International Journal of Pure and Applied Mathematics
Volume 46 No. 4 2008, 311-315

ACM VECTOR BUNDLES ON $\mathbf{P}^{n-1} \times C$ WITH C A SMOOTH CURVE

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Fix a smooth genus $q > 0$ curve, an integral $(n-1)$-dimensional variety Y and an ample $H \in \mathrm{Pic}(Y)$ such that $h^i(Y, H^{\otimes t}) = 0$ for all $1 \le i \le n-2$ and all $t \in \mathbb{Z}$. Let E be a vector bundle on $X := Y \times C$. We will say that E is ACM if $h^i(X, E \otimes L) = 0$ for all $1 \le i \le n-1$ and certain $L \in \mathrm{Pic}(X)$ constructed from H and $\mathrm{Pic}(C)$. Here we study ACM vector bundles on X, mainly when $Y = \mathbf{P}^{n-1}$ and $\mathrm{rank}(E) \le n-2$.

AMS Subject Classification: 14J60
Key Words: vector bundle, ACM bundle, arithmetically Cohen-Macaulay vector bundle

*

Fix an integer $n \ge 2$, an integral $(n-1)$-dimensional projective variety Y and an an ample line bundle H on Y. We make the following assumptions:

(i) $h^i(Y, H^{\otimes t}) = 0$ for all $1 \le i \le n-2$ and all $t \in \mathbb{Z}$;

(ii) $h^{n-1}(Y, H^{\otimes t}) = 0$ for all $t \ge 0$.

Hence condition (i) is empty if $n = 2$, while if $n \ge 3$ it implies $h^i(Y, \mathcal{O}_Y) = 0$ for all $1 \le i \le n-2$. Condition (ii) is equivalent to $h^0(Y, \omega_Y \otimes H^{\otimes z}) = 0$ for all $z \ge 0$ (use the case $p = r$ of the theorem in the preface of [1]). Under

Received: April 5, 2008

very mild assumptions (e.g. if $h^0(Y,H) > 0$), condition (ii) is equivalent to $h^0(Y,\omega_Y) = 0$. Fix a smooth and connected projective curve C of genus $q > 0$ and set $X := Y \times C$. Let $\pi_1 : X \to Y$ and $\pi_2 : X \to C$ denote the projections. Set $\wp_X := \mathbb{Z}\pi_1^*(H) \otimes \pi_2^*(\mathrm{Pic}(C)) \subseteq \mathrm{Pic}(X)$. If H is a generator of $\mathrm{Pic}(Y)$, then $\wp_X = \mathrm{Pic}(X)$ (use that in this case any morphism $Y \to C$ is constant and apply [4], Proposition 2). For any $M \in \mathrm{Pic}(C)$, any $t \in \mathbb{Z}$ and any coherent sheaf F on X set $\mathcal{O}_X(t,M) := \pi_1^*(H^{\otimes t}) \otimes \pi_2^*(M)$ and $F(t,M) := F \otimes \mathcal{O}_X(t,M)$. Set $\Phi := \{\mathcal{O}_X(t,M) \in \wp_X$: either $t \geq 0$ and $\deg(M) \geq 0$ or $t \leq 0$ and $\deg(M) \leq 0\}$ and $\widetilde{\Phi} := \Phi \cup \bigcup_{P\in C}(0,-P)$. Set $\Psi := \{\mathcal{O}_X(t,M) \in \wp_X$: either $t > 0$ and $\deg(M) > 0$ or $t = \deg(M) = 0$ or $t < 0$ and $\deg(M) < 0\}$ and $\widetilde{\Psi} := \Psi \cup \bigcup_{P\in C}(0,-P)$. We will say that the coherent sheaf F is ACM or that it is *arithmetically Cohen-Macaulay* (resp. WACM or *weakly arithmetically Cohen-Macaulay*, resp. SACM, resp. SWACM) with respect to $\wp_X$ if $h^i(X, F \otimes L) = 0$ for all $1 \leq i \leq n-1$ and all $L \in \Phi$ (resp. all $L \in \Psi_X$, resp. all $L \in \widetilde{\Phi}$, resp. all $L \in \widetilde{\Psi}$). A vector bundle A on Y is said to be ACM or *arithmetically Cohen-Macalay* if $h^i(Y, A \otimes H^{\otimes t}) = 0$ for all $1 \leq i \leq n-2$ and all $t \in \mathbb{Z}$. Here we work over an algebraically closed field $\mathbb{K}$ and prove the following results and a few remarks for general Y.

Theorem 1. *Assume* $\mathrm{char}(\mathbb{K}) = 0$, $Y = \mathbf{P}^{n-1}$ *and* $n \geq 3$. *Let* E *be a* rank$r \leq n-2$ *ACM vector bundle on* X. *There are integers* $s > 0$, $r_i > 0$, $1 \leq i \leq s$, *such that* $r_1 + \cdots + r_s = r$, $t_1 > \cdots > t_s$, rankr_i *vector bundles* A_i *on* C *and an increasing filtration* $\{E_i\}_{0\leq i\leq s}$ *of* E *by subbundles such that* $E_0 = 0$, $E_r = E$ *and* $E_i/E_{i-1} \cong \pi_2^*(A_i)(t_i, \mathcal{O}_C)$ *for all* $1 \leq i \leq r$.

Proposition 1. *Take* $Y = \mathbf{P}^{n-1}$, $n \geq 4$, *an ACM* rank$r < n-2$ *vector bundles on* X *and* $s, t_i, r_i, E_i A_i$ *as in the statement of Theorem 1. If* $s = 1$, *then* A_1 *is as described in Example 1 below (case* $q = 1$*) or Example 2 below (case* $q \geq 2$*). Now assume* $s \geq 2$. *Then* $p_2^*(A_s)(t_s, \mathcal{O}_C)$ *is ACM.*

Remark 1. Assume $\mathrm{char}(\mathbb{K}) = 0$, $Y = \mathbf{P}^{n-1}$ and $n \geq 3$. Let E be a rank$r \leq n-2$ ACM vector bundle on X. Since every vector bundle on a smooth curve is an iterated extension of lines bundles, there is an increasing filtration $\{F_i\}_{0\leq i\leq r}$ of E such that $F_0 = 0$, $F_r = E$ and $F_i/F_{i-1} \in \mathrm{Pic}(X)$. The existence of such a filtration gives a strong restriction on the Chern classes of E. Fix any $L \in \mathrm{Pic}(X)$. Künneth formula and Theorem 1 shows that $h^i(X, E \otimes L) = 0$ for all $2 \leq i \leq n-2$.

Theorem 2. *Assume* $\mathrm{char}(\mathbb{K}) = 0$, $Y = \mathbf{P}^{n-1}$ *and* $n \geq 3$. *Let* E *be a* rank$n-1$ *ACM vector bundle on* X. *Either* E *satisfies the thesis of Theorem 1 with respect to the integer* $r := n-1$ *or there are an integer* z, $L \in \mathrm{Pic}(X)$ *and* *a rank$n$ vector bundle B on C such that E fits in one of the following exact sequences*

$$0 \to L \to \pi_2^*(B)(z+1, \mathcal{O}_C) \to E \to 0\,, \tag{1}$$

$$0 \to E \to \pi_2^*(B)(z-1, \mathcal{O}_C) \to L \to 0\,. \tag{2}$$

If (1) occurs, then $B \cong \pi_{2*}(E(-z-1, \mathcal{O}_C))$ and $L \cong \mathcal{O}_X(z, M)$ for some $M \in \mathrm{Pic}(C)$. If (2) occurs, then $B \cong \pi_{2*}(E^*(z-1, \mathcal{O}_C))$ and $L \cong \mathcal{O}_X(z, M)$ for some $M \in \mathrm{Pic}(C)$.

Remark 2. Fix coherent sheaves A on Y and B on C. Set $F := \pi_1^*(A) \otimes \pi_2^*(B)$. Künneth formula gives $h^0(X,F) = h^0(Y,A) \cdot h^0(C,B)$, $h^i(X,F) = h^i(Y,A) \cdot h^0(C,B) + h^{i-1}(Y,A) \cdot h^1(C,B)$ for $1 \le i \le n-1$ and $h^n(X,F) = h^{n-1}(Y,A) \cdot h^1(C,B)$.

Remark 3. Any extension of two ACM (or WACM or SACM or SACM or ...) vectpr bundleas has the same property.

Example 1. Here we assume $q = 1$ and use the characteristic free part of the classification of vector bundles on elliptic curves proved in [2], Part II. Fix an integer $r \ge 1$ and a rank$r$ vector bundle B on C. Set $a := \deg(B)$ and $E_t := \imath_2^*(B)(t, \mathcal{O}_C)$. E_0 is ACM if and only if $h^1(C, B \otimes M) = 0$ for all $M \in \mathrm{Pic}(C)$ with $\deg(M) \ge 0$ (and here it is sufficient to use the degree 0 line bundles M) and $h^0(C, B \otimes M) = 0$ for all line bundles M such that $\deg(M) < 0$ (use (t, M) with $t \le -2$ and Remark 2. Thus E_0 is ACM if and only if it is WACM if and only if in the Harder-Narasimhan filtration of B all semistable graded subquotients of B have degree > 0. Now assume $t > 0$. E_t is ACM if and only if it is WACM if and only if in the Harder-Narasimhan filtration of B all semistable graded subquotients of B have slope $> t$, i.e. if and only if $\mu_-(E) > t$. Now assume $t < 0$. E_t is ACM if and only if it is WACM if and only if in the Harder-Narasimhan filtration of B all semistable graded subquotients of B have slope $< t$, i.e. if and only if $\mu_+(E) < t$.

In the case $q \ge 2$ our results are less complete. Indeed, we may only prove the following almost tautological result.

Example 2. Assume $q \ge 2$. Fix an integer $r \ge 1$ and a rank$r$ vector bundle B on C. Set $a := \deg(B)$ and $E_t := \pi_2^*(B)(t, \mathcal{O}_C)$. E_0 is ACM if and only if $h^1(C, B \otimes M) = 0$ for all $M \in \mathrm{Pic}^0(C)$ and $h^0(C, B \otimes M) = 0$ for all $M \in \mathrm{PIc}^{-1}(C)$ (use (t, M) with $t \le -2$ and Remark 2. Now assume $t > 0$. E_t is ACM if and only if it is WACM if and only if $h^1(C, B \otimes M) = 0$ for all $M \in \mathrm{Pic}^0(C)$ and $h^0(C, B \otimes M) = 0$ for all $M \in \mathrm{PIc}^{-1}(C)$. Now assume $t < 0$. E_t is ACM if and only if it is WACM if and only if $h^0(C, B \otimes M) = 0$ for all

$M \in \mathrm{Pic}^0(C)$ and $h^1(C, B \otimes M) = 0$ for all $M \in \mathrm{PIc}^1(C)$. We will say that B is *positively balanced*, (resp. *negatively balanced* if $h^1(C, B \otimes M) = 0$ for all $M \in \mathrm{Pic}^0(C)$ and $h^0(C, B \otimes M) = 0$ for all $M \in \mathrm{PIc}^{-1}(C)$ (resp. $h^0(C, B \otimes M) = 0$ for all $M \in \mathrm{Pic}^0(C)$ and $h^1(C, B \otimes M) = 0$ for all $M \in \mathrm{PIc}^1(C)$).

As far as we know there is no previously defined invariant of A which captures the conditions listed in Example 2.

Proposition 2. *Let E be a SWACM vector bundle on X. Then $E|Y \times \{P\}$ is ACM for all $P \in C$.*

Proof. Use the exact sequence

$$0 \to E(t, -P \otimes M) \to E(t, M) \to E(t, 0)|Y \otimes \{P\} \to 0\,. \tag{3}$$

In which $M = \mathcal{O}_C$ if $t = 0$, $\deg(M) > 0$ if $t > 0$ and $\deg(M) < 0$ if $t < 0$. □

Proposition 3. *Let E be an ACM vector bundle on X. Fix any $P \in C$. Then $h^i(Y \times \{P\}, E(t, 0)|Y \times \{P\}) = 0$ for all $1 \le i \le n-2$ and all $t \neq 0$. Hence $E|Y \times \{P\}$ is ACM (after the natural identification of $Y \times \{P\}$ with Y induced by π_1).*

Proof. Fix $P \in C$, $M \in \mathrm{Pic}(C)$ and $t \in \mathbb{Z}$ and consider the exact sequence (3) in which we identify. If $t > 0$ take any M with $\deg(M) > 0$. If $t < 0$ take any M with $\deg(M) < 0$. Since E is ACM, ne get that $h^i(Y, E(t, 0)|Y \otimes \{P\}) = 0$ if $1 \le i \le n-2$ and $t \neq 0$. □

Proof of Theorem 1. Fix $P \in C$. Use Proposition 3 to show that we have $h^i(Y \times \{P\}, E(t, 0)|Y \times \{P\}) = 0$ for all $1 \le i \le n-2$ and all $t \neq 0$. By [3], Corollary 1.2, every indecomposable factor G_P of $E|Y \times \{P\}$ is isomorphic to either a line bundle or to Ω_Y^i for some $1 \le i \le n-2$. Since $r \le n-2$, every irreducible factor of $E|T \times \{P\}$ has rank one. Since $n \ge 3$, it is easy to to check that (up to a permutation of the factors) the decomposition into irreducible factors of $E|Y \times \{P\}$ does not depend from the choice of P. We get the existence of integers $s > 0$, $r_i > 0$, $1 \le i \le s$, $t_1 > \cdots > t_s$, such that $r_1 + \cdots + r_s = r$ and $E|Y \times \{P\} \cong \oplus_{i=1}^s \mathcal{O}_{Y \times \{P\}}(t_i)^{\oplus r_i}$ for all $P \in C$ (Harder-Narasimhan filtration). Hence $p_{2*}(E(-t_1, 0))$ is a rankr_1 vector bundles on C. Basing change shows that the natural map $p_2^* p_{2*}(E(-t_1, 0)) \to E(-t_1, 0)$ has constant rankr_1. Basing change shows the existence of s vector bundles A_i, $1 \le i \le s$, on C and an increasing filtration E_i, $0 \le i \le s$, of E by subbundles such that $E_0 = 0$, $E_s = E$ and $E_i/E_{i-1} \cong \pi_2^*(A_i)((t_i, \mathcal{O}_C)$ for all $1 \le i \le s$. □

Proof of Theorem 2. As in the proof of Theorem 1 we apply [3], Corollary 1.2. The new case occurs if there is $P \in C$ and $z \in \mathbb{Z}$ such that either $E|Y \times \{P\} \cong \Omega_Y^1(z)$ or $E|Y \times \{P\} \cong TY(z)$. Looking at the Chern classes we see that in the last two cases $E|Y \times \{Q\}$ does not depend from the choice of Q. First assume $E|Y \times \{P\} \cong TY(z)$. Changing basis shows that $\pi_{2*}(E(-z-1, \mathcal{O}_C))$ is a rank$n$ vector bundle on C and that the natural map $\pi_2^*(\pi_{2*}(E(-z-1, \mathcal{O}_C))) \to E(-z+1, \mathcal{O}_C)$ is surjective. Hence its kernel is a line bundle Hence in this there is $L \in \mathrm{Pic}(X)$ fitting in the exact sequence

$$0 \to L \to \pi_2^*(\pi_{2*}(E(-z-1, \mathcal{O}_C))) \to E(-z-1, \mathcal{O}_C) \to 0\,. \tag{4}$$

Dually, if $E|Y \times \{P\} \cong \Omega^1_Y(z)$ for all $P \in C$, then there is $L \in \mathrm{Pic}(X)$ fitting in the exact sequence

$$0 \to L \to \pi_2^*(\pi_{2*}(E^*(z-1, \mathcal{O}_C))) \to E^*(z-1, \mathcal{O}_C) \to 0\,. \tag{5}$$

Taking dual we get the exact sequence (2). □

Proof of Proposition 1. The case $s = 1$ was checked in Examples 1 and 2. Now assume $s \geq 2$. Recall that $h^x(X, E \otimes L) = 0$ for all $2 \leq x \leq n-2$ and all $L \in \mathrm{Pic}(X)$ (Remark 1). By induction on i as in the second part of Remark 1 we get $h^x(X, E_i \otimes L) = 0$ for all $2 \leq x \leq n-2$ and all $L \in \mathrm{Pic}(X)$. Fix $L \in \mathrm{Pic}(X)$ such that $h^1(X, E \otimes L) = 0$. Since $h^2(X, E_{s-1} \otimes L) = 0$, we get $h^1(X, \pi_2^*(A_s)(t_s, \mathcal{O}_C) \otimes L) = 0$. Since E is ACM, $\pi_2^*(A_s)(t_s, \mathcal{O}_C)$ is ACM. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] A. Altman, S. Kleiman, *Introduction to Grothendieck Duality Theory*, Lect. Notes in Math., **146**, Springer, Berlin (1970).

[2] M. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc.*, **3**, No. 7 (1957), 414-452.

[3] M.-C. Chang, Characterization of arithmetically Buchsbaum subschemes of codimension 2 in $\mathbb{P}^n$, *J. Differential Geometry*, **31**, No. 2 (1990), 321-341.

[4] T. Fujita, Cancellation problem of complete varieties, *Invent. Math.*, **64**, No. 1 (1981), 119-121.