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# PERIODIC SOLUTION OF DIFFUSIVE PREDATOR-PREY SYSTEM TIME DELAY AND TYPE III FUNCTIONAL RESPONSE

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**Abstract:** A nonautonomous diffusive predator-prey model with time delay and type III functional response is studied. By using the continuation theorem of coincidence degree theory, the existence of a positive periodic solution of the investigated system is established.

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**Key Words:** periodic solution, type III functional response, diffusion, time delay, coincidence degree

## 1. Introduction

The permanence of Lotka-Volterra systems is important aspect of the mathematical ecology research. Predator-prey system has been much discussed, the existence of periodic solution of predator-prey system is attractive [4], [3], [9], [8], [11]. On the other hand, for many species spatial factors are important in population dynamics. Recently the influence of diffusion and functional response involving time delays on system dynamics has been discussed by some authors [10], [6], [5], [1].

In this paper, we consider a nonautonomous diffusive predator-prey model with time delay as following:

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$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[a_{10}(t) - a_{11}(t)x_1(t) - \frac{\alpha_1(t)x_1(t)y(t)}{1+\beta_1(t)x_1^2(t)}\right] \\
&\qquad\qquad\qquad\qquad + D_1(t)[x_2(t) - x_1(t)], \\
\dot{x}_2(t) &= x_2(t)[a_{20}(t) - a_{21}(t)x_2(t)] + D_2(t)[x_1(t) - x_2(t)], \\
\dot{y}(t) &= y(t)\left[-a_{30}(t) - a_{31}(t)y(t) + \frac{a_{32}(t)\alpha_1(t)x_1^2(t-\tau_1(t,y(t)))}{1+\beta_1(t)x_1^2(t-\tau_1(t,y(t)))}\right. \\
&\qquad\qquad\qquad\qquad \left. - \frac{\alpha_2(t)y(t)z(t)}{1+\beta_2(t)y^2(t)}\right], \\
\dot{z}(t) &= z(t)\left[-a_{40}(t) - a_{41}(t)z(t) + \frac{a_{42}(t)\alpha_2(t)y^2(t-\tau_2(t,z(t)))}{1+\beta_2(t)y^2(t-\tau_2(t,z(t)))}\right],
\end{aligned}
\tag{1.1}
$$

where $x_1(t), y(t)$ and $z(t)$ are the densities of prey species $x$ and predator species $y$ and $z$ in patch I at time $t$ respectively; $x_2(t)$ is the density of prey species $x$ in patch II at time $t$. Predator species $y$ and $z$ are both confined to patch I, while prey species $x$ can diffuse between two patches $D_i(t)(i=1,2)$ are dispersion coefficients of species $x$. Species $x$ is the prey of species $y$, while species $y$ is the prey of species $z$, then a biological food chain is founded. Our purpose in this paper is, by using continuation theorem of coincidence degree theory which was proposed in [2], [7], to study the existence of positive periodic solution of system (1.1).

## 2. Existence of Periodic Solution of (2.1)

Before the main result, we will make some preparation.

Let $X$ and $Y$ be real Banach spaces, $L : DomL \subset X \to Y$ a Fredholm mapping of index zero and $P : X \to X,\ Q : Y \to Y$ continuous projectors such that $\text{Im } P = KerL, KerQ = \text{Im } L$ and $X = KerL \oplus KerP, Y = \text{Im } L \oplus \text{Im } Q$. Denote by $L_p$ the restriction of $L$ in $DomL \cap KerP,\ K_p : \text{Im } L \to KerP \cap DomL$ the inverse to $L_p$ and $J : \text{Im } Q \to KerL$ an isomorphism of $\text{Im } Q$ onto $KerL$.

Now, we introduce the continuation theorem [2] as follows.

**Lemma 2.1.** *Let $\Omega \subset X$ be an open bounded set and $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$ (i.e., $QN : \overline{\Omega} \to Y$) and $K_p(I-Q)N : \overline{\Omega} \to Y$ are compact). Assume:*

*(a) for each $\lambda \in (0,1), x \in \partial\Omega \cap DomL, Lx \neq \lambda Nx$;*

*(b) for each $x \in \partial\Omega \cap KerL, QNx \neq 0$;*

*(c) $\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$.*

*Then $Lx = Nx$ has at least one periodic solution in $\overline{\Omega} \cap DomL$.*

In the following, we shall use the notations:

$$\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)dt, f^L = \min_{t\in[0,\omega]} |f(t)|, f^M = \max_{t\in[0,\omega]} |f(t)|,$$

where $f$ is a continuous $\omega$-periodic function.

In system (1.1), we always assume the following.

(H1) $a_{i_0}, a_{i_1} (i = 1, 2, 3, 4), D_i(t), \alpha_i, \beta_i (i = 1, 2), a_{j_2}(t)(j = 3, 4)$ are positive periodic continuous functions with period $\omega > 0$.

(H2) $\tau_1(t, y(t))$ and $\tau_2(t, z(t))$ are both continuous and $\omega$-periodic with respect to $t$, and $\tau_1$ and $\tau_2$ due to gestation of $y$ and $z$ respectively.

Now, we are going to prove the main result.

**Theorem 2.1.** *In addition to (H1) and (H2) assume the following:*

*(H3)* $(a_{10} - D_1)^L > 0,$

*(H4)* $(a_{20} - D_2)^L > 0,$

*(H5)* $(-a_{30})^M - a_{32}^L(\frac{\alpha_1}{\beta_1})^L > 0,$

$$(-a_{40})^M - a_{42}^L(\frac{\alpha_2}{\beta_2})^L > 0,$$

*(H6)* $(a_{32}\alpha_1)^L[(a_{10} - D_1)^2]^L > a_{30}^M[a_{11}^2 + \beta_1(a_{10} - D_1)^2]^M.$

*Then system (1.1) has at least one positive $\omega$-periodic solution.*

*Proof.* Let $x_i(t) = \exp^{u_i(t)}, i = 1, 2,\ y(t) = \exp^{u_3(t)}, z(t) = \exp^{u_4(t)}$, then system (1.1) becomes

$$\begin{aligned}
\dot{u}_1(t) &= a_{10}(t) - a_{11}(t)e^{u_1(t)} - \frac{\alpha_1(t)e^{u_1(t)}e^{u_3(t)}}{1 + \beta_1(t)e^{2u_1(t)}} \\
&\qquad + D_1(t)e^{u_2(t)-u_1(t)} - D_1(t)\,, \\
\dot{u}_2(t) &= a_{20}(t) - a_{21}(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)} - D_2(t)\,, \\
\dot{u}_3(t) &= -a_{30}(t) - a_{31}(t)e^{u_3(t)} + \frac{a_{32}(t)\alpha_1(t)e^{2u_1(t-\tau_1(t,e^{u_3(t)}))}}{1 + \beta_1(t)e^{2u_1(t-\tau_1(t,e^{u_3(t)}))}} \\
&\qquad - \frac{\alpha_2(t)e^{u_3(t)}e^{u_4(t)}}{1 + \beta_2(t)e^{2u_3(t)}}\,, \\
\dot{u}_4(t) &= -a_{40}(t) - a_{41}(t)e^{u_4(t)} + \frac{a_{42}(t)\alpha_2(t)e^{2u_3(t-\tau_2(t,e^{u_4(t)}))}}{1 + \beta_2(t)e^{2u_3(t-\tau_2(t,e^{u_4(t)}))}}\,,
\end{aligned} \tag{2.1}$$

where $a_{i_0}(t), a_{i_1}(t)(i = 1, 2, 3, 4), D_i(t), \alpha_i, \beta_i(i = 1, 2), a_{j_2}(t)(j = 3, 4)$ are the same as those in (H1), and $\tau_1, \tau_2$ are the same as those in (H2). It is easy to see that if system (2.1) has one $\omega$-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t))^T$, then $(e^{u_1^*(t)}, e^{u_2^*(t)}, e^{u_3^*(t)}, e^{u_4^*(t)})^T$ is a positive $\omega$-periodic solution of system (1.1).

Then, the next work is to prove that system (2.1) has one $\omega$-periodic solution.

In order to apply the continuation theorem of coincidence degree theory to establish the existence of $\omega$-periodic solution of system (2.1), we take

$$X = Y = \{u = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in C(R, R^4) : u_i(t+\omega) = u_i(t), i = 1, 2, 3, 4\}$$

and

$$\|(u_1(t), u_2(t), u_3(t), u_4(t))^T\| = \sum_{i=1}^{4} \max_{t\in[0,\omega]} |u_i(t)|,$$

here, $|\cdot|$ denotes the Euclidean norm. With this norm $\|\cdot\|$, $X$ and $Y$ are Banach spaces. Set

$L : DomL \cap X, L(u_1(t), u_2(t), u_3(t), u_4(t))^T = (\dot{u}_1(t), \dot{u}_2(t), \dot{u}_3(t), \dot{u}_4(t))^T$, where $DomL = \{u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in C^1(R, R^4)\}$, and $N : X \to X, Lu = Nu$,

$$N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} a_{10}(t) - a_{11}(t)e^{u_1(t)} - \dfrac{\alpha_1(t)e^{u_1(t)}e^{u_3(t)}}{1+\beta_1(t)e^{2u_1(t)}} + D_1(t)e^{u_2(t)-u_1(t)} - D_1(t) \\ a_{20}(t) - a_{21}(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)} - D_2(t) \\ -a_{30}(t) - a_{31}(t)e^{u_3(t)} + \dfrac{a_{32}(t)\alpha_1(t)e^{2u_1(t-\tau_1(t,e^{u_3(t)}))}}{1+\beta_1(t)e^{2u_1(t-\tau_1(t,e^{u_3(t)}))}} - \dfrac{\alpha_2(t)e^{u_3(t)}e^{u_4(t)}}{1+\beta_2(t)e^{2u_3(t)}} \\ -a_{40}(t) - a_{41}(t)e^{u_4(t)} + \dfrac{a_{42}(t)\alpha_2(t)e^{2u_3(t-\tau_2(t,e^{u_4(t)}))}}{1+\beta_2(t)e^{2u_3(t-\tau_2(t,e^{u_4(t)}))}} \end{bmatrix}. \tag{2.2}$$

Define two projectors $P$ and $Q$ as

$$P \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = Q \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{w}\displaystyle\int_0^\omega u_1(t)dt \\ \dfrac{1}{w}\displaystyle\int_0^\omega u_2(t)dt \\ \dfrac{1}{w}\displaystyle\int_0^\omega u_3(t)dt \\ \dfrac{1}{w}\displaystyle\int_0^\omega u_4(t)dt \end{bmatrix}, \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in X. \tag{2.3}$$

Clearly, $KerL = R^4$, Im $L = \{(u_1(t), u_2(t), u_3(t), u_4(t))^T \in X : \int_0^\omega u_i(t)dt =$

$0, i = 1,2,3,4\}$ is closed in $X$ and dim $KerL$ =codim Im $L = 4$, therefore, $L$ is a Fredholm mapping of index zero. Through computation we find that the inverse $K_p$ of $L_p$ has the form

$$K_p : \mathrm{Im}L \to DomL \cap KerP\,,$$

$$K_p \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \int_0^t u_1(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t u_1(s)dsdt \\ \int_0^t u_2(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t u_2(s)dsdt \\ \int_0^t u_3(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t u_3(s)dsdt \\ \int_0^t u_4(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t u_4(s)dsdt \end{bmatrix}. \tag{2.4}$$

Moreover, we have

$$K_p(I-Q)Nu = K_pNu - K_pQNu$$
$$= \int_0^t \dot{u}(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^\eta \dot{u}(s)dsd\eta - \left[\frac{t}{\omega}\int_0^\omega \dot{u}(s)ds - \frac{1}{\omega^2}\int_0^\omega \left(\eta\int_0^\omega \dot{u}(s)ds\right)d\eta\right]$$
$$= \int_0^t \dot{u}(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^\eta \dot{u}(s)dsd\eta - \left(\frac{t}{w} - \frac{1}{2}\right)\int_0^\omega \dot{u}(s)ds\,. \tag{2.5}$$

We can show that $QN$ and $K_p(I-Q)N$ are continuous by Lebesgue convergence theorem and that $QN(\bar{\Omega}), K_p(I-Q)N(\bar{\Omega})$ are relatively compact for any open bounded subset $\Omega$ by Arzela-Ascoli Theorem. Therefore, $N$ is $L$-compact on $\bar{\Omega}$ for any open bounded subset $\Omega \in X$. Corresponding to the operator equation $Lu = \lambda Nu, \lambda \in (0,1)$, we have

$$\begin{aligned}
&\dot{u}_1(t) = \lambda \times \\
&\left[a_{10}(t) - a_{11}(t)e^{u_1(t)} - \frac{\alpha_1(t)e^{u_1(t)}e^{u_3(t)}}{1+\beta_1(t)e^{2u_1(t)}} + D_1(t)e^{u_2(t)-u_1(t)} - D_1(t)\right], \\
&\dot{u}_2(t) = \lambda\left[a_{20}(t) - a_{21}(t)e^{u_2(t)} + D_2(t)e^{u_1(t)-u_2(t)} - D_2(t)\right], \\
&\dot{u}_3(t) = \lambda \times \\
&\left[-a_{30}(t) - a_{31}(t)e^{u_3(t)} + \frac{a_{32}(t)\alpha_1(t)e^{2u_1(t-\tau_1(t,e^{u_3(t)}))}}{1+\beta_1(t)e^{2u_1(t-\tau_1(t,e^{u_3(t)}))}} - \frac{\alpha_2(t)e^{u_3(t)}e^{u_4(t)}}{1+\beta_2 te^{2u_3(t)}}\right], \\
&\dot{u}_4(t) = \lambda\left[-a_{40}(t) - a_{41}(t)e^{u_4(t)} + \frac{a_{42}(t)\alpha_2(t)e^{2u_3(t-\tau_2(t,e^{u_4(t)}))}}{1+\beta_2(t)e^{2u_3(t-\tau_2(t,e^{u_4(t)}))}}\right].
\end{aligned} \tag{2.6}$$

Choose $\xi_i, \eta_i \in [0,\omega]$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), i = 1,2,3,4\,.$$

Then it is clear that

$$\dot{u}_i(\xi_i) = \dot{u}_i(\eta_i) = 0, \quad i = 1,2,3,4\,.$$

From this and system (2.6), we obtain

$$\begin{aligned}
&a_{10}(\xi_1) - a_{11}(\xi_1)e^{u_1(\xi_1)} - \frac{\alpha_1(\xi_1)e^{u_1(\xi_1)}e^{u_3(\xi_1)}}{1+\beta_1(\xi_1)e^{2u_1(\xi_1)}} + D_1(\xi_1)e^{u_2(\xi_1)-u_1(\xi_1)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - D_1(\xi_1) = 0\,, \\
&a_{20}(\xi_2) - a_{21}(\xi_2)e^{u_2(\xi_2)} + D_2(\xi_2)e^{u_1(\xi_2)-u_2(\xi_2)} - D_2(\xi_2) = 0\,, \\
&-a_{30}(\xi_3) - a_{31}(\xi_3)e^{u_3(\xi_3)} + \frac{a_{32}(\xi_3)\alpha_1(\xi_3)e^{2u_1(\xi_3-\tau_1(\xi_3,e^{u_3(\xi_3)}))}}{1+\beta_1(\xi_3)e^{2u_1(\xi_3-\tau_1(\xi_3,e^{u_3(\xi_3)}))}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \frac{\alpha_2(\xi_3)e^{u_3(\xi_3)}e^{u_4(\xi_3)}}{1+\beta_2\xi_3e^{2u_3(\xi_3)}} = 0\,, \\
&-a_{40}(\xi_4) - a_{41}(\xi_4)e^{u_4(\xi_4)} + \frac{a_{42}(\xi_4)\alpha_2(\xi_4)e^{2u_3(\xi_4-\tau_2(\xi_4,e^{u_4(\xi_4)}))}}{1+\beta_2(\xi_4)e^{2u_3(\xi_4-\tau_2(\xi_4,e^{u_4(\xi_4)}))}} = 0\,,
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&a_{10}(\eta_1) - a_{11}(\eta_1)e^{u_1(\eta_1)} - \frac{\alpha_1(\eta_1)e^{u_1(\eta_1)}e^{u_3(\eta_1)}}{1+\beta_1(\eta_1)e^{2u_1(\eta_1)}} + D_1(\eta_1)e^{u_2(\eta_1)-u_1(\eta_1)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - D_1(\eta_1) = 0\,, \\
&a_{20}(\eta_2) - a_{21}(\eta_2)e^{u_2(\eta_2)} + D_2(\eta_2)[e^{u_1(\eta_2)-u_2(\eta_2)} - D_2(\eta_2) = 0\,, \\
&-a_{30}(\eta_3) - a_{31}(\eta_3)e^{u_3(\eta_3)} + \frac{a_{32}(\eta_3)\alpha_1(\eta_3)e^{2u_1(\eta_3-\tau_1(\eta_3,e^{u_3(\eta_3)}))}}{1+\beta_1(\eta_3)e^{2u_1(\eta_3-\tau_1(\eta_3,e^{u_3(\eta_3)}))}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \frac{\alpha_2(\eta_3)e^{u_3(\eta_3)}e^{u_4(\eta_3)}}{1+\beta_2\eta_3e^{2u_3(\eta_3)}} = 0\,, \\
&-a_{40}(\eta_4) - a_{41}(\eta_4)e^{u_4(\eta_4)} + \frac{a_{42}(\eta_4)\alpha_2(\eta_4)e^{2u_3(\eta_4-\tau_2(\eta_4,e^{u_4(\eta_4)}))}}{1+\beta_2(\eta_4)e^{2u_3(\eta_4-\tau_2(\eta_4,e^{u_4(\eta_4)}))}} = 0\,.
\end{aligned} \tag{2.8}$$

If $u_1(t_1) \geq u_2(t_2)$, then $u_1(t_1) \geq u_2(t_2) \geq u_2(t_1)$. From the first formula of (2.7), we have

$$a_{11}(\xi_1)e^{u_1(\xi_1)} < a_{10}(\xi_1), \quad e^{u_1(\xi_1)} < \frac{a_{10}^M}{a_{11}^L} =: \rho_1\,. \tag{2.9}$$

If $u_1(t_1) < u_2(t_2)$, then $u_1(t_2) \leq u_1(t_1) < u_2(t_2)$. By the second formula of (2.7), we get

$$a_{21}(\xi_2)e^{u_2(\xi_2)} < a_{20}(\xi_2) \Rightarrow e^{u_2(\xi_2)} < \frac{a_{20}^M}{a_{21}^L} =: \rho_2\,. \tag{2.10}$$

Set $\rho = \max\{\rho_1, \rho_2\}$, by (2.9) and (2.10), for $\forall t \in [0,\omega]$, we have

$$e^{u_1(\xi_1)} < \rho, e^{u_2(\xi_2)} < \rho\,. \tag{2.11}$$

From the third formula and the last formula of (2.7), we have

$$a_{31}(\xi_3)e^{u_3(\xi_3)} < -a_{30}(\xi_3) + a_{32}(\xi_3)\frac{\alpha_1(\xi_3)}{\beta_1(\xi_3)} := b_3(\xi_3)$$

$$\Rightarrow e^{u_3(\xi_3)} < b_3^M =: \rho_3\,. \tag{2.12}$$

And

$$a_{41}(\xi_4)e^{u_4(\xi_4)} = -a_{40}(\xi_4) + a_{42}(\xi_4)\frac{\alpha_2(\xi_4)}{\beta_2(\xi_4)} := b_4(\xi_4)\,.$$

That is

$$e^{u_4(\xi_4)} < b_4^M =: \rho_4 \tag{2.13}$$

If $u_1(\eta_1) \le u_2(\eta_2)$, then $u_1(\eta_1) \le u_2(\eta_2) \le u_2(\eta_1)$. By the first formula of (2.8), we get

$$a_{11}(\eta_1)e^{u_1(\eta_1)} > a_{10}(\eta_1) - \frac{a_1(\eta_1)e^{u_3(\eta_1)}}{\beta_1(\eta_1)e^{u_1(\eta_1)}},$$

$$\Rightarrow a_{11}(\eta_1)e^{2u_1(\eta_1)} - a_{10}(\eta_1)e^{u_1(\eta_1)} + \frac{\alpha_1(\eta_1)b_3(\eta_1)}{\beta_1(\eta_1)} > 0\,, \tag{2.14}$$

$$e^{u_1(\eta_1)} > \frac{a_{10}(\eta_1) + \sqrt{a_{10}^2(\eta_1) - 4a_{11}(\eta_1)b_3(\eta_1)\frac{\alpha_1(\eta_1)}{\beta_1(\eta_1)}}}{2a_{11}(\eta_1)} \text{ and } a_{10}^2 - 4a_{11}b_3\tfrac{\alpha_1}{\beta_1} \ge 0.$$

Otherwise, $\forall e^{u_1(\eta_1)} \in (0, +\infty)$, (2.14) is true.

So we can find $\delta_1 > 0$, such that $e^{u_1(\xi_1)} > \delta_1$.

If $u_1(\eta_1) > u_2(\eta_2)$, then $u_1(\eta_2) \ge u_1(\eta_1) > u_2(\eta_2)$, by the second formula of (2.8), we obtain

$$a_{21}(\eta_2)e^{u_2(\eta_2)} > a_{20}(\eta_2) \Rightarrow e^{u_2(\eta_2)} > \frac{a_{20}}{a_{21}} := \delta_2\,.$$

Set $\delta = \min\{\delta_1, \delta_2\}$, for $\forall t \in [0, \omega]$, we have

$$e^{u_1(\xi_1)} > \delta, e^{u_2(\xi_2)} > \delta. \tag{2.15}$$

From [8], we have

$$e^{u_3(\eta_3)} > \frac{-a_{30}(\eta_3) + \frac{a_{32}(\eta_3)\alpha_1(\eta_3)}{\frac{1}{\rho_1^2}+\beta_1(\eta_3)} + \sqrt{\left(-a_{30}(\eta_3) + \frac{a_{32}(\eta_3)\alpha_1(\eta_3)}{\frac{1}{\rho_1^2}+\beta_1(\eta_3)}\right)^2 - 4a_{31}(\eta_3)\frac{\alpha_2(\eta_3)\delta_4}{\beta_2(\eta_3)}}}{2a_{13}(\eta_3)} \tag{2.16}$$

and

$$e^{u_4(\eta_4) > \frac{1}{a_{41}(\eta_4)}}\left[-a_{40}(\eta_4) + \frac{a_{42}(\eta_3)\alpha_1(\eta_3)}{\frac{1}{\rho_3^2} + \beta_1(\eta_3)}\right]. \tag{2.17}$$

Therefore, from (2.15), (2.16), (2.17), it follows that there exist three positive constants $\delta, \delta_j (j = 3, 4)$ such that

$$e^{u_i(t)} > \delta, \quad i = 1, 2; \qquad e^{u_j(t)} > \delta, \quad j = 3, 4. \tag{2.18}$$

From (2.11), (2.12), (2.13) and (2.18), we obtain

$$|u_i(t)| < \max\{|\ln\rho|, |\ln\rho_j||\ln\delta|, |\ln\delta_j|\} =: R_i, i = 1, 2, 3, 4, j = 3, 4\,.$$

Obviously, $R_i (i = 1, 2, 3, 4)$ are independence of $\lambda$.

Using the integral mean valued theorem, it follows that there exist some points $\xi_i \in [0, \omega] (i = 1, 2, 3, 4)$ such that when $(u_1, u_2, u_3, u_4)^T$ is a constant vector, we get

$$QN\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \overline{(a_{10} - D_1)} - \overline{a_{11}}e^{u_1} - \dfrac{\overline{\alpha_1}e^{u_1}e^{u_3}}{1 + \beta_1(\zeta_1)e^{2u_1}} + \overline{D_1}e^{u_2 - u_1} \\ \overline{(a_{20} - D_2)} - \overline{a_{21}}e^{u_2} + \overline{D_2}e^{u_1 - u_2} \\ -\overline{a_{30}} - \overline{a_{31}}e^{u_3} + \dfrac{\overline{a_{32}\alpha_1}e^{2u_1}}{1 + \beta_1(\zeta_2)e^{2u_1}} - \dfrac{\overline{\alpha_2}e^{u_3}e^{u_4}}{1 + \beta_2(\zeta_3)e^{2u_3}} \\ -\overline{a_{40}} - \overline{a_{41}}e^{u_4} + \dfrac{\overline{a_{42}\alpha_2}e^{2u_3}}{1 + \beta_2(\zeta_4)e^{u_3}} \end{bmatrix}. \tag{2.19}$$

Denote

$$M = \sum_{i=1}^{4} R_i + R_0,$$

here, $R_0$ is taken sufficient large such that each solution $(\alpha^*, \beta^*, \gamma^*, \nu^*)^T$ of the following system:

$$\begin{aligned} &\overline{(a_{10} - D_1)} - \overline{a_{11}}e^{\alpha} - \frac{\overline{\alpha_1}e^{\alpha}e^{\gamma}}{1 + \beta_1(\zeta_1)e^{2\alpha}} + \bar{D}_1 e^{\beta - \alpha} = 0\,, \\ &\overline{(a_{20} - D_2)} - \overline{a_{21}}e^{\beta} + \overline{D_2}e^{\alpha - \beta} = 0\,, \\ &-\overline{a_{30}} - \overline{a_{31}}e^{\gamma} + \frac{\overline{a_{32}\alpha_1}e^{2\alpha}}{1 + \beta_1(\zeta_2)e^{2\alpha}} - \frac{\overline{\alpha_2}e^{\gamma}e^{\nu}}{1 + \beta_2(\zeta_3)e^{2\gamma}} = 0\,, \\ &-\overline{a_{40}} - \overline{a_{41}}e^{\nu} + \frac{\overline{a_{42}\alpha_2}e^{2\gamma}}{1 + \beta_2(\zeta_4)e^{2\gamma}} = 0\,, \end{aligned} \tag{2.20}$$

satisfies $\|(\alpha^*, \beta^*, \gamma^*, \nu^*)^T\| = |\alpha^*| + |\beta^*| + |\gamma^*| + |\nu^*| < M$, provided that system (2.20) has a solution or a number of solutions. Now we take $\Omega = \{u = (u_1, u_2, u_3, u_4)^T \in X : \|u\| < M\}$. This satisfies condition (a) in Lemma 2.1. When $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap$ Ker $L = \partial\Omega \cap R^4, (u_1, u_2, u_3, u_4)^T$ is a constant vector in $R^4$ with $\sum_{i=1}^{4}|u_i| = M$.

If system (2.20) has a solution or a number of solutions, then

$$QN(u_1, u_2, u_3, u_4)^T \neq (0, 0, 0, 0)^T .$$

If system (2.20) does not have a solution, then naturally

$$QN \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{2.21}$$

This proves that condition (b) in Lemma 2.1 is satisfied.

Finally we will show that condition (c) of Lemma 2.1 is satisfied. So, we define

$$\phi : DomL \times [0, 1] \to X$$

by

$$\phi(u_1, u_2, u_3, u_4, \mu) = \begin{bmatrix} \overline{(a_{10} - D_1)} - \overline{a_{11}}e^{u_1} \\ \overline{(a_{20} - D_2)} - \overline{a_{21}}e^{u_2} \\ -\overline{a_{30}} - \overline{a_{31}}e^{u_3} + \dfrac{\overline{a_{32}\alpha_1}e^{2u_1}}{1 + \beta_1(\zeta_2)e^{2u_1}} \\ -\overline{a_{40}} - \overline{a_{41}}e^{u_4} + \dfrac{\overline{a_{42}\alpha_2}e^{2u_3}}{1 + \beta_2(\zeta_4)e^{u_3}} \end{bmatrix} + \mu \begin{bmatrix} -\dfrac{\overline{\alpha_1}e^{u_1}e^{u_3}}{1 + \beta_1(\zeta_1)e^{2u_1}} + \overline{D_1}e^{u_2 - u_1} \\ \overline{D_2}e^{u_1 - u_2} \\ -\dfrac{\overline{\alpha_2}e^{u_3}e^{u_4}}{1 + \beta_2(\zeta_3)e^{2u_3}} \\ 0 \end{bmatrix}, \tag{2.22}$$

where $\mu \in [0, 1]$ is a parameter.

When $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap KerL = \partial\Omega \cap R^4$, $(u_1, u_2, u_3, u_4)^T$ is a constant vector in $R^4$ with $\sum_{i=1}^{4}|u_i| = M$. We will show that when $(u_1, u_2, u_3, u_4)^T \in \partial\Omega \cap KerL, \phi(u_1, u_2, u_3, u_4, \mu) \neq (0, 0, 0, 0)^T$. Otherwise, constant vector $(u_1, u_2, u_3, u_4)^T$ with $\sum_{i=1}^{4}|u_i| = M$ satisfies

$$\phi(u_1, u_2, u_3, u_4, \mu) = (0, 0, 0, 0, 0)^T ,$$

then from

$$
\begin{aligned}
&\overline{(a_{10}-D_1)} - \overline{a_{11}}e^{u_1} - \frac{\mu\overline{\alpha_1}e^{u_1}e^{u_3}}{1+\beta_1(\zeta_1)e^{2u_1}} + \mu\bar{D}_1 e^{u_2-u_1} = 0\,,\\
&\overline{(a_{20}-D_2)} - \overline{a_{21}}e^{u_2} + \mu\overline{D_2}e^{u_1-u_2} = 0\,,\\
&-\overline{a_{30}} - \overline{a_{31}}e^{u_3} + \frac{\overline{a_{32}\alpha_1}e^{2u_1}}{1+\beta_1(\zeta_2)e^{2u_1}} - \frac{\mu\overline{\alpha_2}e^{u_3}e^{u_4}}{1+\beta_2(\zeta_3)e^{2u_3}} = 0\,,\\
&-\overline{a_{40}} - \overline{a_{41}}e^{u_4} + \frac{\overline{a_{42}\alpha_2}e^{2u_3}}{1+\beta_2(\zeta_4)e^{u_3}} = 0\,,
\end{aligned}
\tag{2.23}
$$

by the above-mentioned argument (2.13), (2.14), (2.15) and (2.21), magnifying $\overline{f}$ into $f^M$ and reducing $\overline{f}$ into $f^L$, here $f$ denotes every function in (H1), and magnifying $\mu$ into 1 and reducing $\mu$ into 0, we get

$$|u_i| < \max\{|\ln\rho|, |\ln\rho_j||\ln\delta|, |\ln\delta_j|\}, \quad i=1,2,3,4,\ j=3,4\,.$$

Then

$$\sum_{i=1}^{4}|u_i| < \sum_{i=1}^{4}R_i < M\,,$$

which contradicts the fact that constant vector $(u_1,u_2,u_3,u_4)^T$ satisfies $\sum_{i=1}^4|u_i| = M$. Therefore,

$$\phi(u_1,u_2,u_3,\mu) \neq (0,0,0,0)^T,\ \text{for } (u_1,u_2,u_3,u_4)^T \in \partial\Omega\cap KerL\,.$$

Using the property of topological degree and take $J = I : \mathrm{Im}L \to KerL$, we have

$$
\begin{aligned}
&\deg\{JQN(u_1,u_2,u_3,\mu)^T, \Omega\cap KerL, (0,0,0,0)^T\}\\
&\qquad = \deg\{\phi(u_1,u_2,u_3,u_4,1), \Omega\cap KerL, (0,0,0,0)^T\}\\
&\qquad = \deg\{\phi(u_1,u_2,u_3,u_4,0), \Omega\cap KerL, (0,0,0,0)^T\}\\
&= \deg\{[\overline{a_{10}-D_1} - \overline{a_{11}}e^{u_1}, \overline{a_{20}-D_2} - \overline{a_{21}}e^{u_2}, -\overline{a_{30}} - \overline{a_{31}}e^{u_3} + \frac{\overline{a_{32}\alpha_1}e^{2u_1}}{1+\beta_1(\zeta_2)e^{2u_1}},\\
&\qquad -\overline{a_{40}} - \overline{a_{41}}e^{u_4} + \frac{\overline{a_{42}\alpha_2}e^{2u_3}}{1+\beta_2(\zeta_4)e^{u_3}}]^T, \Omega\cap KerL, (0,0,0,0)^T\}\,.
\end{aligned}
\tag{2.24}
$$

In view of the conditions of Theorem 2.1, then the system of algebraic equations

$$
\begin{aligned}
&\overline{(a_{10}-D_1)} - \overline{a_{11}}u = 0\,,\\
&\overline{(a_{20}-D_2)} - \overline{a_{21}}v = 0\,,\\
&-\overline{a_{30}} - \overline{a_{31}}m + \frac{\overline{a_{32}\alpha_1}u^2}{1+\beta_1(\zeta_2)u^2} = 0\,,\\
&-\overline{a_{40}} - \overline{a_{41}}n + \frac{\overline{a_{42}\alpha_2}m^2}{1+\beta_2(\zeta_4)m^2} = 0\,,
\end{aligned}
\tag{2.25}
$$

has a unique solution $(u^*, v^*, m^*, n^*)^T$ which satisfies:

$$u^* = \frac{\overline{a_{10} - D_1}}{\overline{a_{11}}} > 0, v^* = \frac{\overline{a_{20} - D_2}}{\overline{a_{21}}} > 0,$$

$$m^* = \frac{1}{\overline{a_{31}}} \left[ -a_{30} + \frac{\overline{a_{32}\alpha_2}\overline{(a_{11} - D_1)^2}}{\overline{a_{11}}^2 + \beta_1 \overline{(a_{11} - D_1)^2}} \right] > 0$$

$$n^* = \frac{1}{\overline{a_{41}}} \left[ -a_{40} + \frac{\overline{a_{42}\alpha_2}\overline{m^*}^2}{\overline{a_{11}}^2 + \beta_2 \overline{m^*}^2} \right] > 0 \,.$$

Therefore

$$\deg\{JQN(u_1, u_2, u_3, u_4, \mu)^T, \Omega \cap KerL, (0,0,0,0)^T\}$$

$$= \text{sgn} \begin{vmatrix} -\overline{a_{11}} u^* & 0 & 0 & 0 \\ 0 & a_{11} v^* & 0 & 0 \\ \dfrac{2\overline{a_{32}\alpha_1} u^*}{[1 + \beta_1(\zeta_2)(u^*)^2]^2} & 0 & -\overline{a_{31}} m^* & 0 \\ 0 & 0 & \dfrac{2\overline{a_{42}\alpha_2} m^*}{[1 + \beta_2(\zeta_4)(m^*)^2]^2} & -\overline{a_{41}} n^* \end{vmatrix}$$

$$= \text{sgn}(\overline{a_{11}} u^*, \overline{a_{21}} v^*, \overline{a_{31}} m^*, \overline{a_{41}} n^*) = 1 \,. \quad (2.26)$$

This completes the proof of condition (c) of Lemma 2.1 and by now we know $\Omega$ satisfies all the requirements of Lemma 2.1. So, system (1.1) has at least one positive $\omega$-periodic solution. □

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