International Journal of Pure and Applied Mathematics
Volume 47 No. 1 2008, 79-88

OPTIMAL BURN-IN TIME UNDER LIFETIME WARRANTY POLICIES

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Abstract: Lifetime warranty is relatively a new concept. It has significant difference with normal warranty. In this paper, the optimal burn-in time to minimize the total mean cost of repairable and nonrepairable products under two typical lifetime warranties (FRLTW and LICLTW) is studied. We assume that the product before undergoing the burn-in procedure has a bathtub shaped failure rate function with change points t_1 and t_2. It is shown that the optimal burn-in time b^* minimizing the cost function is always before t_1.

AMS Subject Classification: 60E15, 60K10
Key Words: burn-in, lifetime warranty, FRLTW, LICLTW, bathtub failure rate

1. Introduction

It is well known that product warranty has started playing an important role in current competitive market. Most durable products are sold with some

Received: May 3, 2008

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type of warranty to protect both the right and the interest of consumers from unexpected early failures and consumers will be satisfactorily when they buy a product that comes with a warranty. Consumers expect that the product has longer warranty period, to manufacturer, however, longer warranty period usually comes with higher expected warranty cost which is a significant portion of the total mean cost. High product reliability in the early period of the product life cycle is crucial to reduce warranty-related costs.

With the development of products market, some of manufacturers have been offering long-term warranty policies, see Chattopadhyay and Rahman [3] for an example, and the warranty period has been becoming longer and longer. Besides, many manufacturers have also started offering lifetime warranties for their products.

Chattopadhyay and Rahman [3] defined Lifetime warranty as the manufacturers' commitment to provide free of cost sharing repair or replacement of the sold product in case of failure due to design, manufacturing defects of quality problems during the useful life of the product or the buyer's ownership period.

In fact, many products are sold to consumers with lifetime warranty police in present commodity market. For example, the warranty period of the relays installed on the circuit system of car is Lifetime of the car and the filter netting on the Range Hood, and so on. Meanwhile, "Lifetime" are defined as the lifetime of the car and Range Hood for relays and filter netting, respectively. It can be seen that the warranty period is a random variable.

Since early failures always result in a high warranty cost, burn-in procedure is employed to reduce the damage from early failures. That is, before delivery to the customers, the products are tested under electrical or thermal conditions that approximate the working conditions in the field operation and those fail to survive the test are removed away. The studies for burn-in testing began with the appearing of transistors in the early 1950s Kuo and Kuo [6], Block and Savits [2], Jensen and Petersen [5]. Nguyen and Murthy [8] first proposed a model to determine the optimal burn-in time for products sold with warranty in 1980s, Chou and Tang [4] and Mi [7] considered two types of warranty policies as same as Nguyen and Murthy [8] in 1990s. One is the failure-free policy, the other is rebate policy. Among these three papers, Mi [7]'s conclusion was most general.

In this paper, we study the following two typical lifetime warranty policies due to Chattopadhyay and Rahman [3].

Policy 1. (Free Rectification Lifetime Warranty (FRLTW)) Under this policy the manufacturer takes the responsibility to rectify all defects and failures of the sold product due to design or manufacturing control problems over the defined lifetime of the product. Rectification is replacement when the product is nonrepairable, otherwise is repair for repairable product. Unlike normal warranty, the coverage period for a lifetime warranty is uncertain and randomly variable.

Policy 2. (Limit on Individual Cost Lifetime Warranty (LICLTW)) Under this policy, if the cost of a rectification on each occasion is below the limit c_I, then it is borne completely by the manufacturer and the customer pays nothing. If the cost of a rectification exceeds c_I, the buyer pays all the cost in excess of c_I. This continues until the termination of warranty based on defined lifetime.

These two type of policies are typical lifetime warranty policies and they have been applying to products of industrial equipments and appliances widely.

In addition, we assume that products have a bathtub failure rate. We give the definition of bathtub shape of failure rate function.

Definition 1. A failure rate function $r\,(t)$ is said to have bathtub shape if there exist $0 \leq t_1 \leq t_2 < \infty$ such that

$$r(t) = \begin{cases} \text{strictly decreases, if } 0 \leq t \leq t_1, \\ \text{constant, if } t_1 \leq t \leq t_2, \\ \text{strictly increases, if } t_2 \leq t\,, \end{cases}$$

where t_1 and t_2 are called the change points of $r(t)$. The time interval $[0, t_1]$ is called the infant mortality period; the interval $[t_1, t_2]$, where $r(t)$ is flat and attains its minimum value is called the normal operating period or useful period; the interval $[t_2, \infty]$ is called the wear-out period.

The rest of this paper is organized as follows. Section 2 briefly describes the assumptions and denotations of the cost models in section 3. Section 3 presents the main results which include four important theorems. Finally, a conclusion is drawn in Section 4.

2. Notations and Assumptions

2.1. Notations

X — lifetime of product;

T — 'Lifetime' in the policy 1 and policy 2;

$F(t)$ — cumulative distribution function of X;

$\bar{F}(t)$ — survival function of X;

$G(t)$ — cumulative distribution function of T;

$\bar{G}(t)$ — survival function of T;

$r(t)$ — hazard rate function of X ;

b — burn-in time;

$F_b(t)$ — cumulative distribution function of product survived burn-in period $b = \frac{F(b+t)-F(b)}{\bar{F}(b)}$;

$B^* = \{b \geq 0 : \ c(b) = \min_{t\geq 0} c(t)\}$;

$b^* = \inf\{B^*\}$;

D_i — the unfixed repair cost of repairable products for i-th failure in the warranty period;

$H(t)$ — cumulative distribution function of D_i's;

c_0 — manufacturing cost per item without burn-in;

c_1 — fixed set up cost of burn-in per item;

c_2 — cost of burn-in per item per unit time;

c_3 — the shop repair or replacement cost per failure during the burn-in period, include the manufacturing cost and set up cost for nonrepairable products;

c_4 — the extra repair or replacement cost per failure during the warranty period;

$h_1(b)$ — random manufacturing cost incurred until the first item survives burn-in period;

$v_1(b) = Eh(b)$;

$W_1(b)$ — random cost for nonrepairable product during the warranty period;

$w_1(b)$ — mean cost for nonrepairable product during the warranty period;

$c_1(b)$ — the total mean cost for nonrepair product;

$N_1(T)$ — the number of failure for nonrepairable product during the warranty period.

In addition, $h_2(b)$, $v_2(b)$, $W_2(b)$, $w_2(b)$, $c_1(b)$ and $N_2(T)$ are corresponding quantities of repairable product.

2.2. Assumptions

Throughout this article, the following assumptions are held.

— The product has bathtub shaped failure rate as definition.

— T is independent of X.

— $c_I > c_4$.

— D_i, $i = 1, 2, \ldots$, have common distribution $H(t)$.

— The repair means minimal repair, and by minimal repair we mean the repair when finished; then the failure rate at the completion of the repair is the same as it was immediately before the failure.

3. Main Results

In this section, the cost model under burn-in and FRLTW and LICLTW is presented. The total mean cost of the product includes its manufacturing cost and warranty cost. Suppose that the burn-in time is b, then random cost is given by

$$h(b) = c_o + c_1 + c_2 b + c_3 \cdot (\text{the number of failures until product survives the burn-in time } b). \tag{1}$$

For nonrepairable products, we have

$$h_1(b) = c_o + c_1 + c_2 \left[\sum_{i=1}^{\eta-1} (X_i | X_i < b) + b \right] + c_3(\eta - 1). \tag{2}$$

Consequently, the mean burn-in cost is given by

$$v_1(b) = c_0 + c_1 + c_2 \frac{\int_0^b \bar{F}(t)\mathrm{d}t}{\bar{F}(b)} + c_3 \frac{F(b)}{\bar{F}(b)}. \tag{3}$$

For repairable products, note that the mean number of failures in the interval $[0, b]$ under minimal repair is $\int_0^b r(t)dt$ (see Block, Borges and Savits [1]). Thus, for the repairable products the mean burn-in cost is given as

$$v_2(b) = c_0 + c_1 + c_2 b + c_3 \int_0^b r(t)\mathrm{d}t. \tag{4}$$

For the proofs of (1)-(4) one can refer to Mi [7]. Thus, the manufacturing cost of product have been obtained, for the warranty cost may be consider for each case in the following.

3.1. Nonrepairable Products

In this subsection, we discuss the optimal burn-in time of nonrepairable products under FRLTW and LICLTW, respectively.

3.1.1. Under FRLTW

First let us consider the optimal burn-in time to minimize the mean total cost of nonrepairable product with FRLTW. The random total cost is given by

$$C_1(b) = h_1(b) + W_1(b)$$

and

$$W_1(b) = N_1(T) \cdot [v_1(b) + c_4].$$

Then the mean total cost is given by

$$c_1(b) = v_1(b) + \mathrm{E}[N_1(T)] \cdot [v_1(b) + c_4],$$

where

$$\mathrm{E}[N_1(T)] = \mathrm{E}[\mathrm{E}[N_1(T)|T]] = \mathrm{E}\left[\sum_{n=1}^{\infty} F_b^{(n)}(T)\right] = \int_0^{\infty} \sum_{n=1}^{\infty} F_b^{(n)}(t)\mathrm{d}G(t), \quad (5)$$

$F^{(n)}(\cdot)$ is n-fold convolution of $F(\cdot)$.

Rewrite the above equations, we have the following result.

Theorem 1. *Under FRLTW, the mean total cost is*

$$c_1(b) = v_1(b) + [v_1(b) + c_4] \int_0^{\infty} \sum_{n=1}^{\infty} F_b^{(n)}(t)\mathrm{d}G(t). \quad (6)$$

Furthermore, the optimal burn-in time b^* must satisfy $b^* \leq t_1$.

Proof. By above discussion, it is easy to obtain equation (6). Assume that burn-in procedure is implemented at certain time $b_0 > t_1$. Then, according to (6), the mean total cost is

$$c_1(b_0) = v_1(b_0) + [v_1(b_0) + c_4] \int_0^{\infty} \sum_{n=1}^{\infty} F_{b_0}^{(n)}(t)\mathrm{d}G(t),$$

and if the burn-in time $b = t_1$, the mean total cost can be written as

$$c_1(t_1) = v_1(t_1) + [v_1(t_1) + c_4] \int_0^{\infty} \sum_{n=1}^{\infty} F_{t_1}^{(n)}(t)\mathrm{d}G(t)\,.$$

Denote by $r_{b_0}(t)$ and $r_{t_1}(t)$ the failure rate functions of $F_{b_0}(t)$ and $F_{t_1}(t)$, respectively. It is easy to see that $r_{b_0}(t) = r(b_0+t)$ and $r_{t_1}(t) = r(t_1+t)$. Hence, we have

$$\bar{F}_{b_0}(t) = \exp\left[-\int_0^t r(b_0+x)\mathrm{d}x\right]$$

and

$$\bar{F}_{t_1}(t) = \exp\left[-\int_0^t r(t_1+x)\mathrm{d}x\right].$$

It holds that $\bar{F}_{b_0}(t) \leq \bar{F}_{t_1}(t)$ for all $t \geq 0$ since $r(t)$ is bathtub shaped, and hence we have

$$F_{b_0}^{(n)}(t) \leq F_{t_1}^{(n)}(t), \quad n \geq 1.$$

Also,

$$\int_0^\infty \sum_{n=1}^\infty F_{b_0}^{(n)}(t)dG(t) \geq \int_0^\infty \sum_{n=1}^\infty F_{t_1}^{(n)}(t)dG(t).$$

On the other hand, $v_1(b)$ is increasing function in $b \geq 0$. Therefore, $c_1(b_0) \geq c_1(t_1)$ for all $b_0 \geq t_1$, the required result follows. □

3.1.2. Under LICLTW

Next, the optimal burn-in time to minimize the mean total cost of nonrepairable products with LICLTW is considered. Based on policy 2 and those assumptions in Section 2, we can derive the following result.

Theorem 2. *If nonrepairable product is sold with LICLTW, then the mean total cost is given by*

$$c_1(b) = v_1(b) + I(b)\int_0^\infty \sum_{n=1}^\infty F_b^{(n)}(t)\mathrm{d}G(t), \tag{7}$$

where $I(b) = \min[v_1(b)+c_4, c_I]$. *Furthermore, the optimal burn-in time* $b^* \leq t_1$.

Proof. The replacement cost at each failure in warranty period is $I(b) = \min(v_1(b)+c_4, c_I)$. Thus, equation (7) can be obtained as in Theorem 2. It suffices to show $I(b_0) \geq I(t_1)$ for $b_0 \geq t_1$ by the proof of Theorem 2, and this is trivial. □

3.2. Repairable Products

In this subsection, we discuss the generalized optimal burn-in time of repairable products under FRLTW and LICLTW, that is, we consider that the cost of the

i-th minimal repair for sold product during warranty period includes two parts, one is fixed cost c_4, the other is unfixed cost D_i which has distribution $H(t)$, $i = 1, 2, \cdots$ It is easy to obtain the cost of i-th repair is $c_4 + D_i$ in warranty period. We have the following result.

3.2.1. Under FRLTW

Theorem 3. *If repairable product is sold with FRLTW, then the mean total cost is given by*

$$c_2(b) = v_2(b) + (c_4 + ED_1) \int_0^\infty r(b+t)\bar{G}(t)\mathrm{d}t\,. \tag{8}$$

In addition, the optimal burn-in time $b^* \leq t_1$.

Proof. Since at each failure the failed product undergoes minimal repair at the manufacturer's cost of $c_4 + ED_1$ during warranty period. Note that the sold products have failure rate $r_b(t) = r(b+t)$, thus, the mean cost during the warranty period is given by

$$\begin{aligned} w_2(b) &= (c_4 + \mathrm{E}D_1)\mathrm{E}[N_2(T)] \\ &= (c_4 + \mathrm{E}D_1)\mathrm{E}\left[\int_0^T r(b+t)dt\right] \\ &= (c_4 + \mathrm{E}D_1)\int_0^\infty \int_0^x r(b+t)\mathrm{d}t\mathrm{d}G(x) \\ &= (c_4 + \mathrm{E}D_1)\int_0^\infty \int_t^\infty r(b+t)\mathrm{d}G(x)\mathrm{d}t \\ &= (c_4 + \mathrm{E}D_1)\int_0^\infty r(b+t)\bar{G}(t)\mathrm{d}t. \end{aligned}$$

Hence, equation (8) holds.

Furthermore, for any $b \geq t_1$ it holds that $w(b)$ is increasing in b since $r(t)$ has bathtub shape, then the desired result follows. □

3.2.2. Under LICLTW

Theorem 4. *Under LICLTW, the mean total cost is*

$$c_2(b) = v_2(b) + w_2(b), \tag{9}$$

where

$$w_2(b) = \int_0^\infty r(b+t)\bar{G}(t)dt \cdot \left[c_4 + \int_0^{c_I - c_4} \bar{H}(t)\mathrm{d}t\right].$$

Furthermore, the optimal burn-in time $b^* \le t_1$.

Proof. It is easy to derive that the warranty random cost is $W_2(b) = \sum_{i=1}^{N_2(T)} \min(c_4 + D_i, c_I)$. Thus, the warranty mean total cost may be computed as

$$w_2(b) = \mathrm{E}[N_2(T)] \cdot [c_4 + \int_0^{c_I - c_4} \bar{H}(t)\mathrm{d}t] \,.$$

By Theorem 3, $\mathrm{E}[N_2(T)] = \int_0^\infty r(b+t)\bar{G}(t)\mathrm{d}t$, then equation (9) holds.

Also note that $v_2(b)$ is strictly increasing in $b \ge 0$ and $w_2(b)$ is increasing in $b \ge t_1$. Hence, the optimal burn-in time $b^* \le t_1$. □

4. Conclusions

Although lifetime warranty is a relatively new concept, it exists widely in various area in the real life, and it is more reasonable warranty policy. In this paper, we chosen two typical lifetime warranty policies and combined burn-in with lifetime warranty as well as have observed that the optimal burn-in time always not exceeded the first change point whether under normal warranty or lifetime warranty.

For the combination of burn-in and lifetime warranty there is huge scope for future research. One can consider general case that products possess two types of failure as it fails or to burn-in products in accelerated environment and so forth. In addition, it is reasonable that other lifetime warranty policies could be considered by researchers.

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