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WEB LIKE SOLUTIONS OF RICCATI TYPE GENERALIZED DIFFERENCE EQUATIONS

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Abstract: In this paper, the authors define ℓ-web solutions of generalized difference equations and discuss the behaviors of ℓ-web solutions of the generalized difference equation

$$\Delta_\ell(p(k)\Delta_\ell u(k)) + r(k)u(k+\ell) = 0, \quad k \in \mathbb{N}\,, \tag{1}$$

where p and r are real valued functions defined on $\mathbb{N}$, ℓ is positive integer, $p(k) > 0$ for all $k \in \mathbb{N}(a)$ and $\Delta_\ell u(k) = u(k+\ell) - u(k)$.

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1. Introduction

In number theory, some applications, like sum of n-th powers of arithmetic progression, the sum of the products of n consecutive terms of arithmetic progression and sum of arithmetico-geometric progression are developed in [2] using the generalized difference operator Δ_ℓ define as $\Delta_\ell u(k) = u(k+\ell) - u(k)$. Generalized Bernoulli's polynomials $B_n(k, \pm\ell)$ are established in [6, 7] using $\Delta_{\pm\ell}$. Qualitative behaviors like rotatory, spiral, boundedness, recessive and dominant properties of the generalized difference equation

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$$p(k)u(k+\ell)+p(k-\ell)u(k-\ell)=q(k)u(k), \quad k\in\mathbb{N}(\ell) \tag{2}$$

for integers ℓ and $-\ell$ are discussed in [3, 4, 5, 8]. Since equation (1) is equivalent to (2) with

$$q(k)=p(k)+p(k-\ell)-r(k-\ell) \tag{3}$$

we analyse the behaviors of equation (1). If $\frac{u(k+\ell)}{u(k)}>0$ for all $k\in\mathbb{N}(a)$ and $u(k)$ is solution of equation (1), then the Riccati type transformation. We let $v(k)=p(k)\frac{\Delta_\ell u(k)}{u(k)}$. Since $v(k)+p(k)=p(k)\frac{u(k+\ell)}{u(k)}>0$, this leads to Riccati type difference equation

$$\Delta_\ell v(k)+\frac{v(k)v(k+\ell)+r(k)v(k)}{p(k)}+r(k)=0, \quad \text{for all k} \in \mathbb{N}(\text{a})\,, \tag{4}$$

which is the same as

$$\Delta_\ell v(k)+\frac{v^2(k)}{v(k)+p(k)}+r(k)=0, \quad k\in\mathbb{N}(a). \tag{5}$$

Results developed in this paper coincides with the corresponding results in [1] when $\ell=1$. Throughout this paper we assume that ℓ is a positive integer, n is the largest non-negative integer for which the corresponding $u(k-n\ell)$ is defined and $k-n\ell\geq 0$, $\mathbb{N}=\{0,1,2,3,\dots\}$, $\mathbb{N}(a)=\{a,a+1,a+2,\cdots\}$, $\mathbb{N}_\ell(a)=\{a,a+\ell,a+2\ell,\cdots\}$, $c_1,c_2,\cdots$ are constants, $\mathbb{Z}$ is the set of all integers.

2. Preliminaries

In this section, we present some definitions and preliminary results which will be used in the subsequent discussions.

Definition 2.1. (see [5]) A sequence of complex numbers $\{w(k)\}$ is called spiral about the line $RP(z)=0$ (about the line $I(z)=0$) by ℓ steps if $w(k)w(k+\ell)<0$ $(w(k)w(k+\ell)>0)$ for all $k\in\mathbb{N}(k_1)$ for some large integer k_1. If all the nontrivial solutions of the generalized difference equation $f(k,u(k),u(k+\ell),\cdots,u(k+n\ell))=0$ are spiral, then the generalized difference equation is called spiral equation.

Definition 2.2. (see [4]) Let $\ell\in\mathbb{N}(1), a\in\mathbb{N}$, b be a complex number and $\{u(k)\}_{k\in\mathbb{N}}$ be a given sequence of complex numbers such that $u(k)\neq b$ for all

$k \in \mathbb{N}$. Define

$$r_\ell(k) = \frac{u(k) - b}{u(k+\ell) - b}.$$

If for any $k_1 \in \mathbb{N}(a)$, there exists $k_2 \in N(k_1)$ such that $r_\ell(k_2) < 0$, then $\{u(k)\}$ is called ℓ-rotatory about b. In $r_\ell(k)$, let $b = 0$.$\{u(k)\}$ is uniform ℓ-rotatory if $r_\ell(k) < 0$ for all $k \in N(a)$, regular ℓ-rotatory if $r_\ell(k)$ is a negative constant for all $k \in N(a)$. In the above cases length of the rotations is 2 ℓ. A real 2-rotatory is oscillatory. $\{u(k)\}$ is ℓ-source, ℓ-sink, ℓ-stable, ℓ-web, ℓ-ray if $0 < r_\ell(k) < 1, r_\ell(k) > 1, r_\ell(k) = 1$, $r_\ell(k) > 0, r_\ell(k) =$ real for all $k \in \mathbb{N}(a)$ respectively. $\{u(k)\}$ is expanding (shrinking) if it is ℓ-source (ℓ-sink) for some ℓ .$\{u(k)\}$ is monotone if it is either expanding or shrinking. Equation (1) is ℓ-web if all its solutions are ℓ-web.

Example 2.3. Let $u(k) = c(k)^{(-1)^k} e^{\frac{i\Pi k}{5}}$, where c is any complex number. Then $\{u(k)\}$ is uniform 5-rotatory and 10-web but neither 10-source nor 10-sink. If $v(k) = ck^m e^{\frac{i\Pi k}{5}}$, then $\{v(k)\}$ is 10-source and uniform 5-rotatory, when m is positive integer and $\{v(k)\}$ is 10-sink and uniform 5-rotatory, when m is negative integer and regular 5-rotatory, 10-stable when $m = 0$.

Lemma 2.4. *If $u(k)$ and $v(k)$ are two functions then*

$$\Delta_\ell[u(k)v(k)] = u(k+\ell)\Delta_\ell v(k) + v(k)\Delta_\ell u(k).$$

3. Main Results

In this section, we present some significant results on the behaviors of solutions of equation (1).

Lemma 3.1. *The difference equation (1) has an ℓ-web solution if and only if there exists a function $v(k)$ salifying $v(k) > 0$ and $\Delta_\ell(p(k)\Delta_\ell v(k)) + r(k)v(k+\ell) \leq 0$ for all sufficiently large $k \in \mathbb{N}$.*

Proof. Since the necessity part is obvious, we need to prove only the sufficient part. Since $v(k) > 0$, if $\Delta_\ell(p(k)\Delta_\ell v(k)) + r(k)v(k+\ell) = 0$, then $v(k)$ is ℓ-web solution of (1). Suppose that $v(k) > 0$ and

$$\Delta_\ell(p(k)\Delta_\ell v(k)) + r(k)v(k+\ell) < 0$$

for all large k $\in \mathbb{N}$, then by (3) $p(k)v(k+\ell) + p(k-\ell)v(k-\ell) < q(k)v(k)$ for all $k \in \mathbb{N}(a)$ for some $a \in \mathbb{N}$ and hence as $p(k) > 0$,

$$0 < p(k)\frac{v(k+\ell)}{v(k)} + p(k-\ell)\frac{v(k-\ell)}{v(k)} < q(k).$$

Choose $w(k) \geq \frac{v(k+1)}{v(k)} > 0$ such that

$$p(k)w(k) + \frac{(k-\ell)}{w(k-\ell)} = q(k) \text{ for all k} \in \mathbb{N}(\mathrm{a}+\ell).$$

Hence, $w(k)$ can be obtained by taking

$$w(a+j) = \frac{v(a+j+\ell)}{v(a+j)} \text{ for j} = 0,1,2,\cdots,\ell-1$$

and

$$w(a+k\ell+j) = \frac{1}{p(a+k\ell+j)}\Big[q(a+k\ell+j) - \frac{p(a+(k-1)\ell+j)}{w(a+(k-1)\ell+j)}\Big]$$

for $k = 1,2,\cdots$. Now the proof follows by taking

$$u(k) = w(k)v(k-\ell),\ u(a+j) = v(a+j) \text{ for j} = 0,1,2,\cdots,\ell-1. \qquad \square$$

Lemma 3.2. *The difference equation (1) has an ℓ-web solution if and only if there exists a function $w(k)$ defined on $\mathbb{N}(0)$ with $w(k) > -p(k), k \in \mathbb{N}(a\ell)$ for some $a \in \mathbb{N}(0)$, satisfying*

$$\Delta_\ell w(k) + \frac{w(k)w(k+\ell) + r(k)w(k)}{p(k)} + r(k) \leq 0,\ k \in N(a) \tag{6}$$

or equivalently

$$\Delta_\ell w(k) + \frac{w^2(k)}{w(k)+p(k)} + r(k) \leq 0. \tag{7}$$

Proof. Necessity part. Take $w(k) = \frac{p(k)\Delta_\ell u(k)}{u(k)}$, where $u(k)$ is ℓ-web solution of (1). Since $\Delta_\ell[p(k)\Delta_\ell u(k)] = \Delta_\ell[u(k)w(k)]$, from lemma 2.4 and (1), we obtain

$$\Delta_\ell w(k) + w(k)\frac{\Delta_\ell u(k)}{u(k+\ell)} + r(k) = 0.$$

Again from (1), we obtain

$$p(k+\ell)\Delta_\ell u(k+\ell) - p(k)\Delta_\ell u(k) = -r(k)u(k+\ell)$$

and hence

$$\frac{w(k+\ell)+r(k)}{p(k)} = \frac{\Delta_\ell u(k)}{u(k+\ell)}$$

which yields (6). Since

$$w(k) + p(k) = \frac{p(k)u(k+\ell)}{u(k)} > 0,$$

(7) follows from

$$\frac{w^2(k)}{w(k)+p(k)} = \frac{p(k)[\Delta_\ell u(k)]^2}{u(k)u(k+\ell)} \text{ and (6).}$$

Sufficiency part. For this, let $z(a\ell+j)=1,\ j=0,1,2,\cdots,\ell-1,$

$$z(s\ell+j) = \prod_{t=a}^{s-1}\Big[\frac{w(t\ell+j)}{p(t\ell+j)}\Big],\ s\in N(a\ell+\ell),$$

then since $w(t\ell+j)+p(t\ell+j)>0$, $z(k)>0$ for all $k\in\mathbb{N}(a\ell)$. Since

$$z(s\ell+\ell+j) = z(s\ell+j)\Big(1+\frac{w(s\ell+j)}{p(s\ell+j)}\Big),$$

we obtain $w(k)=p(k)\frac{\Delta_\ell z(k)}{z(k)}$ which yields from (6),

$$\Delta_\ell\big(\frac{p(k)\Delta_\ell z(k)}{z(k)}\big) + \frac{\Delta_\ell z(k)}{z(k)}\frac{p(k+\ell)\Delta_\ell z(k+\ell)}{z(k+\ell)} + r(k)\frac{\Delta_\ell z(k)}{z(k)} + r(k) \le 0. \quad (8)$$

From Lemma 2.4, we get

$$\Delta_\ell\Big(p(k)\frac{\Delta_\ell z(k)}{z(k)}\Big) = \frac{\Delta_\ell z(k+\ell)}{z(k)}\Delta_\ell p(k) + p(k)\Delta_\ell\Big(\frac{\Delta_\ell z(k)}{z(k)}\Big)$$

and hence (8) becomes

$$p(k+\ell)\frac{\Delta_\ell z(k+\ell)}{z(k+\ell)}\big[1+\frac{\Delta_\ell z(k)}{z(k)}\big] + \frac{\Delta_\ell z(k)}{z(k)}\big(r(k)-p(k)\big) + r(k) \le 0,$$

which yields

$$p(k+\ell)\Delta_\ell z(k+\ell) + r(k)z(k+\ell) - p(k)\Delta_\ell z(k) \le 0. \quad (9)$$

Now the proof follows form (9) and Lemma 3.1. □

Theorem 3.3. *Assume that*

$$\limsup_{k\to\infty}\Big[k^{\frac{-3}{2}}\sum_{t=0}^{k} p(t)\Big] < \infty \quad (10)$$

and the difference equation (1) is ℓ-web. Then the following are equivalent:

(i) $\lim_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k}\sum_{s=0}^{t} r(s)$ exists.

(ii) $\liminf_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k}\sum_{s=0}^{t} r(s) > -\infty$.

(iii) For any ℓ-web solution $\{u(k)\}$ of equation (1) with $\frac{u(k)}{u(k+\ell)}>0, k\in\mathbb{N}(a)$,

the function $v(k) = \frac{p(k)\Delta_\ell u(k)}{u(k)}$, $k \in \mathbb{N}(a)$ *satisfies*

$$\sum_{j=0}^{\ell-1}\sum_{t=a}^{\infty} \frac{v^2(t\ell+j)}{v(t\ell+j)+p(t\ell+j)} < \infty. \tag{11}$$

Proof. Clearly (i) implies (ii). To show that (ii) implies (iii). Suppose to the contrary that there is a solution $u(k)$ of (1) such that $u(k)$ is ℓ-web and $v(k) = \frac{p(k)\Delta_\ell u(k)}{u(k)} > -p(k)$ for all $k \in \mathbb{N}(a\ell)$ and

$$\sum_{t=a}^{\infty} \frac{v^2(t\ell+j)}{v(t\ell+j)+p(t\ell+j)} = \infty, \text{ for some j} \in \{0,1,2,\cdots \ell-1\}. \tag{12}$$

From (4), we have

$$v(k\ell+\ell+j) + \sum_{t=a}^{k} \frac{v^2(t\ell+j)}{v(t\ell+j)+p(t\ell+j)} + \sum_{t=a}^{k} r(t\ell+j) = v(a\ell+j) \tag{13}$$

and therefore for all $k \in \mathbb{N}(a\ell)$

$$\frac{1}{k}\sum_{t=a}^{k}\big[-v(t\ell+\ell+j)\big] = \frac{1}{k}\sum_{t=a}^{k}\sum_{s=a}^{t} \frac{v^2(s\ell+j)}{v(s\ell+j)+p(s\ell+j)} + \frac{1}{k}\sum_{t=a}^{k}\sum_{s=a}^{t} r(ts\ell+j) - \Big(\frac{k-a+1}{k}\Big)v(a\ell+j). \tag{14}$$

From (14), (12) and (11), we obtain

$$\lim_{k\to\infty} \frac{1}{k}\sum_{t=a}^{k}\big[-v(t\ell+j)\big] = \infty$$

and hence

$$\lim_{k\to\infty} \frac{1}{k}\sum_{t=a}^{k}|v(t\ell+j)| = \infty. \tag{15}$$

Let $P(k) = \frac{v^2(k)}{v(k)+p(k)}, k \in \mathbb{N}(a\ell)$. Then $P(k) \geq 0$ and $P(k) = 0$ if and only if $v(k) = 0$. Let $A(k) = \frac{v^2(k)}{P(k)}$ if $v(k) \neq 0$ and $A(k) = 0$ if $v(k) = 0$. Then we have $p(k) \geq A(k) - v(k)$ and hence, for $j = 0, 1, \cdots, \ell-1$,

$$k^{\frac{-3}{2}}\sum_{t=a}^{k} p(t\ell+j) \geq k^{\frac{-3}{2}}\sum_{t=a}^{k} A(t\ell+j) + k^{\frac{-3}{2}}\sum_{t=a}^{k}\big[-v(t\ell+j)\big]. \tag{16}$$

Thus, in view of (10) and $A(k) \geq 0$ it follows that

$$\limsup_{k\to\infty} k^{\frac{-3}{2}} \sum_{t=a}^{k} (-v(t\ell+j)) < \infty. \tag{17}$$

Therefore, on dividing both sides of (14) by $k^{\frac{1}{2}}$, and in the resulting equation using (16) and (17) leads to

$$\limsup_{k\to\infty} k^{\frac{-3}{2}} \sum_{t=a}^{k} \sum_{s=a}^{t} \big[P(s\ell+j)\big] < \infty. \tag{18}$$

Now, since

$$\begin{aligned} k^{\frac{-1}{2}} \sum_{t=a}^{k} P(t\ell+j) = k^{\frac{-3}{2}} k \sum_{t=a}^{k} P(t\ell+j) &\leq k^{\frac{-3}{2}} \sum_{t=a}^{2k} \sum_{s=a}^{t} P(s\ell+j) \\ &= 2^{\frac{3}{2}} (2k)^{\frac{-3}{2}} \sum_{t=a}^{2k} \sum_{s=a}^{t} P(s\ell+j) \end{aligned}$$

from (20), we have

$$\limsup_{k\to\infty} k^{\frac{-1}{2}} \sum_{t=a}^{k} P(t\ell+j) < \infty. \tag{19}$$

On the other hand from (19), there is an $M_j > 0$ such that

$$\begin{aligned} \Big(\sum_{t=a}^{k} |v(t\ell+j)|\Big)^2 &= \Big(\sum_{t=a}^{k} (A(t\ell+j)P(t\ell+j)^{\frac{1}{2}})\Big)^2 \\ &\leq \sum_{t=a}^{k} A(t\ell+j) \sum_{t=a}^{k} P(t\ell+j) \leq M_j k^{\frac{1}{2}} \sum_{t=a}^{k} A(t\ell+j)\,. \end{aligned}$$

Therefore, it follows that

$$k^{\frac{-3}{2}} \sum_{t=a}^{k} A(t\ell+j) \geq \frac{1}{M_j} \Big(\frac{1}{k} \sum_{t=a}^{k} |v(t\ell+j)|\Big)^2,$$

and hence from (15), (17), we have

$$\lim_{k\to\infty} k^{\frac{-3}{2}} \sum_{t=a}^{k} p(t\ell+j) = \infty$$

which contradicts (10). Finally, we shall show that (iii) implies (i). Let $v(k)$ be as in (iii) and let

$$B_j(k) = \sum_{t=a}^{k} |v(t\ell+j)|, \quad j = 0, 1, \cdots \ell-1.$$

Then, we have

$$
\begin{aligned}
\Big[\sum_{t=a}^{k} v(t\ell+j)\Big]^2 &\le B_j^2(k)\\
&= \Big[\sum_{t=a}^{k}\big(P(t\ell+j)\big(v(t\ell+j)+p(t\ell+j)\big)^{\frac{1}{2}}\big)\Big]^2\\
&\le \sum_{t=a}^{k} P(t\ell+j)\sum_{t=a}^{k}\big(v(t\ell+j)+p(t\ell+j)\big)\\
&\le L\big[B_j(k)+\sum_{t=a}^{k} p(t\ell+j)\big]\\
&\le 2L_j \max\{B_j(k),\sum_{t=a}^{k} p(t\ell+j)\},
\end{aligned}
$$

where $L_j=\sum_{t=a}^{k} P(t\ell+j)$. Hence, we have

$$B_j(k)\le \max\{2L_j,\big[2L_j\sum_{t=a}^{k} p(t\ell+j)\big]^{\frac{1}{2}}\}.$$

Thus, form (10) it follows that $\lim_{k\to\infty}\frac{1}{k}B_j(k)=0$, so that

$$\lim_{k\to\infty}\frac{1}{k}\sum_{t=a}^{k}\big[-v(t\ell+j)\big]=0.$$

The result (i) now follows by letting $k\to\infty$ in (14) and sum for $j=0,1,2,\cdots \ell-1$. □

Corollary 3.4. *Let (10) hold. Then every ℓ-ray solution of (1) is ℓ-rotatory in case either of the following satisfied*

$$-\infty<\liminf_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k}\sum_{s=0}^{t} r(s)<\limsup_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k}\sum_{s=0}^{t} r(s) \tag{20}$$

or

$$\lim_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k}\sum_{s=0}^{t} r(s)=\infty=\lim_{k\to\infty}\frac{1}{k}\sum_{j=0}^{\ell-1}\sum_{t=0}^{k}\sum_{s=0}^{t} r(s\ell+j). \tag{21}$$

Remark 3.5. Suppose that (10) and (12) hold. Then if (1) is ℓ-web, we define the constants $C_j=\lim_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k}\sum_{s=0}^{t} r(s\ell+j)$.

Theorem 3.6. *Let (10) and (12) hold.*

(i) If (1) is ℓ-web, then there exists a function $v(k)$ on $\mathbb{N}$ such that $v(k) > -p(k)$, $k \in \mathbb{N}(a\ell)$ for some $a \in \mathbb{N}$ and

$$v(k\ell + j) = C_j - \sum_{t=0}^{k-\ell} r(t\ell + j) + \sum_{t=k}^{\infty} \frac{v^2(t\ell + j)}{v(t\ell + j) + p(t\ell + j)}, \tag{22}$$

for $j = 0, 1, 2, 3, \ldots (\ell - 1)$.

(ii) If there exists a function $v(k)$ on $\mathbb{N}$ such that $v(k) > -p(k),\ k \in \mathbb{N}(a\ell)$ and constants C_j satisfying

$$v(k\ell + j) \geq C_j - \sum_{t=0}^{k-1} r(t\ell + j) + \sum_{t=k}^{\infty} \frac{v^2(t\ell + j)}{v(t\ell + j) + p(t\ell + j)} \geq 0 \tag{23}$$

or

$$v(k\ell + j) \leq C_j - \sum_{t=0}^{k-1} r(t\ell + j) + \sum_{t=k}^{\infty} \frac{v^2(t\ell + j)}{v(t\ell + j) + p(t\ell + j)} \leq 0 \tag{24}$$

for $j = 0, 1, 2, \cdots \ell - 1$ for all $k \in \mathbb{N}(\ell)$, then (1) has an ℓ-web solution.

Proof. (i) Since, for $j = 0, 1, 2, \cdots \ell - 1$,

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{k} \sum_{t=a}^{k} \sum_{s=a}^{t} r(s\ell + j) &= \lim_{k\to\infty} \frac{1}{k} \Big[\sum_{t=0}^{k} \sum_{s=0}^{t} r(s\ell + j) - \sum_{t=0}^{a-1} \sum_{s=0}^{t} r(s\ell + j) \\
&\quad - (k - a) \sum_{t=0}^{a-1} r(t\ell + j) \Big] \\
&= C_j - \sum_{t=0}^{a-1} r(t\ell + j).
\end{aligned}$$

Since, (10) and (12) implies (11), in (14)

$$\sum_{t=a}^{\infty} \big[- v(t\ell + \ell + j) \big] < \infty$$

and from (11), we obtain

$$\lim_{k\to\infty} \frac{1}{k} \sum_{t=a}^{k} \sum_{s=a}^{t} \frac{v^2(s\ell + j)}{v(s\ell + j) + p(s\ell + j)} = \sum_{t=a}^{\infty} \frac{v^2(t\ell + j)}{v(t\ell + j) + p(t\ell + j)}\,.$$

Now (22) follows by letting $k \to \infty$ in (14) and then replacing a by k.

(ii) Suppose that (10) and (12) hold and there exists constants $C_j' s$ such that (23) or (24) holds.

For $j = 0, 1, 2, \cdots, \ell - 1$, let

$$w(k\ell + j) = C_j - \sum_{t=0}^{k-1} r(t\ell + j) + \sum_{t=k}^{\infty} \frac{v^2(t\ell + j)}{v(t\ell + j) + p(t\ell + j)}.$$

Then

$$\Delta_\ell w(k\ell + j) = -r(k\ell + j) - \frac{v^2(k\ell + j)}{v(k\ell + j) + p(k\ell + j)}.$$

But, since

$$v(k\ell + j) \geq w(k\ell + j) \geq 0 \quad \text{(or)} \quad v(k\ell + j) \leq w(k\ell + j) \leq 0,$$

we have

$$\frac{v^2(k\ell + j)}{v(k\ell + j)} \geq \frac{w^2(k\ell + j)}{w(k\ell + j)} \quad \text{(or)} \quad v^2(k\ell + j) \geq w^2(k\ell + j)$$

and in the second case

$$0 < v(k\ell + j) + p(k\ell + j) < w(k\ell + j) + p(k\ell + j)$$

which yields,

$$\frac{v^2(k)}{v(k) + p(k)} \geq \frac{w^2(k)}{w(k) + p(k)}$$

and hence

$$\Delta_\ell w(k) + \frac{w^2(k)}{w(k) + p(k)} + r(k) \leq 0, \;\; w(k) > -p(k), k \in \mathbb{N}(a\ell).$$

By Lemma 3.2, equation (1) has an ℓ-web solution. □

Theorem 3.7. *Assume that*

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k} p(t) < \infty\,, \tag{25}$$

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k} \sum_{s=0}^{t} r(s) = -\infty\,, \tag{26}$$

and

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k} \sum_{s=0}^{t} r(s) > -\infty. \tag{27}$$

Then, the difference equation (1) is not ℓ-web and hence every ℓ-ray solution is ℓ-rotatory.

Proof. Suppose to the contrary that (1) is ℓ-web and let $u(k)$ be any ℓ-web solution. Let $v(k) = \frac{p(k)\Delta_\ell u(k)}{u(k)}$ for $k \in \mathbb{N}(a)$. Since condition (10) follows from (25), Theorem 3.3 and (2) imply that (12) holds. But from (14), we have

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{j=0}^{\ell-1}\sum_{t=a}^{k}\Big[-v(t\ell+\ell+j)\Big]$$
$$\geq \liminf_{k\to\infty}\frac{1}{k}\sum_{j=0}^{\ell-1}\sum_{t=a}^{k}\sum_{s=a}^{t}\frac{v^2(s\ell+j)}{v(s\ell+j)+p(s\ell+j)}$$
$$+\limsup_{k\to\infty}\frac{1}{k}\sum_{j=0}^{\ell-1}\Big[\sum_{t=a}^{k}\sum_{s=a}^{t}r(s\ell+j)-v(a\ell+j)\Big]=\infty,$$

which is impossible from $-v(k) <$ p(t) and (25). □

Theorem 3.8. *Assume that the difference equation (1) is ℓ-web and there exists an*

$$M > 0 \text{ with } 0\leq p(k)\leq M \text{ for all } k\in\mathbb{N}. \tag{28}$$

Then the following are equivalent:

(i) $\sum_{t=0}^{\infty} r(t)$ *exits.*

(ii) (11) holds.

(iii) (12) holds.

(iv) for any ℓ-web solution $u(k)$ of (1) with $\frac{u(k)}{u(k+\ell)}>0$, $k\in\mathbb{N}(a\ell)$, the function $v(k)=p(k)\frac{\Delta_\ell u(k)}{u(k)}$, $k\in\mathbb{N}(a\ell)$ satisfies (11).

Proof. Obviously (i) implies (ii) and (ii) implies (iii). Theorem 3.3 shows that (iii) and (iv) are equivalent. Therefore, we need only to show that (iv) implies (i). But this is immediate by letting k $\to\infty$ in (13) and observing that (iv) implies $v(k)\to 0$ as $k\to\infty$. □

Corollary 3.9. *If the assumptions of Theorem 3.8 hold, then the following are equivalent:*

(i) $\sum_{t=0}^{\infty} r(t)=-\infty.$

(ii) (26) holds.

(iii) There exists an ℓ-web solution u(k) of (1) with $\frac{u(k)}{u(k+\ell)}>0$, on $\mathbb{N}(a\ell)$ for some $a\in\mathbb{N}$ such that the function $v(k)=p(k)\frac{\Delta_\ell u(k)}{u(k)}>-p(k)$, $k\in\mathbb{N}(a\ell)$ satisfies (12).

Corollary 3.10. *Let (12) and (28) hold. If* $\sum_{t=0}^{\infty} r(t)$ *does not exits, then* *every ℓ-ray solution of (1) is ℓ-rotatory.*

Corollary 3.11. *Let (30) hold. If* $-\infty = \liminf_{k\to\infty} \sum_{t=0}^{\infty} r(t) < \limsup_{k\to\infty} \sum_{t=0}^{\infty} r(t)$, *then every ℓ-ray solution of (1) is ℓ-rotatory.*

Theorem 3.12. *If there exist two sequences $\{k_t\}$ and $\{m_t\}$ of integers with $m_t \geq k_t + 1$ such that $k_t \to \infty$ as $t \to \infty$ and for $j = 0, 1, 2, \cdots \ell - 1$,*

$$\sum_{s=k_t}^{m_t-1} r(s\ell + j) \geq p(k_t\ell + j) + p(m_t\ell + j), \tag{29}$$

then every ℓ-ray solution of (1) is ℓ-rotatory.

Proof. Suppose that (1) has an ℓ-ray solution which is not ℓ-rotatory. Then there exists an ℓ-web solution $u(k)$ such that $\frac{u(k)}{u(k+\ell)} > 0$ for all $k \in \mathbb{N}(a)$ for some $a \in \mathbb{N}$. Let $v(k) = p(k)\frac{\Delta_\ell u(k)}{u(k)}$. Then $v(k)$ satisfies (5) and $v(k) > -p(k)$ for all $k \in \mathbb{N}(a)$. We will show that

$$\sum_{t=a}^{k-1} r(t\ell + j) < p(a\ell + j) + p(k\ell + j), \quad j = 0, 1, 2, \cdots, \ell - 1, \tag{30}$$

holds for all $k \in \mathbb{N}(a + 1)$ and then this contradiction will prove the theorem.

From (5), we have

$$\begin{aligned}
r(a\ell + j) &= v(a\ell + j) - v(a\ell + \ell + j) - \frac{v^2(a\ell + j)}{v(a\ell + j) + p(a\ell + j)} \\
&< p(a\ell + \ell + j) + \frac{v(a\ell + j)p(a\ell + j)}{v(a\ell + j) + p(a\ell + j)} \\
&= p(a\ell + \ell + j) + p(a\ell + j) - \frac{p^2(a\ell + j)}{v(a\ell + j) + p(a\ell + j)} \\
&< p(a\ell + \ell + j) + p(a\ell + j), \quad j = 0, 1, 2, \cdots, \ell - 1.
\end{aligned}$$

Therefore (30) holds for $k = a + 1$. For any $k \in \mathbb{N}(a + 2)$, from (5) we have

$$\begin{aligned}
\sum_{t=a+1}^{k-1} r(t\ell + j) &= v(a\ell + \ell + j) - v(k\ell + j) - \sum_{t=a+1}^{k-1} \frac{v^2(t\ell + j)}{v(t\ell + j) + p(t\ell + j)} \\
&< v(a\ell + \ell + j) + p(k\ell + j).
\end{aligned}$$

However from (5), since

$$v(a\ell + \ell + j) = p(a\ell + j)\Big[1 - \frac{u(a\ell + j)}{u(a\ell + \ell + j)}\Big] - r(a\ell + j)$$

$$< p(a\ell + j) - r(a\ell + j), \quad j = 0, 1, 2, \cdots, \ell - 1,$$

(30) follows immediately. □

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