( X \) such that \( h^i(X, F(t)) = 0 \) for all \( 1 \leq i \leq n - 1 \) and all \( t < 0 \). Fix any torsion free sheaf \( F \) on \( X \). Let \( s(F) \) be the maximal integer \( t \) such that there is \( M \in \text{Pic}(C) \) with \( h^0(X, F(-t) \otimes M^*) > 0 \). Now assume that \( E \in A(2) \). For that integer \( t := s(F) \) take \( M \) with maximal degree. The maximality of \( \text{deg}(M) \) and of \( t \) implies that any non-zero map \( \pi^*(M)(t) \to F \) drops rank at most in codimension 2. Hence \( F \) fits in an exact sequence

\[
0 \to \pi^*(M)(t) \to F \to \mathcal{I}_Z \otimes \det(F)(-t) \otimes \pi^*(M^*) \to 0 \tag{1}
\]

in which either \( Z = \emptyset \) or \( Z \) is a locally complete intersection closed subscheme of \( X \) with pure codimension 2. We have \( c_2(F) = Z - \pi^*(M)(t) \cdot \det(F)(-t) \otimes \pi^*(M^*) \) in the Chow ring of \( X \). Set \( z(F) := Z \cdot \text{Pic}(1)^{n-2} \). The integer \( z(F) \) is well-defined, i.e. it does not depend from the choice of \( M \) and the injection in (1), because we took \( M \) with maximal degree. If either \( n = 2 \) or \( E \) is ample, then the integer \( z(F) \) is non-negative and \( z(F) = 0 \) if and only if \( Z = \emptyset \). We will say that a rank \( r \geq 2 \) torsion free \( F \) on \( X \) is \textit{balanced} (resp. \textit{semibalanced}) if \( s(G)/s \leq s(F)/r \) (resp. \( s(G)/s \leq s(F)/r \)) for all integers \( 1 \leq s \leq r - 1 \) and all rank \( s \) subsheaves \( G \) of \( F \). Obviously, in the previous definition it is sufficient to test the saturated torsion free sheaves \( G \) of \( F \), i.e. the subsheaves \( G \) such that \( F/G \) has no torsion. Here we prove the following result.

**Theorem 1.** Assume that \( E \) satisfies Condition \( \diamond \). Fix \( F \in A(2) \) which is not balanced. If \( n \geq 3 \) assume that \( E \) is spanned. Then \( z(F) = 0 \) and \( F \) is an extension of two line bundles.

We work over an algebraically closed field \( \mathbb{K} \).

**Remark 1.** We have \( \omega_X \cong \pi^*(\det(E) \otimes \omega_C)(-n) \). Notice that \( R^i \pi_*\mathcal{O}_X(t) \) \( = 0 \) for all \( 0 \leq i \leq n - 2 \) and all \( t < 0 \). Hence the Leray spectral sequence of \( \pi \) gives \( h^i(X, \mathcal{O}_X(t)) = 0 \) for all \( 0 \leq i \leq n - 2 \) and all \( t < 0 \). Now assume that \( \det(E) \otimes \omega_C \) is spanned. Since \( g > 0 \), his condition is satisfied if \( \det(E) \) is spanned. Serre duality gives \( h^i(X, \mathcal{O}_X(t)) = h^{n-i}(X, \pi^*(\det(E) \otimes \omega_C)(-t-n)) \) for all \( i, t \). Since \( \det(E) \otimes \omega_C \) is spanned, Condition \( \diamond \) gives \( h^1(C, \omega_C \otimes S^t(E)) = 0 \)
for all $t > 0$. Hence $h^{n-1}(X, \mathcal{O}_X(t)) = 0$ for all $t < 0$ and $h^n(X, \mathcal{O}_X(t)) = 0$ for all $t \geq -n + 1$.

**Remark 2.** Any extension of an element of $A(r)$ by an element of $A(s)$ is an element of $A(r + s)$.

**Remark 3.** Fix $L \in \text{Pic}(X)$, say $L \cong M(a)$ with $M \in \text{Pic}(C)$ and $a \in \mathbb{Z}$. $h^1(X, L(-a)) = 0$ if and only if $h^1(C, M) = 0$. If $h^1(C, M) = 0$, then $\deg(M) \geq g - 1$; if $g - 1 \leq \deg(M) \leq g - 2$ then $h^1(C, M)$ may vanish or not, depending on $M$. $h^1(X, L(t)) = 0$ for all $t > -a$ if and only if $S^m(E) \otimes M = 0$ for all $m > 0$. $h^1(X, L(-a - 1)) = 0$ for any $M$. Serre duality gives that $h^{n-1}(X, L(t)) = 0$ for all $t \leq -a - 2$ if and only if $h^1(C, S^m(E) \otimes M^* \otimes \det(E)) = 0$ for all $m > 0$. Now we normalize $\mathcal{O}_X(1)$ so that $0 \leq b := \deg(E) \leq n - 1$. First assume $g = 1$. If $E$ is not semistable, then $\mu_-(E) < 0$. Hence $h^1(X, L(t)) > 0$ for $t > 0$ and any $M$. Now assume $E$ semistable. We assume $h^1(C, M) = 0$ and in particular $\deg(M) \geq 0$. If $\deg(M) \geq b + 1$, then $h^1(C, L(-a - 2)) > 0$. If $b > 0$ we get $h^1(C, L(t)) = 0$ for all $t > 0$. If $b = 0$ and $\deg(M) = 0$ is sufficiently general in $\text{Pic}^0(C)$ (for a fixed $E$), then $h^1(X, L(t)) = 0$ for all $t \in \mathbb{Z}$.

**Remark 4.** Assume $g \geq 2$. Recall that $h^0(X, \mathcal{O}_X(t)) = h^0(C, S^t(E))$ and that $S^t(E)$ has rank $\binom{n+t-1}{n}$ and slope $t \cdot \mu(E)$. for all $t > 0$. Since $\mu_-(E) \geq g - 1$, $E$ is ample if $\text{char}(\mathbb{K}) = 0$ (see [8]). In positive characteristic assume $\mu_-(E) > 2g - 1$, use the spannedness of $E$ (see below), and then apply [9], Cor. III.1.9. Hence $h^1(C, S^t(E))$ for $t \gg 0$ (and even for $t \geq 3$ if $\text{char}(\mathbb{K}) = 0$). Riemann-Roch gives $h^0(C, S^t(E)) = \binom{n+t-1}{n} t \mu(E) + \binom{n+t-1}{n} (1 - g)$. $h^0(C, E) = h^0(X, \mathcal{O}_X(1))$. Notice that $h^0(C, E) > 0$ (i.e. $h^0(X, \mathcal{O}_X(1)) > 0$) if and only if $\mu(E) > g - 1$. $E$ is spanned if and only if $\mathcal{O}_X(1)$ is spanned. If $\mu_-(E) > 2g - 1$, then $E$ is spanned. Hence $\mathcal{O}_X(1)$ is spanned if $\mu(E) > 2g - 1$ and this is true in arbitrary characteristic.

**Proof of Theorem 1.** First assume $n = 2$. Fix an exact sequence (1) with $t = s(F)$ and twist it by $\mathcal{O}_X(-t - 1)$. Since $h^2(X, M_1(-1)) = 0$, we get $h^1(X, I_Z \otimes M_2(\det(F) - 2t - 1)) = 0$. Since $t = s(F)$ and $F$ is not balanced, $t(\det(F)) - 2t - 1 < 0$. Hence $h^0(X, M_2(t(\det(F)) - 2t - 1)) = 0$. Since $Z$ is zero-dimensional, and $h^1(X, I_Z \otimes M_2(t(\det(F)) - 2t - 1)) = 0$, we get $Z = \emptyset$. Now assume $n \geq 3$. By assumption $\mathcal{O}_X(1)$ is ample and spanned. Since $Z$ has pure codimension 2, $Z = \emptyset$ if and only if it does not intersect a general intersection of $n - 2$ members of $|\mathcal{O}_X(1)|$. Using this intersection we reduce to the case $n = 2$ just proved. □

What happens if $F$ has higher rank or if $F$ is balanced? Even in the case $g = 1$ we do not have any result, even a very partial one.
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References


