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PARTICLE DISPERSION IN STOCHASTIC FLOWS WITH CONSTANT DRIFT GRADIENT

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Abstract: Absolute and relative dispersion are investigated for two types of stochastic flows with a linear drift, the Brownian flow, which implies delta-correlated velocity fluctuations, and the first order Markov flow with memory characterized by finite correlation time. It is shown that anisotropy of absolute dispersion is completely determined by anisotropy of the drift while its magnitude depends on both, drift and velocity fluctuation statistics. In contrast, anisotropy of the relative dispersion is strongly affected by the fluctuation statistics and the crucial parameter is the normal correlation length.

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1. Introduction

We address Lagrangian motion in a velocity field given by

$$\mathbf{u}(t,\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(t,\mathbf{x}), \quad \mathbf{x},\mathbf{u} \in R^d$$

with the steady mean flow linearly depending on the space coordinate, i.e.

$$\bar{\mathbf{u}}(\mathbf{x}) \equiv E\{\mathbf{u}(t,\mathbf{x})\} = \mathbf{U} + \mathbf{G}\mathbf{x} \tag{1}$$

and velocity fluctuations described by a random velocity field $\mathbf{u}'(t,\mathbf{x})$. Here $E\{\}$ means averaging over an ensemble (expectation), $\mathbf{U}$ and $\mathbf{G} = (g_{ij})$ are

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constant vector and matrix respectively.

Define the Lagrangian trajectory $\mathbf{r}(t) = \mathbf{r}(t, \mathbf{a})$ starting at point $\mathbf{a}$ in the usual way

$$d\mathbf{r}/dt = \mathbf{u}(t, \mathbf{r}), \quad \mathbf{r}(0, \mathbf{a}) = \mathbf{a}.$$

The goal is to investigate the absolute dispersion

$$\mathbf{D} = \mathbf{D}(t) = E\{(\mathbf{r}(t) - \bar{\mathbf{r}}(t))(\mathbf{r}(t) - \bar{\mathbf{r}}(t))^T\}, \quad \bar{\mathbf{r}}(t) = E\{\mathbf{r}(t)\} \tag{2}$$

and the relative dispersion

$$\mathbf{D}_r = \mathbf{D}_r(t, \mathbf{r}_0) = E\{(\mathbf{z}(t) - \bar{\mathbf{z}}(t))(\mathbf{z}(t) - \bar{\mathbf{z}}(t))^T\}, \quad \mathbf{z}(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t)\,, \tag{3}$$

where $\mathbf{r}_1(t), \mathbf{r}_2(t)$ are trajectories of two particles with initial separation $\mathbf{r}_0$. For those purposes two models of the stochastic part $\mathbf{u}'(t, \mathbf{x})$ are considered. In the first one $\mathbf{u}'(t, \mathbf{x})$ is assumed to be a Gaussian white noise in time and homogeneous isotropic in space with a smooth covariance tensor $\mathbf{B}(\mathbf{x})$. This flow is known in mathematical literature as Brownian flow, e.g. Baxendale and Harris [3] and referred to as Kraichnan model in physics of fluids, e.g. Falkovich et al [6]. In the second considered model the fluctuation part is not Gaussian and has finite memory characterized by a dissipation matrix $\mathbf{A}$ (defined in (16,22)). Instead, its forcing (acceleration) is a Gaussian white noise with the space covariance tensor denoted by the same symbol $\mathbf{B}(\mathbf{x})$. We call the second model by the first order Markov (FOM). A remarkable fact we exploit here is that FOM, written originally in form of a non-linear equation for the Eulerian velocity field, leads to closed equations for the positions and velocities of any number of Lagrangian particles. In the case of zero drift the model was first formulated and developed in Piterbarg [10], [11]. The specific issues and quantities we focus on are as follows.

(i) Necessary and sufficient conditions on the gradient $\mathbf{G}$, variance $\mathbf{B}(\mathbf{0})$ and dissipation $\mathbf{A}$ for existence of the inertial regime defined as

$$\lim_{t \to \infty} \frac{1}{t}\mathbf{D}(t) = \mathbf{K}. \tag{4}$$

In this case $\mathbf{K}$ is called the diffusion tensor or simply diffusivity.

(ii) Anisotropy of $\mathbf{K}$ and its relation to anisotropy of the mean stream characterized by tensor $\mathbf{G}$.

(iii) Exact expressions for diffusivity in some specific physically important cases such as shear flow and gyre.

(iv) Second Lyapunov moment defined as

$$\Lambda = \lim_{t \to \infty} \lim_{|\mathbf{r}_0| \to 0} \frac{1}{t} \log \frac{1}{|\mathbf{r}_0|} tr(\mathbf{D}_r(t, \mathbf{r}_0))\,.$$

Loosely speaking that means

$$\mathbf{D}_r \sim \mathbf{K}_r e^{\Lambda t} \tag{5}$$

for small enough initial separation. Thus, Λ characterizes the mixing rate of a tracer in the fluid while the diffusivity is responsible for its spreading only.

(v) Numerical modeling $\mathbf{D}_r(t, \mathbf{r}_0)$ for different model parameters including Hurst exponent of the space covariance. This exponent discriminates between the local and non-local dynamics, e.g. Babiano et al [1].

(vi) Anisotropy of mixing characterized by anisotropy of $\mathbf{K}_r$ in (5).

This work is partially motivated by applications to upper ocean turbulence which is usually viewed as a two-dimensional phenomenon. For this reason we further focus on the case $d = 2$ and our main findings are the following.

(i) In the framework of FOM the inertial regime (4) exists if the following four conditions are satisfied: first, the mean flow is incompressible, second, the origin is its elliptic stationary point, then, the real parts of the matrix $\mathbf{G} + \mathbf{A}$ are positive, and, finally, the velocity correlation time is small enough comparing to a time scale related to the mean flow, see inequality (26).

(ii) Orientation and excentrisitet of the anisotropy ellipse for absolute dispersion coincide with that of the drift for both, Brownian flow and FOM. Thus, anisotropy of absolute dispersion is completely determined by anisotropy of the mean flow while the diffusivity magnitude depends on parameters of both, the mean flow and velocity fluctuations.

(iii) In contrast to the absolute dispersion, anisotropy of the relative dispersion is determined by both factors, gradients of the mean flow and statistics of the velocity fluctuations in the framework of both models. The crucial impact on the mixing ellipse orientation is produced by the normal component of the covariance $\mathbf{B}(\mathbf{r})$.

The paper is organized as follows. In Section 2 all the indicated problems are treated for the Brownian flow. We guess that our results for this model can be overlapped with that of other authors, but we were not able to find exact references.

In Section 3 FOM is presented in detail. Notice that this model is intensively used in the theory of inertial particles, e.g. Balkovsky et al [2]. Section 4 is devoted to investigation of the absolute dispersion for FOM. Finally in Section 5 we address the relative dispersion for FOM.

2. Brownian Flow

The Brownian flow is determined by the assumption that the velocity fluctuation $\mathbf{u}'(t, \mathbf{r})$ is a Gaussian white noise in t, i.e.

$$E\{\mathbf{u}'(t_1, \mathbf{x}_1)\mathbf{u}'(t_2, \mathbf{x}_2)^T\} = \delta(t_1 - t_2)\mathbf{B}(\mathbf{x}_1 - \mathbf{x}_2) \ ,$$

where $\delta(t)$ is the Dirac delta-function. As a consequence the one-particle motion is given by the following ordinary stochastic differential equation

$$d\mathbf{r} = (\mathbf{U} + \mathbf{G}\mathbf{r})dt + \mathbf{B}(\mathbf{0})^{1/2}d\mathbf{w},$$

where $\mathbf{w}(t)$ is the standard Brownian motion in two dimensions. Direct computations, which are given in detail for FOM in Section 4, show that the limit in (4) exists if and only if

$$tr(\mathbf{G}) = 0 \ \text{ and } \ r^2 = \det(\mathbf{G}) > 0, \tag{6}$$

i.e. the mean flow is incompressible and the origin is an elliptic point of the mean circulation. Under conditions (6)

$$\mathbf{K} = \frac{g_{12}b_{22} - g_{21}b_{11} + 2g_{11}b_{12}}{r^2}\mathbf{G}^{\perp} \ , \tag{7}$$

where

$$\mathbf{G}^{\perp} = \begin{pmatrix} g_{12} & -g_{11} \\ g_{22} & -g_{21} \end{pmatrix}$$

and $(b_{ij}) = \mathbf{B}(\mathbf{0})$.

Define the diffusivity ellipse by

$$\mathbf{K}^{-1}\mathbf{x} \cdot \mathbf{x} = 1$$

and notice that a stream function for the mean circulation can be expressible as

$$\psi(\mathbf{x}) = \left(\mathbf{G}^{\perp}\right)^{-1}\mathbf{x} \cdot \mathbf{x}.$$

Comparing the latter to (7) we conclude that if $r \neq 0$, then the diffusion ellipse and the mean circulation ellipse defined as $\psi(\mathbf{x}) = 1$ have the same direction and same excentrisitet regardless of the properties of the fluctuation circulation ellipse. This conclusion is not true in the case $r = 0$ as happening in the classical shear model given by

$$g_{ij} = \alpha\delta_{i2}\delta_{1j}. \tag{8}$$

This case requires some comments. As was mentioned, say, in Shen and Yeung [13], the absolute dispersion in the shear flow grows as t^2 as $t \to \infty$. Thus, no inertial regime for $\mathbf{r}(t)$ is possible. In this case the quantity of interest is the displacement $\mathbf{r}'$ with respect to the mean flow defined by $d\mathbf{r}' = \mathbf{B}(\mathbf{0})^{1/2}d\mathbf{w}$ for which the conclusion is trivial

$$\lim_{t\to\infty}\frac{1}{t}E\{\mathbf{r}'\mathbf{r}'^T\}=\mathbf{B}(\mathbf{0}).$$

However the following below consideration of the relative dispersion is more reach in content even for this degenerate case.

A closed equation for the separation can be written as

$$d\mathbf{z}=\mathbf{G}\mathbf{z}dt+(2(\mathbf{B}(\mathbf{0})-\mathbf{B}(\mathbf{z})))^{1/2}d\mathbf{w}. \tag{9}$$

Accept the additional assumption of isotropy of velocity fluctuations under which entries $b_{ij}(\mathbf{x})$ of $B(\mathbf{x})$ are given in Monin and Yaglom [9]

$$b_{ij}(\mathbf{r})=b_N(r)\delta_{ij}+\frac{x_ix_j}{r^2}(b_L(r)-b_N(r)), \tag{10}$$

where $r=|\mathbf{x}|, \mathbf{x}=(x_1,x_2)$ and $b_L(r), b_N(r)$ are longitudinal and normal covariances respectively. Assume smoothness of the covariance function at zero

$$b_L(r)=b_0-\frac{1}{2}\beta_Lr^2+o(r^2),\quad b_N(r)=b_0-\frac{1}{2}\beta_Nr^2+o(r^2))\,,$$

where $b_0, \beta_N, \beta_L>0$. Parameters $l_L=\sqrt{b_0/\beta_L)}$, $l_N=\sqrt{b_0/\beta_N}$ are interpreted as longitudinal and normal correlation length respectively while

$$R=\left(\frac{4b_0}{\beta_N+\beta_L}\right)^{1/2}$$

is treated as the velocity space correlation radius. Notice that for the solenoidal velocity fluctuations $\Gamma=\beta_N/\beta_L=3$ and for the potential fluctuations $\Gamma=1/3$.

Now we proceed to computing the second Lyapunov exponent defined in (5) and investigating the relative dispersion anisotropy. Assuming small initial separation $r_0\ll R$ we linearize equation (9) and arrive at

$$d\mathbf{z}=\mathbf{G}\mathbf{z}dt+\mathbf{\Phi}(\mathbf{z})d\mathbf{w}\,, \tag{11}$$

where

$$\mathbf{\Phi}(\mathbf{z})=\begin{pmatrix}\beta_L^{1/2}x & \beta_N^{1/2}y\\ \beta_L^{1/2}y & -\beta_N^{1/2}x\end{pmatrix},\quad \mathbf{z}=(x,y).$$

Then applying Ito formula for system (11) we obtain for $\rho_1=E\{x^2\}$, $\rho_{12}=E\{xy\}$, $\rho_2=E\{y^2\}$ the following closed system

$$\begin{aligned}\dot{\rho}_1&=(2g_{11}+\beta_L)\rho_1+2g_{12}\rho_{12}+\beta_N\rho_2\,,\\ \dot{\rho}_{12}&=g_{21}\rho_1+(\beta_L-\beta_N)\rho_{12}+g_{12}\rho_2\,,\\ \dot{\rho}_2&=\beta_N\rho_1+2g_{21}\rho_{12}+(2g_{22}+\beta_L)\rho_2.\end{aligned} \tag{12}$$

Thus, the second Lyapunov moment is given by $\Lambda=\mu_m+\beta_L$, where μ_m is the greatest real root of the following cubic equation

$$(\mu + \beta_N)(\mu^2 - \beta_N^2 - 4g_{11}^2) - 4g_{12}g_{21}(\mu - \beta_N) - 2\beta_N(g_{12} + g_{21})^2 = 0,$$

where the incompressibility condition (6) is taken into account. Hence, the principal asymptotic terms are given by

$$\rho_1 \sim e_1 e^{\Lambda t}, \quad \rho_{12} \sim e_{12} e^{\Lambda t}, \quad \rho_1 \sim e_2 e^{\Lambda t}, \tag{13}$$

where (e_1, e_{12}, e_2) is the eigenvector corresponding to the maximum eigenvalue. The cubic equation for μ can be exactly solved, but we restrict ourselves to the following helpful bounds

$$\mu_1 \wedge \mu_2) \le \mu \le \mu_1 \vee \mu_2, \quad \mu_1 = -2g_{11} - \frac{g_{21}}{g_{12}}\beta_N, \quad \mu_2 = 2g_{11} - \frac{g_{12}}{g_{21}}\beta_N$$

which follows from $e_1, e_2 > 0$. Next, we define the mixing ellipse by equation

$$x^2/e_1 - 2xy/e_{12} + y^2/e_2 = 1. \tag{14}$$

The complete characteristics of mixing anisotropy are the excentricitet $\varepsilon = \sqrt{e_1/e_2}$ and the angle φ between an axis of (14) and the x-axis. Easy to find

$$\varepsilon^2 = \frac{g_{12}\mu_m + 2g_{11}g_{12} + \beta_N g_{21}}{g_{21}\mu_m - 2g_{11}g_{21} + \beta_N g_{12}}$$

and

$$\tan 2\varphi = \frac{\mu_m^2 - \beta_N^2 - 4g_{11}^2}{(\beta_N - \mu_m)(g_{12} - g_{21}) - 2g_{11}(g_{12} + g_{21})}$$

which does not coincide with the corresponding angle for the mean circulation which is

$$\tan 2\varphi_0 = \frac{2g_{11}}{g_{12} + g_{21}}.$$

Another important observation is that the anisotropy parameters do not depend of β_L and hence the normal correlation length l_N determines the relative dispersion anisotropy in full (under fixed the mean flow parameters).

In particular for the shear flow both anisotropy parameters are completely defined by the ratio

$$c = \alpha/\beta_N.$$

Namely

$$\varepsilon^2 = (y_c - 1)/c, \quad \tan 2\varphi = y_c/c\,,$$

where

$$y_c = \frac{4}{9u} + u + \frac{2}{3}, \quad u = \sqrt[3]{c^2 + \frac{8}{27} + c\sqrt{c^2 + \frac{16}{27}}}\,.$$

As follows from the latter, both parameters are monotone functions of c, the excentricitet changes from ∞ to 0 and the angle φ from $\pi/4$ to 0 as c varies from 0 to ∞. The second Lyapunov moment for the shear flow is given by

$$\Lambda = \beta_L + (y_c - 1)\beta_N \, .$$

For the gyre defined as

$$\mathbf{G} = \begin{pmatrix} \gamma & \Omega \\ -\Omega & -\gamma \end{pmatrix} \tag{15}$$

one obtains

$$\varepsilon^2 = \frac{2\gamma + \mu_m - \beta_N}{2\gamma - \mu_m + \beta_N}, \quad \tan 2\varphi = \frac{\beta_N^2 + 4\gamma^2 - \mu_m^2}{2\Omega\mu_m} \, .$$

3. First Order Markov Flow

The goal of this section is to formulate an appropriate Lagrangian stochastic model covering motion of any number of particles in the velocity field with a finite correlation time. It would be perfect to have a simple closed equation for $\mathbf{r}(t)$ when the velocity field is covered by the classical hydrodynamics equations such as the Navier-Stokes or Euler equations. Unfortunately, it is not possible and for this reason other approaches have been attempted to get efficient description of Lagrangian motion in turbulence. For example, one can postulate stochastic differential equations covering evolution of $\mathbf{r}(t)$ and Lagrangian velocity $\dot{\mathbf{r}}\ (t)$ in a general form and then to determine the drift and diffusion coefficients to match the Eulerian statistics obtained from some general principles and direct numerical simulations. This method proved to be very efficient in considering a single particle motion even under non-stationary inhomogeneous environments, Thomson [14] and Griffa [7]. However, some difficulty arises in this approach when describing particle pairs, called the 2-1 correspondence violation, which means that the statistics of a single particle obtained from the particle pair distribution do not match that of derived from the corresponding single-particle model, Thomson [15], Borgas and Sawford [4]. The simplest way to avoid such an ambiguity is to explicitly bring forward an Eulerian velocity field, where the particles live. Following this direction we developed a multi particle Lagrangian model mostly in the case of zero (or constant) mean flow in Piterbarg [10], [11]. Here we use the same idea to incorporate a linear mean flow. Thus, to get simple closed equations for the particle positions and velocities we start with a non-linear equation for $\mathbf{u}(t, \mathbf{x})$ resembling the indicated hydrodynamics equations, but not equivalent them at all. Namely, let

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{A}\mathbf{u} = \mathbf{f}(t, \mathbf{x}), \tag{16}$$

where $\mathbf{A}$ is a constant matrix characterizing friction and $\mathbf{f}(t, \mathbf{x})$ is a forcing. The following statement gives, first, a condition for the mean flow of form (1) and, second, linear stochastic differential equations for the particle positions and their velocities.

Proposition 1. *Assume that*

$$\mathbf{f}(t, \mathbf{x}) = \mathbf{AU} + (\mathbf{A} + \mathbf{G})\mathbf{Gx} + \mathbf{f}'(t, \mathbf{x}),$$

where $\mathbf{f}'(t, \mathbf{x})$ *is a Gaussian white noise in time, i.e.*

$$E\{\mathbf{f}'\} = 0, \quad E\{\mathbf{f}'(t, \mathbf{x})\mathbf{f}'(s, \mathbf{y})^T\} = \delta(t - s)\mathbf{B}(\mathbf{x}, \mathbf{y}),$$

$\mathbf{B}(\mathbf{x}, \mathbf{y})$ *is a given covariance tensor. Let* $\mathbf{u}$ *be the solution of (16) with the initial condition*

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{U} + \mathbf{Gx}.$$

Then, first, for all t *its mean remains the same as given in (1) and, second, the following equations hold true*

$$d\mathbf{r} = (\mathbf{U} + \mathbf{Gr} + \mathbf{v})dt, \quad d\mathbf{v} = -(\mathbf{A} + \mathbf{G})\mathbf{v}dt + d\mathbf{w}(t, \mathbf{r}), \tag{17}$$

where $\mathbf{w}(t, \mathbf{r})$ *is a Brownian motion in Hilbert space whose distribution is fully determined by the forcing covariance tensor, more exactly*

$$E\{d\mathbf{w}(t, \mathbf{r}_1)d\mathbf{w}(t, \mathbf{r}_2)^T\} = \mathbf{B}(\mathbf{r}_1, \mathbf{r}_2)dt.$$

Proof. The standard averaging procedure of (16) leads to

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + E\{\mathbf{u}' \cdot \nabla \mathbf{u}'\} + \mathbf{A}\bar{\mathbf{u}} = \mathbf{AU} + (\mathbf{A} + \mathbf{G})\mathbf{Gx}.$$

The usual parameterization of the turbulent diffusion

$$E\{\mathbf{u}' \cdot \nabla \mathbf{u}'\} = -\mathbf{K}\nabla^2 \bar{\mathbf{u}}, \tag{18}$$

where $\mathbf{K}$ is a constant turbulent viscosity, leads to a closed equation for $\bar{\mathbf{u}}$. It is easy to check that its solution is given by (1). Finally, substitute $\mathbf{x} = \mathbf{r}(t)$ in (16) and introduce $\mathbf{v} = \dot{\mathbf{r}} - \mathbf{Gr}$ which results in (3). □

Let us comment on Proposition 1 and its proof.

First, the derivation of (17) from (16) is not rigorous since we use the closing conjecture (18). This hypothesis implies in particular homogeneous turbulence ($\mathbf{B}(\mathbf{x}, \mathbf{y}) = \mathbf{B}(\mathbf{x} - \mathbf{y})$). Since the framework of validity of (18) is more or less understood, one can get an idea to which extent model (17) is relevant to the particle motion in a stochastic flow with linear drift. An alternative to the hypothesis (18) could be the assumption that the non-linear term in the averaged equation is small and can be neglected. This assumption again leads to the same equations (17).

Second, it can be shown that equation (16) loses the uniqueness in finite time say t_f, while the flow (17) is well determined for all t although collisions of particles are allowed. In this sense (16) and (17) are equivalent only for $t < t_f$. It also can be shown that $\log t_f \sim -1/K^2$, where $K = \sigma\tau/R$ is the Kubo number, σ^2 the velocity variance, τ the velocity correlation time and R the space correlation radius. Thus, for small K typical, for example, for the upper ocean turbulence, t_f is practically infinite. Summing up with the first comment, one can view (17) as an independent Lagrangian stochastic model rather than a consequence of (16) and just (17) is the framework of our consideration while equation (16) is presented only for better understanding the model (17).

Third, physical properties of the Eulerian velocity field given by (16) and that of described by Navier-Stokes are quite different. In particular, the former is always compressible. So, the model (17) is not intended to encompass the real physics of fluids in full, it is rather an approximator mimicking some features of real turbulence. Nevertheless such models are popular in different fields at least for a constant mean flow, Eckhardt and Schumacher [5], Mehlig and Wilkinson [8], Shaw [12].

Then, as we show below the reduced Lagarngian velocity $\mathbf{v}$ is a stationary stochastic process under some natural conditions while the actual Lagrangian velocity $\dot{\mathbf{r}}$ is not so except the trivial case $\mathbf{G} = \mathbf{0}$. An interesting observation we present below is that the single particle dispersion still has an inertial asymptotics under some mild assumptions including the mean flow incompressibility and ellipticity in the vicinity of the origin.

Finally, it follows from (17) that motion of any M particles is described by the following system of $2M$ equations

$$d\mathbf{r}_m = (\mathbf{U} + \mathbf{G}\mathbf{r}_\mathbf{m} + \mathbf{v}_m)dt, \quad d\mathbf{v}_m = -(\mathbf{A} + \mathbf{G})\mathbf{v}_m dt + d\mathbf{w}_m(t), \tag{19}$$

where $\mathbf{w}_m,\ m = 1, \dots M$ are dependent $2D$ zero mean Wiener processes with the conditional cross correlation given the particle positions

$$E\{d\mathbf{w}_m d\mathbf{w}_n^T\} = \mathbf{B}(\mathbf{r}_m, \mathbf{r}_n)dt.$$

Thus, positions and velocities of different particles are correlated through the noise terms in (19).

Denote

$$\mathbf{C} = \mathbf{A} + \mathbf{G}$$

and for further purposes consider a statistically homogeneous forcing only, i.e.

$$\mathbf{B}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{B}(\mathbf{x} - \mathbf{y}).$$

In this case the single particle motion is described by

$$d\mathbf{r} = (\mathbf{U} + \mathbf{G}\mathbf{r} + \mathbf{v})dt, \quad d\mathbf{v} = -\mathbf{C}\mathbf{v} + \mathbf{B}(\mathbf{0})^{1/2}d\mathbf{w} \tag{20}$$

as follows from (19), where $\mathbf{w}$ is the standard $2D$ Wiener process. Model (20) and its extension to non-homogeneous and non-stationary environments has been first developed in the atmospheric physics by Thomson [14] and is widely exploited in the physical oceanography, Griffa [7].

The separation process for two particles

$$\mathbf{z} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{q} = \mathbf{v}_1 - \mathbf{v}_2$$

in the framework of (19) is described by

$$d\mathbf{z} = (\mathbf{G}\mathbf{z} + \mathbf{q})dt, \quad d\mathbf{q} = -\mathbf{C}\mathbf{q} + (2(\mathbf{B}(\mathbf{0}) - \mathbf{B}(\mathbf{z})))^{1/2}d\mathbf{w}. \tag{21}$$

In the case of constant mean flow these equations have been repeatedly used for different purposes such as investigating the relative dispersion [4] and senior Lyapunov exponent in Mehlig and Wikinson [8] and Piterbarg [10], [11].

Finally, aiming oceanographic applications we focus on the specific form of the dissipation matrix accepted in most of Lagarngian stochastic models, e.g. Veneziani et al [17], namely

$$\mathbf{A} = \begin{pmatrix} \theta & -\omega \\ \omega & \theta \end{pmatrix}, \tag{22}$$

where $1/\theta$ is the Lagrangian correlation time of the velocity fluctuations in the case of zero shear and ω is the spin. About the physical meaning of the spin and its estimates in the ocean see Veneziani et al [16].

4. Absolute Dispersion in FOM

Here we establish conditions under which the inertial regime (4) for the particle dispersion is realized and give an explicit expression for the diffusion tensor. Obviously, for that purpose the single particle motion equations (20) are sufficient. To analyze the absolute dispersion we can assume with no loss of generality that $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{U} = \mathbf{0}$ and hence

$$\mathbf{D} = \int_0^t \int_0^t e^{s_1\mathbf{G}}\mathbf{R}_v(s_1, s_2)e^{s_2\mathbf{G}^T} ds_1 ds_2 \tag{23}$$

with

$$\mathbf{R}_v(s_1, s_2) = E\{\mathbf{v}(s_1)\mathbf{v}(s_2)^T\}\,.$$

Proposition 2. *Assume that the deterministic part of the flow is solenoidal and elliptic, i.e.*

$$tr(\mathbf{G}) = 0, \quad \det(\mathbf{G}) > 0, \tag{24}$$

the eigenvalues λ_1 *and* λ_2 *of* $\mathbf{C}$ *have positive real parts*

$$Re(\lambda_j) > 0, \quad j = 1, 2 \tag{25}$$

and finally

$$\theta^2 > \frac{(g_{12} + g_{21})^2}{4} - g_{11}g_{22}. \tag{26}$$

Then $\mathbf{v}(t)$ *is a stationary process with*

$$\mathbf{R}_v(s_1, s_2) \equiv \mathbf{R}_v(s_1 - s_2) = e^{(s_1 - s_2)\mathbf{C}}\Sigma, \quad s_1 < s_2, \quad \Sigma = \mathbf{R}_v(0) \tag{27}$$

and (4) holds with

$$\mathbf{K} = c \begin{pmatrix} g_{12} & -g_{11} \\ g_{22} & -g_{21} \end{pmatrix}, \tag{28}$$

where c *is a positive scalar depending on the entries of* $\mathbf{G}, \mathbf{A}$ *and* $\mathbf{R}_v(0)$

Proof. To prove (27) under condition (25) it is enough to show that the entries of the spectral tensor defined by

$$\mathbf{R}_v(t) = \int_{-\infty}^{\infty} e^{ikt}\mathbf{E}(k)dk$$

satisfy

$$f(k) \equiv E_{11}(k)E_{22}(k) - |E_{12}(k)|^2 > 0.$$

Under condition (24) one can find that

$$e^{\mathbf{G}t} = \cos(rt)\mathbf{I} + \frac{\sin(rt)}{r}\mathbf{G}, \tag{29}$$

where r^2 =det($\mathbf{G}$) and $\mathbf{I}$ is the identity matrix. Notice that from (29) it follows that

$$\mathbf{G}^2 = -r^2\mathbf{I} \tag{30}$$

Using (29) and (30) it is easy to find

$$E_{11}(k) = \frac{\sigma^2((\theta + g_{11})k^2 + (\theta + g_{22})(\theta^2 + q^2))}{\pi Q},$$

$$E_{22}(k) = \frac{\sigma^2((\theta + g_{22})k^2 + (\theta + g_{11})(\theta^2 + q^2))}{\pi Q},$$

$$E_{12}(k) = \frac{\sigma^2(-g(\theta^2 + q^2 - k^2) - 2ik\theta(\alpha + 2\omega)}{2\pi Q},$$

$$E_{21}(k) = E_{12}(-k),$$

where

$$Q = (\theta^2 + r^2 + q^2)^2 - 4r^2q^2, \quad q^2 = \det(\mathbf{C} - \theta\mathbf{I}), \quad g = g_{12} + g_{21}.$$

Then the difference $f(k)$ is expressible up to a positive factor as

$$f(k) = (\theta^2 - g_{11}^2 - g^2/4)(k^4 + 2k^2(\theta^2 - q^2) + (\theta^2 + q^2)^2).$$

Thus the necessary and sufficient condition for stationarity is given by (26) provided with (24) and (25).

Now we proceed to proving (28). From (23)

$$\frac{d}{dt}\mathbf{D}(t) = e^{t\mathbf{G}} \int_0^t e^{-s\mathbf{G}} e^{-s\mathbf{C}} ds \boldsymbol{\Sigma} e^{t\mathbf{G}^T} + Transposed\,.$$

Let γ_1, γ_2 be the eigenvalues of $\mathbf{G}$, then $e^{t\mathbf{G}} = \mathbf{E}\boldsymbol{\Gamma}(t)\mathbf{E}^{-1}$ and $e^{t\mathbf{C}} = \mathbf{F}\boldsymbol{\Lambda}(t)\mathbf{F}^{-1}$, where the columns of $\mathbf{E}$ and $\mathbf{F}$ are the eigenvectors of $\mathbf{G}$ and $\mathbf{C}$ respectively, and

$$\boldsymbol{\Gamma} = \begin{pmatrix} e^{\gamma_1 t} & 0 \\ 0 & e^{\gamma_2 t} \end{pmatrix}, \quad \boldsymbol{\Lambda} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

Thus

$$\frac{d}{dt}\mathbf{D}(t) = \mathbf{E}\mathbf{S}(t)\mathbf{E}^T + Transposed\,,$$

where

$$\mathbf{S}(t) = \boldsymbol{\Gamma}(t) \int_0^t \boldsymbol{\Gamma}(-s)\mathbf{H}\Lambda(-s) ds \tilde{\mathbf{H}}\boldsymbol{\Gamma}(t)$$

and $\mathbf{H} = \mathbf{E^{-1}F}$, $\tilde{\mathbf{H}} = \mathbf{F^{-1}}\boldsymbol{\Sigma}(\mathbf{E^{-1}})^T$. After integrating in s the entries of $\mathbf{S}$ become

$$s_{11} = \frac{h_{11}\tilde{h}_{11}}{\gamma_1 + \lambda_1}\left(e^{2\gamma_1 t} - e^{(\gamma_1 - \lambda_1)t}\right) + \frac{h_{12}\tilde{h}_{21}}{\gamma_1 + \lambda_2}\left(e^{2\gamma_1 t} - e^{(\gamma_1 - \lambda_2)t}\right),$$
$$s_{12} = \frac{h_{11}\tilde{h}_{12}}{\gamma_1 + \lambda_1}\left(e^{(\gamma_1+\gamma_2) t} - e^{(\gamma_2 - \lambda_1)t}\right) + \frac{h_{12}\tilde{h}_{22}}{\gamma_1 + \lambda_2}\left(e^{(\gamma_1+\gamma_2) t} - e^{(\gamma_2 - \lambda_2)t}\right),$$
$$s_{21} = \frac{h_{21}\tilde{h}_{11}}{\gamma_2 + \lambda_1}\left(e^{(\gamma_1+\gamma_2) t} - e^{(\gamma_1 - \lambda_1)t}\right) + \frac{h_{22}\tilde{h}_{21}}{\gamma_2 + \lambda_2}\left(e^{(\gamma_1+\gamma_2) t} - e^{(\gamma_1 - \lambda_2)t}\right),$$
$$s_{22} = \frac{h_{21}\tilde{h}_{12}}{\gamma_2 + \lambda_1}\left(e^{2\gamma_2 t} - e^{(\gamma_2 - \lambda_1)t}\right) + \frac{h_{22}\tilde{h}_{22}}{\gamma_1 + \lambda_2}\left(e^{2\gamma_2 t} - e^{(\gamma_2 - \lambda_2)t}\right).$$

From the above expressions it can be seen that the inertial regime exists if and only if, both γ_1 and γ_2 are imaginary $\gamma_1 = ri$, $\gamma_2 = -ri$ and real parts of λ_1, λ_2 are not negative, i.e. under conditions (24), (25).

Using again (29), (30) one can find

$$\mathbf{P} \equiv \int_0^\infty e^{-s\mathbf{G}} e^{-s\mathbf{C}} ds = (\mathbf{F}\boldsymbol{\Lambda_1} - \mathbf{GF}\boldsymbol{\Lambda_2})\mathbf{F^{-1}},$$

where

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \frac{\lambda_1}{\lambda_1^2+r^2} & 0 \\ 0 & \frac{\lambda_1}{\lambda_1^2+r^2} \end{pmatrix}, \quad \mathbf{\Lambda}_2 = \begin{pmatrix} \frac{1}{\lambda_1^2+r^2} & 0 \\ 0 & \frac{1}{\lambda_2^2+r^2} \end{pmatrix},$$

and hence (4) holds true with

$$\mathbf{K} = \mathbf{M} + \frac{1}{r^2}\mathbf{G}\mathbf{M}\mathbf{G}^{\mathbf{T}}, \tag{31}$$

where

$$\mathbf{M} = \mathbf{P}\mathbf{\Sigma} + \mathbf{\Sigma}\mathbf{P}^{\mathbf{T}}.$$

From (31) it follows that $\mathbf{K}$ satisfies

$$\mathbf{G}\mathbf{K}\mathbf{G}^{\mathbf{T}} = \mathbf{K}r^2.$$

Any solution of this equation is represented in form (28) where c is a scalar to be defined say from (31). □

Let us make some remarks on Proposition 2.

First, the absolute dispersion ellipse is identical to the mean circulation one exactly like in the case of the Brownian flow. That can be seen from (28).

Then, in the case $\mathbf{\Sigma} = \sigma^2\mathbf{I}$ and $\mathbf{A}$ given by (22) one gets after some algebra

$$c = \frac{4\sigma^2\theta(h(\theta^2 - r^2 + q^2) - 4\omega r^2)}{rQ},$$

where $h = g_{12} - g_{21}$. Let us demonstrate that $c > 0$ under condition (26). Indeed, the minimum value of c with respect to ω up to a positive factor is given by

$$c_{\min} = 4\theta^2h^2 - (h^2 - 4r^2)^2.$$

It is easy to check that (26) implies $c_{\min} > 0$ under conditions (24), (25).

Finally, from the proof one can figure out that in fact conditions (24)-(26) are also necessary for existence of the inertial regime.

The statement of Proposition 2 is not applicable to the shear flow given by (8) since the ellipticity condition (24) fails. In this case the absolute dispersion grows as t^2 exactly like in the case of the Brownian flow and the quantity of interest is the displacement $\mathbf{r'}$ with respect to the mean flow defined by $d\mathbf{r'} = \mathbf{v}dt$. Simple calculations leads to the following expression for the reduced diffusivity

$$\mathbf{K'} = \lim_{t\to\infty} \frac{1}{t}E\{\mathbf{r'}(t)\mathbf{r'}(t)^T\} = \frac{1}{\theta^2 + \omega^2 - \alpha\omega}\begin{pmatrix} \theta & \alpha \\ \alpha & \theta \end{pmatrix}.$$

As one can see in this case the dispersion anisotropy ellipse does not coincide with stream lines of the mean flow.

On the other side in the hyperbolic case, $\det(\mathbf{G}) < 0$, the dispersion grows exponentially

$$\mathbf{D}(t) \sim e^{2rt}\mathbf{D}, \tag{32}$$

where

$$r = \sqrt{-det(\mathbf{G})}, \quad \mathbf{D} = \frac{1}{4}((\mathbf{I}+\mathbf{G}/r)\int_{-\infty}^{\infty} \frac{Re\,(\mathbf{E}(k))\,dk}{k^2+r^2}(\mathbf{I}+\mathbf{G^T}/r).$$

Indeed

$$\mathbf{D}(t) = \int_{-\infty}^{\infty} \mathbf{Q}(t,k)\mathbf{E}(k)\mathbf{Q}(t,k)^H dk,$$

where

$$\mathbf{Q}(t,k) = \int_0^t e^{s\mathbf{G}+iks} ds = \mathbf{Q}_1(t,k) + i\mathbf{Q}_2(t,k)$$

and

$$\begin{aligned} \mathbf{Q}_1 &\sim \frac{e^{rt}(r\cos kt + k\sin kt)}{2(k^2+r^2)}(\mathbf{I}+\mathbf{G}/r), \\ \mathbf{Q}_2 &\sim \frac{e^{rt}(r\sin kt - k\cos kt)}{2(k^2+r^2)}(\mathbf{I}+\mathbf{G}/r). \end{aligned} \tag{33}$$

Let $\mathbf{E}(k) = \mathbf{E}_1(k) + i\mathbf{E}_2(k)$, then

$$\mathbf{D}(t) = \int_{-\infty}^{\infty} (\mathbf{Q_1E_1Q_1^T} + \mathbf{Q_2E_1Q_2^T} + \mathbf{Q_1E_2Q_2^T} - \mathbf{Q_2E_2Q_1^T})dk. \tag{34}$$

The latter with (33) implies (32). Notice that the approach based on representation (34) leads to the same results (28) in the elliptic case.

Dependence of c on the dimensionless parameters γ/Ω, $\omega\tau$ is complex enough and illustrated in Figure 1. Typical dispersion curves in Figure 2 clearly demonstrate the ballistic and inertial regimes with possible oscillations which are essential only in the elliptic case with $\gamma \gg r$.

5. Relative Dispersion in FOM

Here we study the relative dispersion (3) for model (19) mostly focusing on the Lyapunov second moment Λ and the relative dispersion anisotropy. Unlike in the case of Brownian flow most of results of this section are obtained by numerical means based on an obtained closed system for the second moments of position/velocity for the linearized separation equations (21). We concentrate on the isotropic forcing, i.e. entries $b_{ij}(\mathbf{r})$ of $\mathbf{B}(\mathbf{r})$ are given by (10). Notice that this assumption does not imply any isotropy of the velocity fluctuations

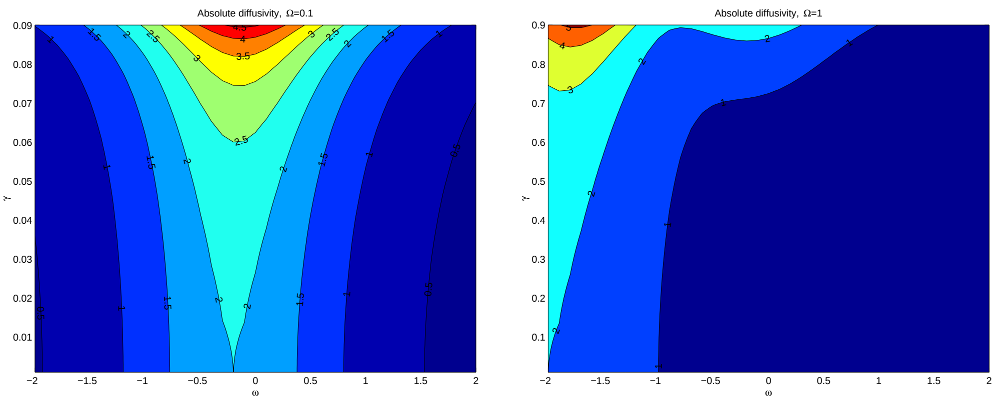

Figure 1: Diffusivity vs γ and ω for different Ω

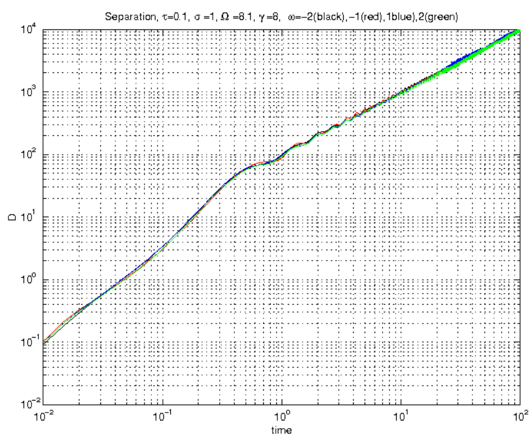

Figure 2: Dispersion vs time for $\omega = -2, -1, 1, 2$ and $\gamma = 9, \Omega = 10, \tau = 0.05, \sigma = 100$. All four curves are almost indistinguishable

themselves. Here we assume more general behavior of the covariance function at zero

$$b_L(r) = 1 - \frac{1}{2}\beta_L r^h + o(r^h), \quad b_N(r) = 1 - \frac{1}{2}\beta_N r^h + o(r^h)\,,$$

where $0 < h \leq 2$ and $\beta_N, \beta_L > 0$ to treat both non-local ($h < 2$) and local ($h = 2$) dynamics. The second Lyapunov moment is well defined only in the last case and by this the following below study of the second Lyapunov moment is carried out only for the smooth case ($h = 2$) while examples of the relative dispersion curve

$$\rho(t) = E\{|\mathbf{r}_1(t) - \mathbf{r}_2(t)|^2$$

are given for the non-smooth case as well.

Under the local dynamics assumption let us linearize the separation equations (21) assuming as before $\mathbf{\Sigma} = \sigma^2\mathbf{I}$

$$d\mathbf{z} = (\mathbf{Gz} + \mathbf{q})dt, \quad d\mathbf{q} = -\mathbf{Cq} + \mathbf{\Phi}(\mathbf{z})d\mathbf{w}\,, \tag{35}$$

where

$$\mathbf{\Phi}(\mathbf{z}) = \begin{pmatrix} \beta_L^{1/2}x & \beta_N^{1/2}y \\ \beta_L^{1/2}y & -\beta_N^{1/2}x \end{pmatrix}, \quad \mathbf{z} = (x, y)\,.$$

Set $\mathbf{s} = (\mathbf{z}, \mathbf{q})$, then applying Ito formula for system (35) we obtain for $\mathbf{R} = E\{\mathbf{ss^T}\}$

$$\frac{d\mathbf{R}}{dt} = \mathbf{PR} + \mathbf{RP^T} + \mathbf{QRQ^T}\,, \tag{36}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{G} & \mathbf{I} \\ -\mathbf{C} & \mathbf{0} \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{\Psi} & \mathbf{0} \end{pmatrix}, \quad \mathbf{\Psi} = \begin{pmatrix} \beta_L^{1/2} & \beta_N^{1/2} \\ \beta_L^{1/2} & -\beta_N^{1/2} \end{pmatrix}.$$

Hence, Λ is the maximum eigenvalue of the system (36) containing 16 equations. Using the symmetry of covariance matrices this system reduces to the system of 10 equations. Namely, set $\mathbf{q} = (u, v)$ and introduce the following column vector-functions of time

$$\mathbf{f} = (E\{x^2\}, E\{xy\}, E\{y^2\}), \ \mathbf{g} = (E\{xu\}, E\{xv\}, E\{yu\}, E\{yv)\},$$
$$\mathbf{h} = (E\{u^2\}, E\{uv\}, E\{v^2\}).$$

Then (36) can be rewritten as

$$\frac{d\mathbf{f}}{dt} = \mathbf{A}_{11}\mathbf{f} + \mathbf{A}_{12}\mathbf{g}, \ \frac{d\mathbf{g}}{dt} = \mathbf{A}_{22}\mathbf{g} + \mathbf{A}_{23}\mathbf{h}, \frac{d\mathbf{h}}{dt} = \mathbf{A}_{31}\mathbf{f} + \mathbf{A}_{33}\mathbf{h}\,, \tag{37}$$

where

$$\mathbf{A}_{11} = \begin{pmatrix} 2g_{11} & 2g_{12} & 0 \\ g_{21} & g_{11} + g_{22} & g_{12} \\ 0 & 2g_{21} & 2g_{22} \end{pmatrix}, \quad \mathbf{A}_{12} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\mathbf{A}_{22} = \begin{pmatrix} g_{11} - c_{11} & -c_{12} & g_{12} & 0 \\ -c_{21} & g_{11} - c_{22} & 0 & g_{12} \\ g_{21} & 0 & g_{22} - c_{11} & -c_{12} \\ 0 & g_{21} & -c_{21} & g_{22} - c_{22} \end{pmatrix}, \quad \mathbf{A}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

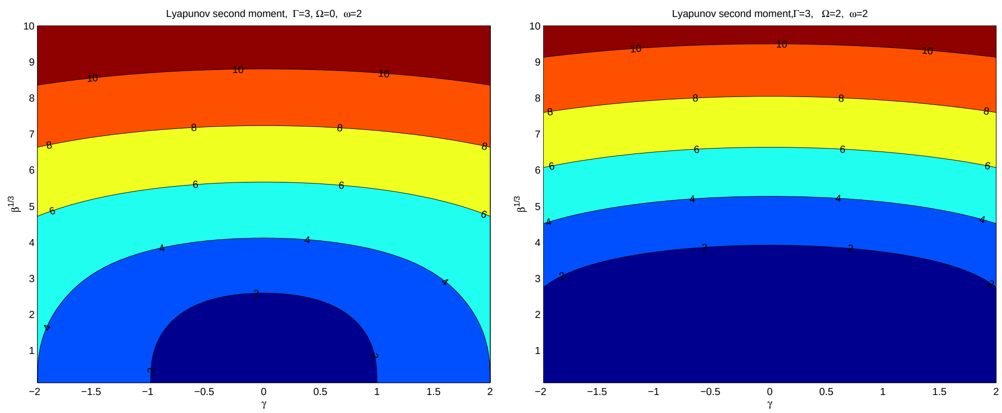

Figure 3: Lyapunov second moment vs γ and $\beta^{1/3}$ for $\Omega = 0$ (left) and $\Omega = 2$ (right) in the case of solenoidal forcing ($\Gamma = 3$) for gyre (15)

$$\mathbf{A}_{31} = \begin{pmatrix} \beta_L & 0 & \beta_N \\ 0 & \beta_L - \beta_N & 0 \\ \beta_N & 0 & \beta_L \end{pmatrix}, \quad \mathbf{A}_{33} = -\begin{pmatrix} 2c_{11} & 2c_{12} & 0 \\ c_{21} & c_{11} + c_{22} & c_{12} \\ 0 & 2c_{21} & 2c_{22} \end{pmatrix}.$$

To reduce the number of parameters further we focus on three particular cases, the gyre defined in (15), shear flow defined by (8) and the compressible mean flow given by

$$\mathbf{G} = \begin{pmatrix} \gamma & \Omega \\ -\Omega & \gamma \end{pmatrix}. \tag{38}$$

For the shear flow and solenoidal drift we did not find any reduction of system (37) (except the gyre with $\gamma = 0$) and give the results of numerical investigations. All the simulations are done under fixed values $\theta = 1,\ \sigma^2 = 1000$. Figure 3 shows the dependence of Λ on γ and $\beta^{1/3}$ for two values of $\Omega = 0, 2$ in the cases of the solenoidal forcing ($\Gamma = 3$), where $\Gamma = \beta_L/\beta_N$. The exponent 1/3 in β is taken to make the dimension of the parameter (T^{-1}) the same as of γ.

As one can see Λ grows in β and $|\gamma|$ for both instances. In other experiments not shown here the same behavior was observed. The greater Ω the less dependence of Λ on γ is manifested and the dependence on $\beta^{1/3}$ is almost linear. Finally, other experiments have shown that even though the ratio Γ influences the general picture it causes no dramatic effects.

Figure 4 illustrates the dependence of Λ on γ and Ω for two values of ω for the same model (18).

Again, no essential changes in this dependence were observed when Γ varied.

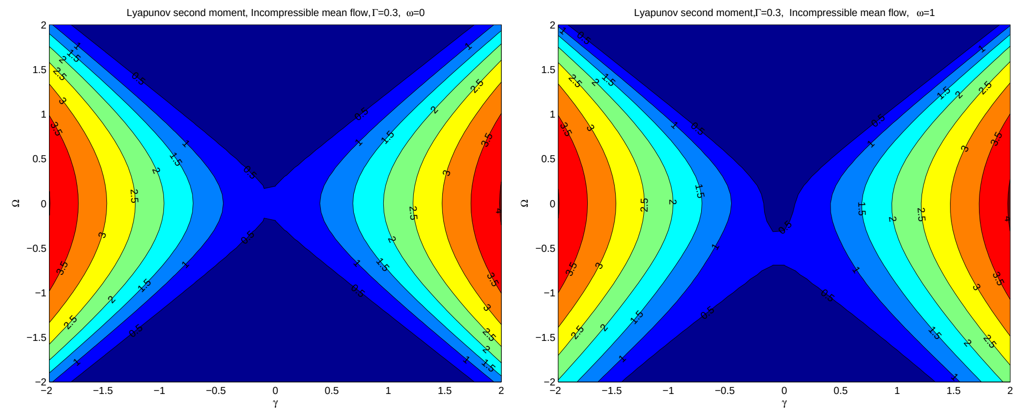

Figure 4: Lyapunov second moment vs γ and Ω for $\omega = 0$ (left) and $\omega = 1$ (right) in the case of potential forcing ($\Gamma = 1/3$) for model (18)

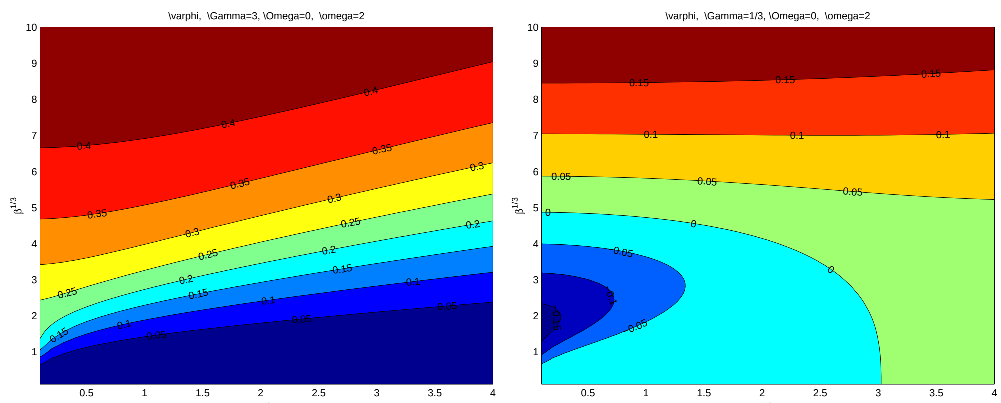

Figure 5: Anisotropy angle vs γ and $\beta^{1/3}$ for $\Gamma = 3$ (left) and $\Gamma = 1/3$ (right)

Anisotropy effects are illustrated in Figure 5, where the dependence of the anisotropy angle φ on γ and $\beta^{1/3}$ is shown for two values of Γ. As one can see, this parameter is crucial in the mixing anisotropy exactly like in the case of Brownian flow.

To finish with the gyre, we give examples of the relative dispersion curves vs time in Figure 5 for different values of the parameters γ, Ω, and Hurst exponent $h = 0.1, 1, 1.9$ controlling the forcing energy slope on large wave numbers. We fixed $\omega = 2$, the correlation radius $R = 100$ and initial separation $r_0 = 1$.

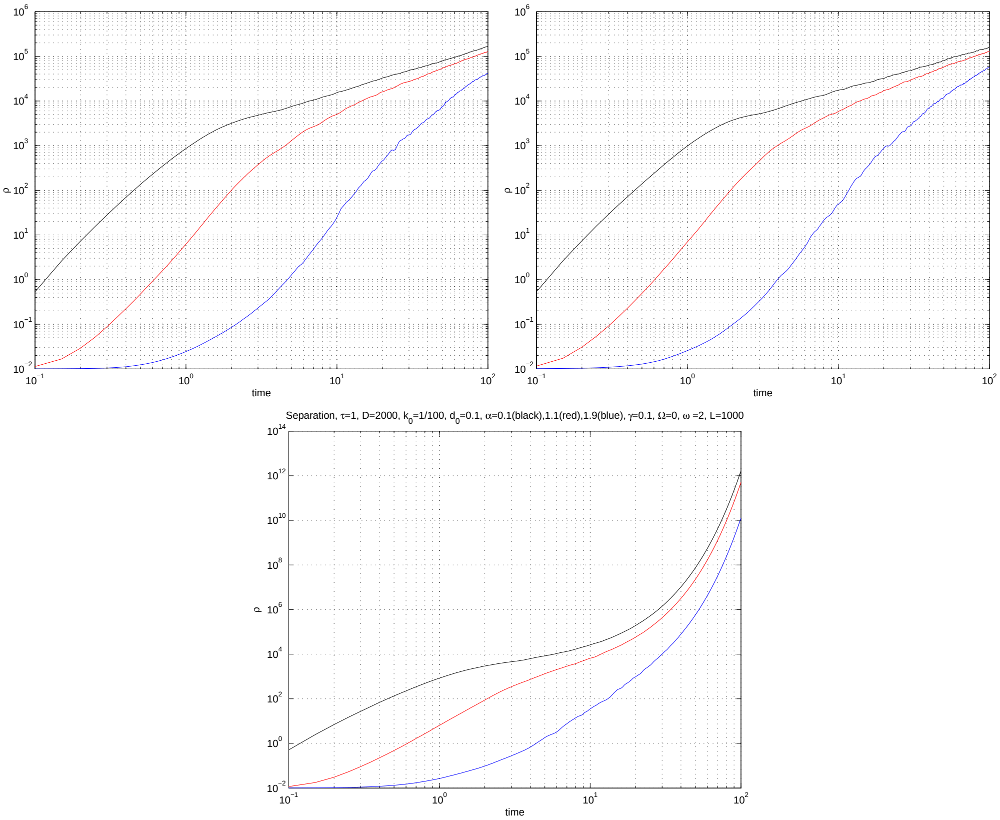

Figure 6: Examples of relative dispersion vs time for gyre (15) with different parameters $\gamma = 0, \Omega = 0$ (left), $\gamma = 0.1, \Omega = 0$ (center), $\gamma = 0.1, \Omega = 2$ (right), and different values of Hurst exponent $h = 0.1$ (upper curve, black), $h = 1$ (middle curve, red), $h = 2$ (lower curve, blue)

As one can see, the ballistic regime is similar for all the instances while the inertial regime is observed only for the zero mean flow (left) and in the elliptic case (right). In the hyperbolic case (center) an exponential explosion is well seen. The intermediate exponential regime (with the corresponding Λ) is observed only for $h = 2$, i.e. in the case of local dynamics. Indeed the difference between the zero mean flow (left) and mean flow with $\gamma = 0.1, \Omega = 2$ (right) is very subtle. However, one can find that the exponential stage for the blue curve is slightly longer in the latter.

For the shear flow (8) no reduction of full system (37) was found either.

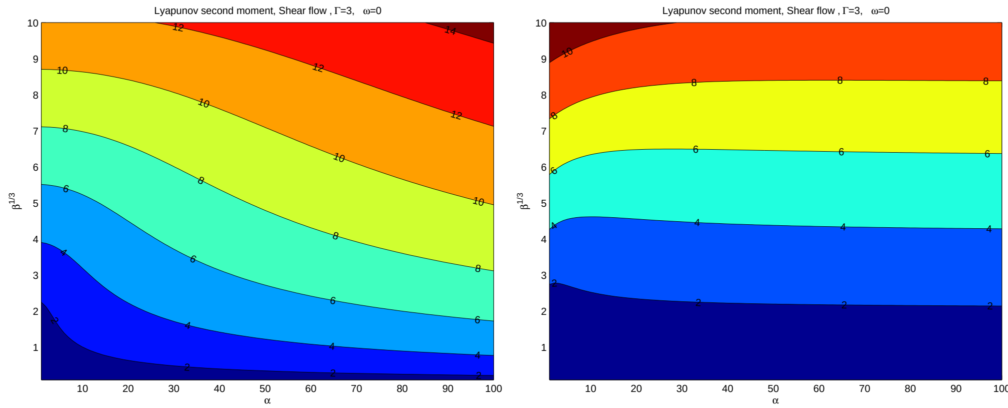

Figure 7: Lyapunov second moment vs α and $\beta^{1/3}$ for $\omega = 0$ (left) and $\omega = 2$ for the shear flow, model (8)

Nevertheless, asymptotics of Λ can be derived as $\alpha \to \infty$. For this we used *Maple* to find the characteristic polynomial of system (37) which is of 10-th degree in λ, third degree in α if $\omega \neq 0$ and second degree in α if $\omega = 0$. From an explicit expression for this polynomial it can be found that in the case $\omega \neq 0$, Λ goes to the greatest root of

$$\lambda^4\omega^2 - 2\lambda(\beta_L\omega^2 - \theta^2\beta_N) + 4\theta^3\beta_N = 0$$

while if $\omega = 0$ the limiting second Lyapunov moment is the greatest root of

$$5\lambda^3 + 15\theta\lambda^2 + 14\theta^2\lambda - 2(\beta_L - \beta_N + 2\theta^3) = 0\,.$$

The difference between two cases is illustrated in Figure 7. Notice that the case $\alpha = 0$ was considered in detail in [4].

For the compressible model (38) the system (37) reduces to a system for only four variables

$$\rho_1 = E\{x^2 + y^2\}, \quad \rho_{12} = E(\{xu + yv\}, \quad \rho_{21} = E\{xv - yu\}, \quad \rho_2 = E\{u^2 + v^2\}.$$

Namely

$$\begin{aligned} \frac{d\rho_1}{dt} &= 2\gamma\rho_1 + 2\rho_{12}\,, \\ \frac{d\rho_{12}}{dt} &= -\theta\rho_{12} + c\rho_{21} + \rho_2\,, \\ \frac{d\rho_{21}}{dt} &= -c\rho_{12} - \theta\rho_{21}\,, \\ \frac{d\rho_2}{dt} &= \beta\rho_1 - 2\theta\rho_2\,, \end{aligned} \tag{39}$$

where $c = \omega - 2\Omega$. Introducing $\mu = \lambda + \theta$, where λ is an eigenvalue of (39) we

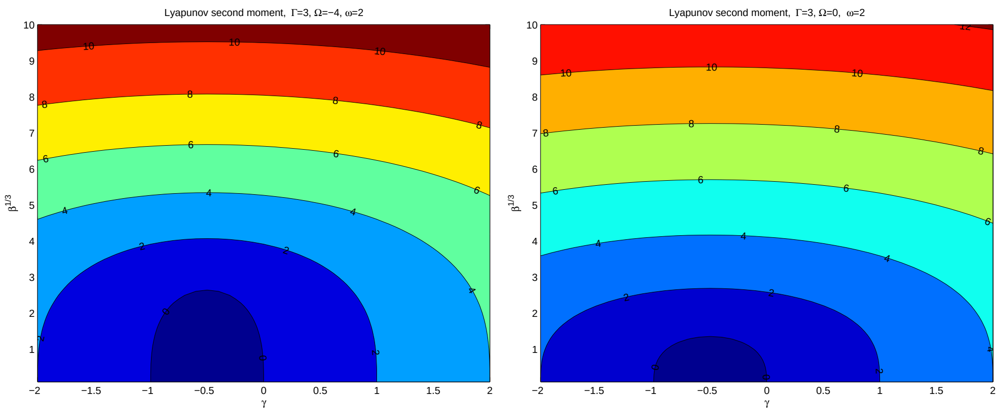

Figure 8: Lyapunov second moment vs γ and $\beta^{1/3}$ for $\Omega = 0$ (left) and $\Omega = 2$ (right) for the compressible mean flow, model (38)

get the following equation

$$(\mu - 2\gamma - \theta)(\mu + \theta)(\mu^2 + c^2) - 2\beta\mu = 0. \tag{40}$$

Equation (40) can be solved exactly, but we do not give the explicit formula since it is too lengthy and restrict ourselves by the following asymptotics

$$\Lambda \cong \sqrt[3]{2\beta}, \quad \text{as} \quad \beta \to \infty$$

and limit

$$\lim_{\beta \to 0} \Lambda = \max\{-2\theta, 2\gamma\}\,.$$

A careful look at (40) leads to the conclusion that $\Lambda > 0$ if and only if $\beta > -2\gamma(c^2 + \theta^2)$. Thus, under negative γ, Λ can assume negative values as well which is illustrated in Figure 7. This figure shows also that Λ grows with both γ and β as before, however there is no symmetry in γ at all by the obvious reason. Notice also that Λ is not sensitive to Γ in this case at all. Dependence on γ and Ω is shown in Figure 9 in order to compare with the gyre

Acknowledgments

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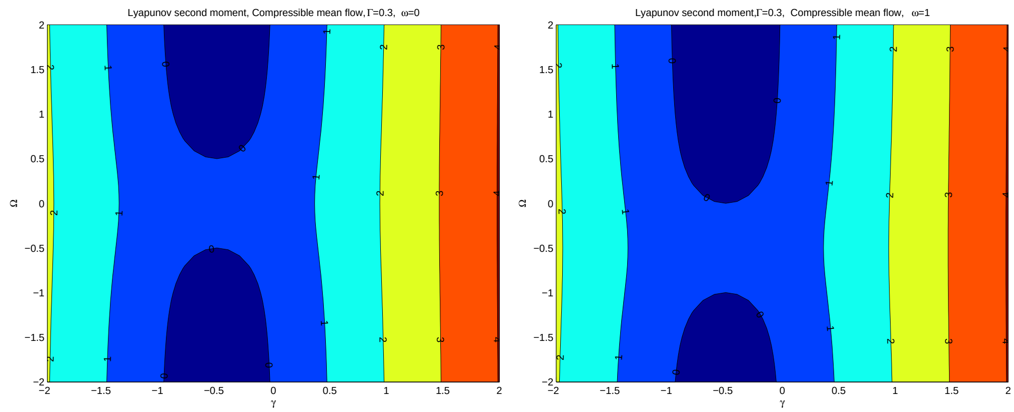

Figure 9: Lyapunov second moment vs γ and Ω for $\omega = 0$ (left) and $\omega = 1$ for the compressible mean flow, model (38)

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