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MONADS ON PROJECTIVE VARIETIES

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Abstract: Here we study existence or non-existence of certain linear monads (or the semistability of their cohomology sheaves) on certain projective varieties, mainly if $\mathrm{Num}(X) \cong \mathbb{Z}$. We use in an essential way the proofs in a paper by Fløystad and in a paper by Jardin and Miró-Roig: we just easily extend their proofs to our more general set-up.

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1. Introduction

Here we study existence or non-existence of certain linear monads (or the semistability of their cohomology sheaves) on certain projective varieties, mainly if X has a unique polarization, up to numerical equivalence, i.e. if $\mathrm{Num}(X) \cong \mathbb{Z}$. We use in an essential way the proofs in a paper by Fløystad and in a paper by Jardin and Miró-Roig: we just easily extend their proofs to our more general set-up. We were motivated by the extension to odd-dimensional smooth quadrics of the notion of instanton bundle to the case of od-dimensional smooth quadrics given in [2] and for the study of linear sheaves made in [2] and [8]. We start with an easy existence theorem, which just use that any variety admits a map to a projective space and the applying the existence part of [3], Main Theorem.

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Proposition 1. *Let X be an integral projective variety and $R \in \mathrm{Pic}(X)$. Assume that R is spanned. Set $n := \dim(X)$. Fix positive integers a, b, c. There is monad on X of the form*

$$R^{\oplus a} \to \mathcal{O}_X^{\oplus b} \to R^{\oplus c} \tag{1}$$

if one of the following conditions is satisfied:

(i) $b \geq 2c + n - 1$ *and* $b \geq a + c$*;*

(ii) $b \geq a + c + n$*.*

If either (ii) is satisfied or n is odd, $a = c$, and $b + a + c + n - 1$, then there is a monad (1) whose cohomology is locally free.

The necessity part in [3], Main Theorem, was extended to monads of arbitrary direct sums of line bundles under the restriction that the cohomology sheaf of the monad is locally free (see [9], Theorem 1 and its proof). To extend some parts of [8] we introduce the following definition.

Definition 1. Let X be an integral projective variety such that $\mathrm{Num}(X) \equiv \mathbb{Z}$. We will say that $L \in \mathrm{Pic}(X)$ has degree d if d is its equivalence class in $\mathrm{Num}(X)$. We take the convection that "degree > 0" is equivalent to "ample". Fix an integer t A rank r vector bundle A on X will be said to be *purely filtrable with weight t* if it has an increasing filtration by subsheaves $\{A_i\}_{0 \leq i \leq r}$ such that $A_0 = 0$, $A_r = A$ and A_i/A_{i-1} is a line bundle of degree t. Fix positive integers a, b, r, c, x. A monad

$$A \xrightarrow{u} \mathcal{O}_X^{\oplus b} \xrightarrow{v} D \tag{2}$$

will be said of linear and of type $(a, b, c; x)$ if A (resp. D) is a rank a (resp. rank c) purely filtrable vector bundle with weight $-x$ (resp. weight x). A rank r torsion free sheaf E on X will be called an *instanton sheaf with charge c and weight x* if it is the cohomology of a linear monad of type $(c, r + 2c, c; x)$.

Take any integral variety X as in Definition 1. For any two ample line bundles R, L on X there are positive integers a, b such that $R^{\otimes a}$ is numerically equivalent to $L^{\otimes b}$. Hence the notion of slope stability and slope semistability for torsion free sheaves on X does not depend from the choice of a polarization on X. We will prove the following result.

Theorem 1. *Let X be a smooth and projective variety such that $\mathrm{Num}(X) \equiv \mathbb{Z}$. Assume $n := \dim(X) \geq 2$. Let E be a locally free instanton sheaf on X with rank $r \leq 2n - 1$. Then E is slope semistable.*

Theorem 2. *Let X be a smooth and projective variety such that* $\mathrm{Num}(X) \equiv \mathbb{Z}$. *Assume $n := \dim(X) \geq 2$. Let E be a rank $r \leq n$ locally free sheaf on X. Assume that E is linear and of type $(a, r+a+c, c; x)$. If $\deg(E) \neq 0$, i.e. if $a \neq c$, then E is stable.*

We work over an algebraically closed field $\mathbb{K}$. Except in the set-up of Proposition 1 we assume $\mathrm{char}(\mathbb{K}) = 0$ to apply Kodaira vanishing.

2. The Proofs and Related Results

Proof of Proposition 1. Since R is spanned, there is a linear subspace $V \subseteq H^0(X, E)$ such that V spans R and $k := \dim(V) - 1 \leq n$. If $k = 0$, then $R \cong \mathcal{O}_X$ and everything is obvious. If $1 \leq k \leq n$, then the pair (R, V) induces a morphism $f : X \to \mathbb{P}^k$ such that $R \cong f^*(\mathcal{O}_{\mathbb{P}^k}(1))$. We may take as monad (1) the pull-back of a monad given by the statement of [3], Main Theorem. The same result gives a monad with locally free cohomology (and hence whose pull-back has locally free cohomology) if $b \geq a + b + k$ and in particular if $b \geq a + b + n$. If n is odd, $a = c$, and $b = a + c + n - 1$, then use the pull-back of the monad given any instanton bundle on $\mathbb{P}^n$ with Chern number c (see [10], [1] or [3] Corollary 1). □

Remark 1. Let X be an integral projective variety such that $\mathrm{Num}(X) \equiv \mathbb{Z}$. Note that $h^0(X, R) = 0$ for every line bundle R such that $\deg(R) < 0$.

Remark 2. Let X be a smooth and connected projective variety such that $\mathrm{Num}(X) \equiv \mathbb{Z}$. Let A be a purely filtrable vector bundle on X with weight $y < 0$. Applying Kodaira vanishing for line bundles to the subquotients of the filtration of A we get $h^i(X, A) = 0$ for all $0 \leq i < \dim(X)$. Notice that each symmetric power of A and each alternating power of A satisfies the same vanishing for the same reason.

Remark 3. Let X be a locally factorial integral projective variety such that $\mathrm{Num}(X) \equiv \mathbb{Z}$. For any rank s reflexive sheaf A on X the sheaf $\bigwedge^s(F)$ is reflexive and with rank 1. Since X is locally factorial, $\bigwedge^s(F)$ is locally free. Its degree is called the degree of A or of $\det(A)$. There is a unique integer k_A such that $-s + 1 \leq \deg(A) + s \cdot k_A \leq 0$. Let E be a rank $r \geq 2$ locally free sheaf on X. There is a unique integer k_E such that $-r + 1 \leq \deg(E) + r \cdot k_E \leq 0$. Here we want to show that Hoppe's criterion of slope stability or slope stability ([6], Lemma 2.6, or [8], Proposition 4) extends to our set-up. Fix an integer q such that $1 \leq q \leq r - 1$ and take a rank q subsheaf F of E saturated in X and with maximal slope. Since E is locally free and F is saturated in E, F is reflexive.

The inclusion $F \hookrightarrow E$ induces and inclusion $\bigwedge^q(F) \hookrightarrow \bigwedge^q(E)$. Since $\bigwedge^q(F)$ is reflexive and with rank 1 and X is locally factorial, $R := \bigwedge^q(F)$ is a line bundle. Thus we get $h^0(X, \bigwedge^q(E) \otimes R^*) > 0$. Notice that $\mu(\bigwedge^q(F)) = q \cdot \mu(F)$ and $\mu(\bigwedge^q(E)) = q \cdot \mu(E)$. Hence to see that E is not destabilized by rank q subsheaves, it is sufficient to prove $h^0(X, \bigwedge^q(E) \otimes M) = 0$ for all $M \in \mathrm{Pic}(X)$ such that $\deg(M) \leq k_{\bigwedge^q(A)}$.

Look at the monad (2). The vector bundle $K := \mathrm{Ker}(v)$ is called the kernel bundle of the monad (2). For any $R \in \mathrm{Pic}(X)$ the monad (2) gives the exact sequences

$$0 \to A \otimes R \to K \otimes R \to E \otimes R \to 0 \tag{3}$$

$$0 \to K \otimes R \to R^{\oplus b} \to D \otimes R \to \tag{4}$$

Proposition 2. *Let X be a smooth and projective variety such that* $\mathrm{Num}(X) \equiv \mathbb{Z}$. *Assume $n := \dim(X) \geq 2$. Let E be a rank 2 instanton sheaf on X. Then E is slope semistable.*

Proof. Since X is smooth, every rank 1 reflexive sheaf on X is locally free ([5], Proposition 1.9). Take the monad (2) with $a = c$ and $b = r + 2c$. Since $\deg(\det(E)) = 0$, it is sufficient to prove $h^0(X, E \otimes R) = 0$ for every $R \in \mathrm{Pic}(X)$ such that $\deg(R) < 0$. Fix any $R \in \mathrm{Pic}(X)$ such that $\deg(R) < 0$. From (1) we get $h^0(X, K \otimes R) = 0$. Remark 2 gives $h^1(X, A \otimes R) = 0$. Hence $h^0(X, E \otimes R) = 0$. □

Proofs of Theorems 1 and 2. Remark 2 and (2) are exactly what is needed to follow line by line respectively the proofs of [8], Theorem 3 and Theorem 7. □

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