International Journal of Pure and Applied Mathematics
Volume 47 No. 3 2008, 397-403

GENERALIZED THUE-MORSE SEQUENCES AND THE VON KOCH CURVE

L. Kennard[1], M. Zaremsky[2], J. Holdener[3 §]

[1]Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104, USA
e-mail: kennardL@math.upenn.edu

[2]Department of Mathematics
University of Virginia
Charlottesville, VA 22904
e-mail: mcz5r@virginia.edu

[3]Department of Mathematics
Kenyon College
Gambier, OH 43022, USA
e-mail: holdenerj@kenyon.edu

Abstract: In a recent paper, Ma and Holdener used turtle geometry and polygon maps to show that the Thue-Morse sequence encodes the von Koch curve. In the final paragraph of this same paper, they ask whether or not there exist certain generalized Thue-Morse sequences that also encode the curve. Here we answer this question in the affirmative, providing an infinite family of words that generate generalized Thue-Morse sequences encoding the von Koch curve.

AMS Subject Classification: 68R15, 28A80
Key Words: Thue-Morse sequence, von Koch curve, recurrent curves, fractals

1. Introduction

Let Σ^* be the monoid of words on the alphabet $\Sigma = \{F, L\}$ under the operation of concatenation (i.e. $(w_1, w_2) \mapsto w_1 w_2$ for all $w_1, w_2 \in \Sigma^*$). Here "F" denotes a

Received: May 26, 2008

[§]Correspondence author

forward motion of the turtle by one unit in the plane and "L" a counterclockwise rotation by the fixed angle $\theta = 2\pi/6$. For each $w = \prod_{i=1}^{k} F^{m_i} L^{n_i} \in \Sigma^*$, define $\overline{w} \in \Sigma^*$ by $\overline{w} = \prod_{i=1}^{k} L^{m_i} F^{n_i}$.[1] We call $\overline{w}$ the negation of w, because it is obtained from w by replacing L's with F's and vice versa. Next define $\sigma : \Sigma^* \to \Sigma^*$ by $\sigma(w) = w\overline{w}$ for all $w \in \Sigma^*$. The *Thue-Morse turtle programs*, $TM_n \in \Sigma^*$, are defined iteratively by setting $TM_0 = F$ and letting $TM_{n+1} = \sigma(TM_n) = TM_n\overline{TM_n}$. We obtain

$$\begin{aligned} TM_0 &= F \\ TM_1 &= FL \\ TM_2 &= FLLF \\ TM_3 &= FLLFLFFL \\ TM_4 &= FLLFLFFLLFFLFLLF \\ &\vdots \end{aligned}$$

In the proper closure of Σ^*, it can be seen that $\lim_{n\to\infty} \sigma^n(F)$ exists; the limit is called the Thue-Morse sequence:

$$FLLFLFFLLFFLFLLFLFFLFLLFFLLFLFFL\ldots$$

In 2005 J. Ma and J. Holdener used turtle geometry and polygon maps to realize the Thue-Morse sequence as the limit of polygonal curves in the plane [4]. After scaling, the sequence of turtle trajectories encoded by the even iterates $\{TM_{2n}\}_{n\geq 1}$ converges to the Koch snowflake in the Hausdorff metric (see Theorem 5.0.14 of [4] and Figure 1 below). Moreover, Ma and Holdener showed that certain "generalized Thue-Morse sequences" also encode turtle trajectories converging to the Koch snowflake.

In this paper we revisit the notion of generalized Thue-Morse turtle programs. In particular, we answer the question posed in the final paragraph of, see [4]:

> *Is it possible to find a $w \in \Sigma^*$ not of the form TM_{2n} or $\overline{TM}_{2n}$ such that w generates turtle programs $\{\sigma^{2n}(w)\}_{n\geq k}$ that encode turtle trajectories converging to the Koch snowflake?*

Answering this question in the affirmative, we provide an infinite family of words of the form $w = F^a L^b F^a$ that encode the von Koch curve via the iteration map σ.

[1] If we assume each m_i and n_i (except perhaps m_1 and n_k) is positive, then each $w \in \Sigma^*$ has a unique representation, and this operation is well-defined, see [2].

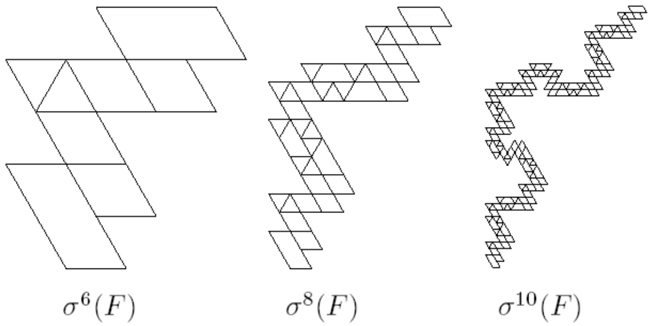

Figure 1: Even Thue-Morse turtle trajectories converging to the von Koch curve

2. Preliminaries

Turtle geometry arose in the early 1980's with the work of the Logo group at MIT, see [1], [3]. A turtle state is any pair $(z, \eta) \in \mathbb{C} \times \mathbb{S}^1$, where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = \sqrt{z\overline{z}} = 1\}$. This can be interpreted as a position and a heading in the complex plane. Throughout this paper, let $\epsilon = e^{2\pi i/6}$. We define turtle transformations $T_w : \mathbb{C} \times \mathbb{S}^1 \to \mathbb{C} \times \mathbb{S}^1$ for all $w \in \Sigma^*$ recursively as follows: For all $(z, \eta) \in \mathbb{C} \times \mathbb{S}^1$, set

$$\begin{aligned} T_F(z, \eta) &= (z + \eta, \eta), \\ T_L(z, \eta) &= (z, \epsilon\eta). \end{aligned}$$

To complete the definition, define $T_{w_1 w_2} = T_{w_2} T_{w_1}$ for all $w_1, w_2 \in \Sigma^*$. Since L and F generate Σ^*, this sufficiently defines T_w for all words $w \in \Sigma^*$.

It will be convenient to let $g : \Sigma^* \to \mathbb{C}$ be the mapping defined by $g(w) = z$ where $(z, \eta) = T_w(0, 1)$. We will also refer to the mapping $\vec{g} : \Sigma^* \to \mathbb{R} \times \mathbb{R}$, which is related to g via the natural isomorphism between $\mathbb{C}$ and $\mathbb{R} \times \mathbb{R}$.

We say that $w \in \Sigma^*$ *generates* the von Koch curve via the map σ if the sequence of turtle trajectories encoded by the turtle programs $\{\sigma^{2n}(w)\}_{n \geq 0}$ converges to the von Koch curve in the Hausdorff metric (see [4] for more clarification of this definition). Ma and Holdener provided the following characterization of words generating the von Koch curve [4].

Theorem 1. *A word w_1 generates the Koch snowflake if and only if, for some $n \in \mathbb{N} \cup \{0\}$, $w = \sigma^n(w_1)$ satisfies the following:*

C1: $|w|_F \equiv_6 |w|_L \equiv_6 \pm 2$.

C2: *$\overrightarrow{g(w)}$ and $\overrightarrow{g(\overline{w})}$ point in opposite directions.*

Associating the vector $\overrightarrow{g(w)} \in \mathbb{R} \times \mathbb{R}$ with $g(w) \in \mathbb{C}$, C2 implies that $g(w) = re^{i\phi}$ and $g(\overline{w}) = se^{i(\phi+\pi)}$ for some $r, s > 0$ and $\phi \in \mathbb{R}$. Taking this one step further, we see that the second condition is equivalent to $g(\overline{w})/g(w) = -s/r < 0$, which in turn is equivalent to $\overline{g(w)}g(\overline{w}) < 0$. Hence our conditions can be restated as:

C1: $|w|_F \equiv_6 |w|_L \equiv_6 \pm 2$.
C2: $\overline{g(w)}g(\overline{w}) < 0$.

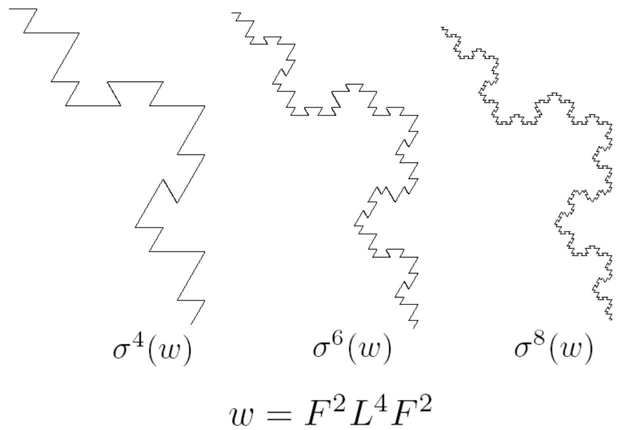

Figure 2: Two-switch case #1, where $w = F^2L^4F^2$

3. An Infinite Family that Works

We wish to determine whether there exists $w \in \Sigma^*$ not of the form TM_n that satisfies C1 and C2. By the symmetry of C1 and C2 we can assume, without loss of generality, that the first letter in each word is F. Each result which follows can then be easily translated into the general result allowing for words which begin with an L.

For each $w = \prod_{i=1}^{k} F^{m_i} L^{n_i} \in \Sigma^*$, define $|w|_S$, the number of switches in w, by

$$|w|_S = \begin{cases} 2k-2, & \text{if } n_k = 0; \\ 2k-1, & \text{if } n_k > 0. \end{cases}$$

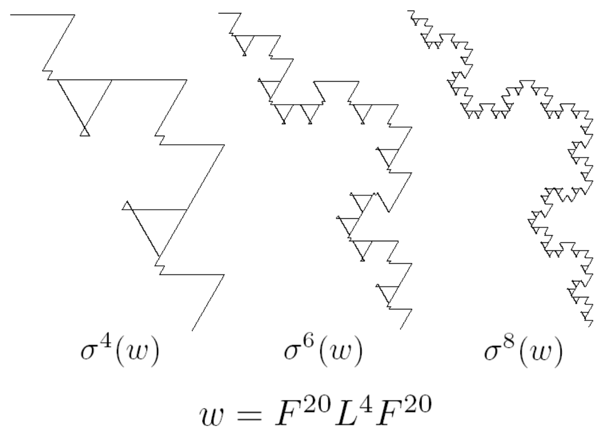

Figure 3: Two-switch case #2, where $w = F^{26}L^4F^{26}$

The number of switches in w is the number of times the letters in the word "switch" from L to F or from F to L when reading from left to right.

For small switch numbers, we can characterize which words satisfy C1 and C2. If $|w|_S = 0$, for example, then $w = F^a$ for some $a > 0$. In this case, $g(w) = a$ and $g(\overline{w}) = 0$. So $\overline{g(w)}g(\overline{w}) = 0$, contradicting C2. Thus there are no words w with $|w|_S = 0$ such that C1 and C2 are satisfied.

Moving on to the case, where $|w|_S = 1$, we have $w = F^aL^b$ for some $a, b > 0$. So $g(w) = a$ and $g(\overline{w}) = b\epsilon^a$. If w satisfied C2, then $0 = \mathrm{Im}(\overline{g(w)}g(\overline{w})) = ab\,\mathrm{Im}(\epsilon^a)$, from which we conclude that $3 \mid a$. But then w fails to satisfy C1 since $|w|_F = a \not\equiv_6 \pm 2$. Thus there are no words w with $|w|_S = 1$ such that C1 and C2 are satisfied.

Having shown that no word with fewer than two switches satisfies C1 and C2, we now fully characterize words that have exactly two switches.

Theorem 2. *A word w of the form $w = F^aL^bF^c$ satisfies C1 and C2 if and only if $a = c \equiv_6 -b \equiv_6 \pm 2$.*

Proof. First suppose $w = F^aL^bF^c$ where $a = c \equiv_6 -b \equiv_6 2$ (the case, where $a = c \equiv_6 -b \equiv_6 -2$ is similar). Then

$$\begin{aligned} g(w) &= a + c\epsilon^b = a(1+\epsilon^{-2}) = -a\epsilon^2, \\ g(\overline{w}) &= b\epsilon^a = b\epsilon^2, \end{aligned}$$

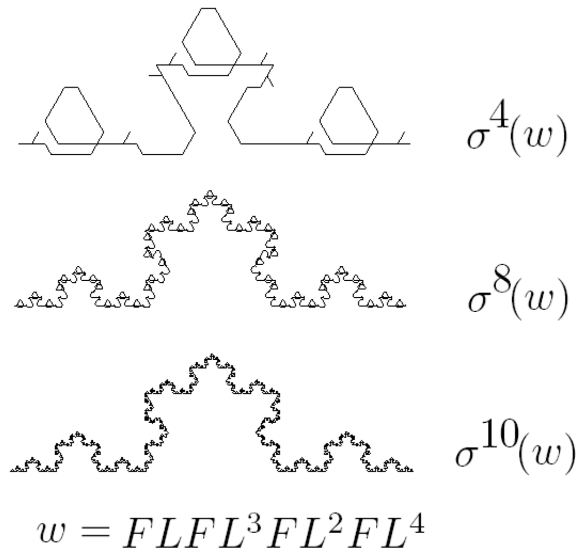

Figure 4: Seven-switch case

and hence $\overline{g(w)}g(\overline{w}) = -ab < 0$. Since

$$|w|_F = a + c \equiv_6 4 \equiv b = |w|_L,$$

w satisfies conditions C1 and C2.

Now suppose $w = F^a L^b F^c$ satisfies C1 and C2. By C1, $b = |w|_L \equiv_6 \pm 2$. We assume $b \equiv_6 2$; proving the case $b \equiv_6 -2$ is similar. C2 implies

$$\begin{aligned} 0 = \mathrm{Im}(\overline{g(w)}g(\overline{w})) &= \mathrm{Im}((a + c\epsilon^{-b})(b\epsilon^a)) \\ &= \mathrm{Im}((a + c\epsilon^{-2})(b\epsilon^a)) = -\frac{\sqrt{3}}{2}bc\cos\frac{\pi a}{3} + \left(a - \frac{c}{2}\right) b \sin\frac{\pi a}{3}. \end{aligned}$$

We know that $2a \neq c$ since otherwise $|w|_F = a + c \equiv_3 \pm 0$, contradicting C1. Also, $b > 0$, so

$$\sqrt{3}\left(\frac{c}{2a - c}\right) = \tan\frac{\pi a}{3}.$$

Now since $c > 0$, we have $\tan(\pi a/3) \neq 0$ and, consequently, that a is congruent to 1, 2, 4, or 5 modulo 6. In each of these cases $\tan(\pi a/3) = \pm\sqrt{3}$, so we have $c = \pm(2a - c)$. The only positive integer solution to this is $a = c$, in which case $\tan(\pi a/3) = \sqrt{3}$. Thus a is congruent to 1 or 4 modulo 6. Using the full statement of C2, we have

$$0 > \overline{g(w)} g(\overline{w}) = (a\epsilon^{-1})(b\epsilon^{a}) = ab\epsilon^{a-1},$$

from which we obtain $a \equiv_6 4 \equiv_6 -2$, as desired. □

We see that there are infinitely many words not of the form TM_n that encode the von Koch curve under iteration of the σ map. Some examples of two-switch words that generate the Koch snowflake can be seen in the pictures below.

4. Discussion and Future Work

Having completely mapped out the two-switch case, we turn our attention to words with more than two switches. One can show that no word with three switches satisfies C1 and C2, and we believe that for any number of switches larger than three, a word exists that satisfies C1 and C2. For example, in the four switch case, consider a word of the form

$$w = F^a L^b F^c L^b F^a,$$

where $a \equiv_6 -b \equiv_6 \pm 1$ and $c \equiv_6 2$. This word satisfies C1 and C2. Also, the words $FLF^4L^2F^5L$, $FLF^3L^3F^9L^4F$, and $FLFL^3FL^2FL^4$, which have 5, 6, and 7 switches, respectively, satisfy C1 and C2. One seven-switch case can be seen below.

Future work could involve rigorously proving these facts and classifying such words that have more than three switches.

References

[1] H. Abelson, A.A. diSessa, *Turtle Geometry: The Computer as a Medium for Exploring Mathematics*, MIT Press, Cambridge, MA (1981).

[2] J. Holdener, A. Wagaman, A classification of periodic turtle sequences, *Int. J. Math. Math. Sci.*, **34** (2003), 2193-2201.

[3] J. King, Geometry and computers, *Computers and Mathematics*, MAA Notes **9**, Washington, DC (1981), 101-114.

[4] J. Ma, J. Holdener, When Thue-Morse meets Koch, *Fractals*, **13** (2005), 191-206.