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BEST SIMULTANEOUS APPROXIMATION ON THE UNIT SPHERE

E. Aghdassi[1 §], S.F. Rzaev[2]

[1]Department of Mathematics
Faculty of Mathematical Sciences
University of Tabriz
Tabriz, IRAN
e-mail: esaghdassi@@tabrizu.ac.ir

[2]Department of Mathematics
Khazar University
Baku, AZERBAIJAN
e-mail: rzseymur@hotmail.com

Abstract: In this paper, with the help of Laplace–Beltrami operator the best simultaneous approximation on the unit sphere is defined and some theorems analogue has been 2π-periodical functions are proved.

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1. Introduction

In the fifties and sixties a number of classical problems of the constructive theory of functions were completely solved. More details can be found in the monographs [6], [27] and others.

We note that trigonometric polynomials are the eigenfunctions of the shift operator on the circle spherical polynomials are the eigenfunction of the Laplace–Beltrami operator on the unit sphere. From this point of view they have set of "close relatives". Apart from the circle, there are other homogeneous spaces, on which a group of motions acts. On such homogeneous spaces there are special functions, similar to the harmonics on the circle.

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§Correspondence author

For example, there is one class of homogeneous spaces, where productive work on the investigation of the approximative methods of harmonic analysis was done a long time ago. These are the n–dimensional sphere S^n and more generally compact globally symmetric spaces of rank one. The general theory of approximation on the sphere was formed in our time, although, for S^n a lot of researches were done in the beginning of this century.

An idea of the size of the literature on approximation in the sphere is given by the bibliographical index of the [31]. We refer the reader, interested in the further development of approximation on the sphere and its application, to the monograph [18]. There are many papers devoted to the study of approximation by spherical polynomials on the unit sphere S^n. History as well as references to further works dealing with this circle of ideas can be found e.g. in the papers [2], [3], [5], [10], [16], [23] and others.

The results of S.M. Nikolskii and D.I. Lizorkin on this circle of questions laid the foundation of an entire direction on approximation theory on the unit sphere. Among the most active mathematicians working on problems of the approximation theory on the sphere we name A.G. Babenko, Z. Ditzian, Dai Feng, A.P. Terekhin, Wang Kunjang and Li Luoqing, Yuan Xu (this list can be extended).

The present paper is divided into 3 sections. Section 2 contains basic notations, definitions and auxiliary theorems. The main results of the paper are given in Section 3. This section is devoted to the investigation the best simultaneous approximation in the metric of the space L_p, $1 \le p \le \infty$, on the sphere S^n. Furthermore Freud and Garkavi type estimates are established.

2. Preliminaries and Auxiliary Theorems

Let S^n be the unit sphere in $\mathbb{R}^{n+1}, n \ge 1$, about the origin, let $L_p(S^n)$, $1 \le p \le \infty$, be the space of function $f(\theta)$ integrable to the p-th power on S^n, with the norm

$$\|f\|_p = \left(\int_{S^n} |f(\theta)|^p d\theta \right)^{\frac{1}{p}},$$

where $d\theta$ stands for the elementary area on the sphere S^n and for simplicity let $L_\infty(S^n)$ be the space of continuous functions on S^n with the usual norm $\|f\|_\infty = \max\{|f(\theta)| \,:\, \theta \in S^n\}$.

Denote by δ the Laplace–Beltrami operator on S^n, i.e. the spherical part of the Laplace operator Δ in $\mathbb{R}^{n+1}$, and by H_m, $m = 0, 1, \ldots$, the space of homogeneous spherical harmonic of degree m, which is the eigensubspace of δ corresponding to the eigenvalue $m(m+n-1)$ (see [25])

$$H_m = \{f \in C^\infty(S^n); \delta f = m(m+n-1)f\}, \quad m = 0, 1, \dots .$$

The spaces H_m and H_k are mutually orthogonal in $L_2(S^n)$ for $m \neq k$, and we have the orthogonal decomposition $L_2(S^n) = \sum_0^\infty \oplus H_m$.

We note that r–power δ^r of the operator, $r \in (-\infty, \infty)$, acts on a function $f \in H_m$ $m = 0, 1, 2, \dots$, in the following way:

$$\delta^r f = m^r (m+n-1)^r f \tag{1}$$

($m \neq 0$ is isolated of $r < 0$). Then for natural numbers $r = 1, 2, \dots$, the differential operator δ^r defined by (1), has locality properties. For fractional $r > 0$, δ^r are pseudo–differential operators. In this paper we will consider only r–power δ^r–Laplase–Beltrami operators for $r \in \mathbb{N}$.

The orthogonal projection $Y_m : L_2(S^n) \to H_m$ is given by the formula

$$Y_m(f)(\theta) = \frac{\Gamma(\lambda+1)}{2\pi^{\lambda+1}} \frac{m+\lambda}{\lambda} \int_{S^n} C_m^\lambda(<\theta, \sigma>) f(\sigma) d\sigma, \quad m \in \mathbb{N},$$

where $2\lambda = n-1$, $<\theta, \sigma>$ means the scalar product of the unit vectors pointing from the origin to the points θ and σ of the sphere S^n, and $C_m^\lambda(\cos\phi)$ is the ultraspherical (or Gegenbueer) polynomial of order m, with index λ, normalized by the condition

$$C_m^\lambda(1) = \Gamma(m+2\lambda)/(\Gamma(m).\Gamma(2\lambda)),$$

and Γ is the Euler gamma function.

The aforementioned elements of harmonic analysis on the sphere can be found, for example in [18] and [25]. $C, C_1, C_2, \dots$ will denote positive constants, which may be different in different formulas, and depend on incidental parameters. For the sake of simplicity, these parameters will not be written out.

Let $E_m(f)_p$; $m = 0, 1, \dots$, be the best approximation of a function $f \in L_p(S^n)$, $1 \le p \le \infty$, by spherical polynomials of degree $\le m$, i.e.,

$$E_m(f)_p = \inf\left\{ \|f - P_m\|_p \ : \ P_m \in \sum_{k=0}^m \oplus H_k \right\}, \quad m = 0, 1, \dots .$$

The presence of the Laplace–Beltrami operator gives the possibility to define classes of Sobolev type

$$W_p^{2r}(S^n) := \{f \in L_p(S^n) \ : \ \|f\|_{W_p^{2r}} := \|f\|_p + \|(-\delta)^r f\|_p < \infty\},$$

where $\delta^r f = \delta(\delta^{r-1} f)$, $r \in \mathbb{N}$, $\delta^0 = id$ (the identity operator).

Theorem 1. *Suppose* $1 \le p \le \infty$*; and* $r \in \mathbb{N}$*, the following conditions are equivalent:*

1. $f \in W_p^{2r}(S^n)$.

2. There exists a function $\psi(\theta) \in L_p(S^n)$ *such that* f *can be represented as the spherical convolution*

$$f(\theta) = (K_{2r} * \psi)(\theta) \ : \ \int_{S^n} K_{2r}(<\theta, \sigma>)\psi(\sigma)d\sigma,$$

where

$$K_{2r}(\cos t) = \frac{\Gamma(\lambda)}{2\pi^{\lambda+1}}\left[1 + \sum_{m=1}^{\infty} \frac{m+\lambda}{m(m+2\lambda)^r} C_m^{\lambda}(\cos t)\right].$$

For arbitrary $r > 0$ this result was obtained by Kh.P. Rustamov [22], for r even this theorem is contained in [26] and [29], see also [24]. There are analogues theorems in periodic version: for example see [1], [8] and [14]. From this we could have the following:

Theorem 2. *Let* $f \in W_p^{2r}(S^n)$, $r \in \mathbb{N}$, *then*

$$E_m(f)_p \le \frac{C_1}{m^{2r}} E_m(\delta^r f)_p \, .$$

For $r \in \mathbb{N}$ and $n = 2$ it was proved by Ar.S. Dzafarov [7] and for $r > 0$ by Kh.P. Rustamov [23] (see also [13] and [20]).

We assert that on the sphere S^n the beginnings of constructive theorem of functions has been constructed by G.G. Kushnirenko [15], Ap.S. Dzafarov [7], S.V. Topuriya [31], D.L. Ragozin [21], A.I. Kamzolov [12], S. Pawelke [20], P.I. Lizorkin and S.M. Nikolskii [16, 17], also variety of versions of the Bernstein inequality was proved.

Theorem 3. *Let* $1 \le p \le \infty$. *For any spherical polynomial* P_m *the inequality*

$$\|\delta^r P_m\|_p \le C_2 m^{2r} \|P_m\|_p,$$

is true.

Various special cases and versions of this theorem have been discussed by A.P. Shaginyan, G.G. Kushnirenko, e.g. Gol'shtein, S.V. Topuriya, Ar.S. Dzafarov, A.I. Kamzolov, S. Pawelke, G.V. Zidkov, E.M. Stein, D.L. Butzer, and H. Jonen, D.L. Ragozin, A.I. Kamzolov, S.M. Nikoloskii and P.I. Lizorkin, Kh.P. Rustamov and others. More common and exact form is contained in A.I. Kamzolov [13].

3. Basic Results

Let $f \in W_p^{2r}(S^n)$ and put

$$\xi_{mr}(f)_p = \inf_{P_m} \max_{s\in[0:r]} \frac{\|\delta^s f - \delta^s P_m\|_p}{E_m(\delta^s f)_p}, \tag{2}$$

where $s, r \in \mathbb{N} \cup \{0\} := \mathbb{Z}_+$ and $[0 : r] := \mathbb{Z}_+ \cap [0, r]$.

We assume that $E_m(f) > 0$. In this case we will show that, $E_m(\delta^s f)_p > 0$ for all $s \in [0 : r]$.

The spherical polynomials P_m^*, for which

$$\max_{s\in[0:r]} \frac{\|\delta^s f - \delta^s P_m^*\|_p}{E_m(\delta^s f)_p} = \xi_{mr}(f)_p, \tag{3}$$

is called the best simultaneous approximation spherical polynomial on the sphere S^n.

The possibility of simultaneous approximation on circle was introduced by R. M. Trigub [28] although the first property was suggested by A.F. Timan [27]. Simple proofs for this type theorems is given by V.N. Malozemov [19], see also [6].

Theorem 4. *Let $f \in W_p^{2r}(S^n)$. If $E_m(f)_p > 0$, then $E_m(\delta^s f)_p > 0$ all the $s \in [0 : r]$.*

Proof. Indeed, we assume that $E_m(\delta^s f)_p = 0$ for some $s \in [0 : r]$. If P_m^* stands for the spherical polynomial, it is a best approximation to $\delta^s f$. Then we have

$$\|\delta^s f - P_m^*\|_p = E_m(\delta^s f)_p = 0,$$

since $\delta^s f \equiv P_m^*$. Hence by Theorem 1

$$f(\theta) = (K_{2r} * P_m^*)\ .$$

From this we obtain that $f(\theta)$ is spherical polynomial of degree at most m which implies that $E_m(f) = 0$. This establishes the theorem. □

The following results play a significant role in our construction: Let $\eta \in C^\infty[0, +\infty)$ be a function defined by $\eta(x) = 1$ for $0 \le x \le 1$ and $\eta(x) = 0$ if $x \ge 1$.

A sequence of operators η_m for $m \in \mathbb{N}$ is defined by (see [23])

$$\eta_m f := \sum_{k=0}^{\infty} \eta(\frac{k}{m}) Y_k(f)(\theta).$$

Since $\eta(\frac{k}{m}) = 0$ if $k \ge 2m$, the infinite series terminates $k = 2m - 1$ so that

η_m is spherical polynomial of degree at most $2m-1$. The main properties of η_m are given in the following proposition.

Proposition 5. (see [23]) *Let $f \in L_p(S^n)$, $1 \le p \le \infty$. For $m \in \mathbb{N}$:*

a) $\eta_m f$ is a spherical polynomial of degree at most $2m-1$,

b) $\|\eta_m f\|_p \le C_3 \|f\|_p$,

c) $\|f - \eta_m f\|_p \le C_4 E_m(f)_p$.

Proposition 6. *Let $f \in W_p^{2r}(S^n)$, $1 \le p \le \infty$ and $P_m(\theta)$ is spherical polynomials of degree m in the $L_p(S^n)$ best approximation of $\eta_m f$, then*

$$\max_{s\in[0;r]} \frac{\|\delta^s f - \delta^s P_m\|_p}{E_m(\delta^s f)_p} \le O(1)(1+q), \tag{4}$$

where $q = \min\{4^m, 4^r\}$, and $O(1)$ means bounded by absolute constant.

Proof. It is easy to see from (1) that for an arbitrary $s \in [0:r]$ we have

$$\delta^s \eta_m f = \eta_m \delta^s f.$$

Then

$$\|\delta^s f - \delta^s P_m\|_p \le \|\delta^s f - \eta_m \delta^s f\|_p + \|\delta^s \eta_m f - \delta^s P_m\|_p. \tag{5}$$

By c) of the Proposition 5,

$$\|\delta^s f - \eta_m \delta^s f\|_p \le C_4 E_m(\delta^s f)_p. \tag{6}$$

Also we note that by a) of Proposition 5, $\eta_m f - P_m$ is a spherical polynomial of degree at most $2m-1$, then by Theorem 3 we obtain

$$\|\delta^s \eta_m f - \delta^s P_m\|_p \le C_2(2m-1)^{2s}\|\eta_m f - P_m\|_p = C_2(2m-1)^{2s} E_m(\eta_m f)_p.$$

Then for spherical polynomials $P_m^* \in \sum_{k=0}^m H_k$, we have

$$\begin{aligned} E_m(\eta_m f)_p &= \inf_{(P_m^*)} E_m(\eta_m f - P_m^*)_p = \inf_{P_m^*} E_m(\eta_m f - \eta_m P_m^*)_p \\ &\le \inf_{P_m^*} \|\eta_m f - \eta_m P_m^*\|_p \le C_3 \inf_{P_m^*} \|f - P_m^*\|_p = C_3 E_m(f)_p. \end{aligned} \tag{7}$$

Now using inequality (6) and Theorem 2 we have

$$\begin{aligned} \|\delta^s \eta_m f - \delta^s P_m\|_p &\le C_2 C_3 (2m-1)^{2s} E_m(f)_p \\ &\le C_5 (\frac{2m-1}{m})^{2s} E_m(\delta^s f)_p, \end{aligned}$$

where $C_5 = C_1.C_2.C_3$. Hence putting inequalities (6) and (7) in (5), we get

$$\|\delta^s f - \delta^s P_m\|_p \le [C_4 + C_5(\frac{2m-1}{m})^{2s}] E_m(\delta^s f)_p.$$

If $r > m$, then

$$\|\delta^s f - \delta^s P_m\|_p \le [C_4 + C_5(2 - \frac{1}{m})^{2r}] E_m(\delta^s f)_p \le (C_4 + C_5 2^{2r}) E_m(\delta^s f)_p$$

$$\leq (C_4 + C_5 q) E_m(\delta^s f)_p.$$

If $r \leq m$, then

$$\|\delta^s f - \delta^s P_m\|_p \leq [C_4 + C_5(2 - \frac{1}{m})^{2m}] E_m(\delta^s f)_p \leq (C_4 + C_5 2^{2m}) E_m(\delta^s f)_p$$
$$= (C_4 + C_5 q) E_m(\delta^s f)_p.$$

Finally, for all $s \in [0 : r]$, we have

$$\|\delta^s f - \delta^s P_m\|_p \leq (C_4 + C_5 q) E_m(\delta^s f)_p.$$

From this it follows (4) and the Proposition 6 is proved. □

Theorem 7. *Let $f \in W_p^{2r}(S^n)$, then*

$$\xi_{mr}(f) \leq O(1)(1 + q),$$

where $q = \min\{4^m, 4^r\}$.

Proof. This theorem follows from (2) and Proposition 6. □

Theorem 8. *Let $f \in W_p^{2r}(S^n)$ and P_m be arbitrary spherical polynomial of degree not greater than m. Then*

$$\|\delta^r f - P_m^r\|_p \leq C_2 m^{2r} \|f - P_m\|_p + \xi_{mr}(f)_p [C_2 m^{2r} E_m(f)_p + E_m(\delta^r f)_p]$$
$$\leq C_3 [m^{2r} \|f - P_m\|_p + \xi_{mr}(f) E_m(\delta^r f)_p], \qquad (8)$$

where $C_3 = \max\{C_2, 1 + C_1.C_2\}$.

Proof. Let P_{mr}^* be best simultaneous approximation spherical polynomial of the function $f(\theta) \in W_p^{2r}(S^n)$, then for all $s \in [0 : r]$, by (3) we have

$$\|\delta^s f - \delta^s P_{mr}^*\|_p \leq \xi_{mr}(f)_p E_m(\delta^s f)_p. \qquad (9)$$

Using (9), by Theorem 3, we get

$$\|\delta^r f - \delta^r P_m\|_p \leq \|\delta^r f - \delta^r P_{mr}^*\|_p + \|\delta^r P_{mr}^* - \delta^r P_m\|_p$$
$$\leq \xi_{mr}(f)_p E_m(\delta^r f)_p + C_2 m^{2r} \|P_{mr}^* - P_m\|_p$$
$$\leq \xi_{mr}(f)_p E_m(\delta^r f)_p + C_2 m^{2r} \{\xi_{mr}(f)_p E_m(f)_p + \|f - P_m\|_p\}$$
$$= C_2 m^{2r} \|f - P_m\|_p + \xi_{mr}(f)_p [C_2 m^{2r} E_m(f)_p + E_m(\delta^r f)_p],$$

which proves first inequality in (8).

The second inequality follows from Theorem 2. □

There are analogues of these theorems in periodic version; for example see [9], [11], [19].

Now using inequality (8) we have following:

Corollary 9. *Let $f \in W_p^{2r}(S^n)$ and spherical polynomials $\{P_m\}_{m=0}^{\infty}$ sat-*

isfying inequality

$$\|f - P_m\|_p \leq \frac{A}{C_1} E_m(f)_p,$$

then

$$\|\delta^r f - \delta^r P_m\|_p \leq C_3\{\xi_{mr}(f)_p + A\} E_m(\delta^r f)_p\,.$$

From this corollary and Theorem 6 we have following result: If the sequence of spherical polynomials $\{P_m\}_{m=0}^{\infty}$ given for functions $f \in W_p^{2r}(S^n)$ orders best approximation, then $\{\delta^r P_m\}$ given for $\delta^r f$ also order best approximation.

We note that the converse of above result is not true.

Proposition 10. *For any $r \in \mathbb{N}$ any $f \in W_p^{2r}(S^n)$ and arbitrary sequence of positive numbers $\{\alpha_n\}$ we can show that for some sequences of spherical polynomials $\{Q_m\}$, satisfying*

$$\|\delta^r f - \delta^r Q_m\|_p \leq A_r E_m(\delta^r f),$$

we have

$$\|f - Q_m\|_p \geq \alpha_n,$$

where A_r depends only on r, and $m = 0, 1, \ldots$.

Proof. Let P_m^*–spherical polynomial of best approximation order m for function f, i.e.

$$\|f - P_m^*\|_p = E_m(f)_p\,.$$

By Corollary 9

$$\|\delta^r f - \delta^r P_m^*\|_p \leq A_r E_n(\delta^r f)_p\,.$$

We introduce the sequence of spherical polynomials

$$Q_m(\theta) = P_m^*(\theta) + \alpha_n + E_m(f)_p \quad (m = 0, 1, 2, \ldots)\,.$$

We have

$$\|\delta^r f - \delta^r Q_m\|_p = \|\delta^r f - \delta^r P_m^*\|_p \leq A_r E_m(\delta^r f)_p\,.$$

On the other hand

$$\|f - Q_m\|_p \geq \alpha_n + E_m(f)_p - \|f - P_m^*\|_p = \alpha_n\,.$$

The proposition is proved. □

Remark 11. Using the results in [4] we can get analogous results for compact globally symmetric spaces of rank 1. Also using results in [30] we get analogous results for h-harmonic functions on sphere and unit ball.

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