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NUMERICAL INTEGRATORS FOR FOURTH ORDER INITIAL AND BOUNDARY VALUE PROBLEMS

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Abstract: In this paper, a continuous k-step linear multistep method (LMM) is developed and used to generate new finite difference methods (NFDMs), which are assembled and applied as simultaneous numerical integrators to solve fourth order initial and boundary value problems without reducing them to an equivalent first order system. The NFDMs are analyzed for convergence via consistency and zero-stable by conveniently expressing them as block methods. The initial value problems (IVPs) are solved without the need for either predictors or starting values from other methods, while the boundary value problems (BVPs) are solved by assembling the NFDMs into a single block matrix equation. We illustrate our process using a specific example for $k = 4$. Numerical examples are given to show the efficiency of the methods.

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1. Introduction

Fourth order initial and boundary value problems frequently arise in several areas of science and engineering. For instance, in practice, the bending of an elastic beam is described with a fourth-order BVP. It is commonly known that several fourth order ordinary differential equations (ODEs) do not have theoretical solutions, hence looking for numerical solutions is imperative. Therefore, we propose NFDMs for solving the fourth-order ordinary differential equation

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of the form

$$\aleph y \equiv y^{(iv)} = f(x, y, y', y'', y'''), \tag{1}$$

subject to the initial conditions

$$y(a) = y_0,\ y'(a) = \delta_0,\ y''(a) = \gamma_0,\ y'''(a) = \eta_0$$

or boundary conditions

$$y(a) = y_0,\ y'(a) = \delta_0,\ y(b) = y_N,\ y'(b) = y_{N1}\,,$$

where f satisfies a Lipschitz condition subject to initial conditions, as given in Henrici [9]. Moreover, Keller [15] has given the theorem and the proof of the general conditions which ensure that the solution to (1) subject to boundary conditions will exist and be unique. We note that the NFDMs can be extended without any difficulty to solve other problems with boundary conditions such as $y(a) = y_0,\ y''(a) = \gamma_0,\ y'(b) = y_{N1},\ y''(b) = y_{N2}$ and $y(a) = y_0,\ y'''(a) = \eta_0,\ y'(b) = y_{N1},\ y'''(b) = y_{N3}$.

Conventionally, problem (1) is solved by first reducing it to an equivalent system of first-order ODEs and then applying the various methods available for solving systems of first order IVPs or BVPs. This approach is extensively discussed in the literature, for instance, see Lambert [16], [17], Brugnano and Trigiante [3], Onumanyi et al [20], [19], Fatunla [6], and Jennings [14]. It was shown in Awoyemi [2] that a direct solution approach to (1) involves less computational work. However, in [2], the direct approach to (1) subject to initial conditions was implemented in a predictor-corrector mode using the Taylor series algorithm to supply the starting values. Although, the implementation of the method yielded reasonable accuracy, the procedure is more costly to implement since the subroutines for incorporating starting values lead to longer computer time and more human effort. Furthermore, the method in [2] is not applicable to BVPs. Thus, in this paper, we are motivated to develop NFDMs which are applicable to both IVPs and BVPs, self-starting, and hence cheaper to implement.

Recently, Jator and Li [13], Jator [10], [12] proposed LMMs for the direct solution of the general second and third order IVPs, which were shown to be zero stable and implemented without the need for either predictors or starting values from other methods. In Jator [11] the LMMs developed for IVPs were used with additional methods obtained from the same continuous scheme to solve two-point third order BVPs. In this paper, we developed NFDMs, which are assembled and applied as simultaneous numerical integrators to solve fourth order initial and boundary value problems without reducing them to an equivalent first order system. We also show that the NFDMs are zero-stable, consistent, and hence convergent. The main continuous k-step LMM is derived through interpolation and collocation, see Lie and Norsett [18], Atkinson [1], Onumanyi et al [19], and Gladwell and Sayers [8].

The paper is organized as follows. In Section 2, we derive an approximation $Y(x)$ for the exact solution $y(x)$ which is continuous. Section 3 is devoted to the specification of the methods and how the NFDMs are obtained. The analysis and implementation of the methods are discussed in Section 4. Numerical examples are given in Section 5 to show the efficiency of the methods. Finally, the conclusion of the paper is discussed in Section 6.

2. Derivation of the Methods

In this section, we approximate the exact solution $y(x)$ by seeking the continuous method $Y(x)$ of the form

$$Y(x) = \sum_{j=0}^{r+s-1} \lambda_j \Upsilon_j(x)\,, \tag{2}$$

where $x \in [a,b]$, λ_j's are unknown coefficients and $\Upsilon_j(x)$'s are polynomial basis functions of degree $r+s-1$. The number of interpolation points r and the number of distinct collocation points s are chosen to satisfy $4 \leq r \leq k$, and $0 < s \leq k+1$ respectively. The positive integer $k \geq 4$ denotes the step number of the method. We then construct a k-step multistep collocation method by imposing the following conditions.

$$Y\left(x_{n+j}\right) = y_{n+j}, \quad j = 0,1,2,\ldots,r-1\,, \tag{3}$$

$$\aleph Y\left(x_{n+j}\right) = f_{n+j}, \quad j = 0,1,2,\ldots,s-1\,. \tag{4}$$

Equations (3) and (4) lead to a system of $(r+s)$ equations, which is solved to obtain the λ_j's. We proceed by considering the following notations.

We define the interpolation/collocation matrix $\bigvee$ of dimension $(r+s)\times(r+s)$ as

$$\bigvee = \begin{pmatrix} P_0(x_n) & \cdots & P_{r+s-1}(x_n) \\ P_0(x_{n+1}) & \cdots & P_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ P_0(x_{n+r-1}) & \cdots & P_{r+s-1}(x_{n+r-1}) \\ \aleph P_0(x_n) & \cdots & \aleph P_{r+s-1}(x_n) \\ \aleph P_0(x_{n+1}) & \cdots & \aleph P_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ \aleph P_0(x_{n+s-1}) & \cdots & \aleph P_{r+s-1}(x_{n+s-1}) \end{pmatrix}$$

and consider further notations by defining the following vectors:

$$\bigwedge = (y_n, y_{n+1}, \ldots, y_{n+r-1}, f_n, f_{n+1}, \ldots, f_{n+s-1})^T\,,$$
$$\Upsilon(x) = (P_0(x), P_1(x), \ldots, P_{r+s-1}(x))^T\,,$$
$$W = (\lambda_0, \lambda_1, \ldots, \lambda_{n+r-1})^T\,,$$

where T denotes the transpose of the vectors. It worth noting that $\Upsilon(x)$ represents a vector of arbitrary basis functions. The collocation points are selected from the extended set Ψ, where

$$\Psi = \{x_n, \ldots, x_{n+k}\} \bigcup \{x_{n+k-1}, x_{n+k}\}\,.$$

Theorem 2.1. *Let $Y(x)$ satisfy conditions (3) and (4), then, the continuous k-step LMM is constructed from the following equation:*

$$Y(x) = \overset{T}{\bigwedge} \left(\overset{-1}{\bigvee}\right)^T \Upsilon(x)\,.$$

Proof. We begin the proof by first writing (2) in vector form as follows:

$$Y(x) = W^{\,T}\Upsilon(x)\,. \tag{5}$$

We also write the system obtained from (3) and (4) in matrix form as follows:

$$\bigvee W = \bigwedge\,. \tag{6}$$

We assume that $\bigvee$ is non-singular and hence invertible, it follows from (6) that

$$W = \bigwedge \bigvee^{\,-1}\,. \tag{7}$$

Thus, from (5) and (7) we obtain

$$Y(x) = \overset{T}{\bigwedge} \left(\overset{-1}{\bigvee}\right)^T \Upsilon(x)\,,$$

hence, the proof is complete. □

The k-step LMM is obtained from Theorem 2.1 after some manipulation and expressed in the form

$$Y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^4 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j}\,, \tag{8}$$

where the $\alpha_j(x)$'s and $\beta_j(x)$'s are the continuous coefficients. The continuous k-step LMM (8) is used to generate NFDMs, which are applied as simultaneous numerical integrators to provide the discrete solution to (1). In this light, we seek a solution on the mesh

$$\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \cdots < x_N = b\,,$$

where π_N is a partition of $[a,b]$ and $h=(b-a)/N$ is the constant step-size of the partition of π_N.

3. Specification of the Methods

In this section, we obtain the NFDMs from (8) and the formulas

$$Y'(x)=\frac{1}{h}\left(\sum_{j=0}^{r-1}\alpha_j'(x)y_{n+j}+h^4\sum_{j=0}^{s-1}\beta_j'(x)f_{n+j}\right), \tag{9}$$

$$Y''(x)=\frac{1}{h^2}\left(\sum_{j=0}^{r-1}\alpha_j''(x)y_{n+j}+h^4\sum_{j=0}^{s-1}\beta_j''(x)f_{n+j}\right), \tag{10}$$

$$Y'''(x)=\frac{1}{h^3}\left(\sum_{j=0}^{r-1}\alpha_j'''(x)y_{n+j}+h^4\sum_{j=0}^{s-1}\beta_j'''(x)f_{n+j}\right), \tag{11}$$

which are generated by imposing that

$$Y'(x)=\delta(x),\quad Y''(x)=\gamma(x),\quad Y'''(x)=\eta(x)\,, \tag{12}$$

$$Y'(a)=\delta_0,\quad Y''(a)=\gamma_0,\quad Y'''(a)=\eta_0\,. \tag{13}$$

generate MFDMs.

In particular, we use (8) to obtain a continuous k-step LMM by specifying r, s, k, and $\Upsilon_j(x)$. We emphasize that the main method is obtained by evaluating (8) at $x=x_{n+k}$. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of z for convenience, where $z=(x-x_{n+k-1})/h$. The coefficients $\alpha_j'(x)$, $\alpha_j''(x)$, $\alpha_j'''(x)$, $\beta_j'(x)$, $\beta_j''(x)$ and $\beta_j'''(x)$ are easily obtained from the first, second and third derivatives of $\alpha_j(x)$ and $\beta_j(x)$. We discuss details of a specific method next.

Case $k=4$. We use (8) to obtained a continuous 4-step method with the following specifications: $r=4$, $s=5$, $k=4$, $\Upsilon_i(x)=x^i$, $i=0,1,\ldots,8$. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of z, where $z=(x-x_{n+3})/h$ in what follows.

$$\alpha_0(z)=\frac{1}{6}(-2z-3z^2-z^3);\;\; \alpha_1(z)=\frac{1}{2}(3z+4z^2+z^3),$$
$$\alpha_2(z)=\frac{1}{2}(-6z-5z^2-z^3);\quad \alpha_3(z)=\frac{1}{6}(6+11z+6z^2+z^3),$$
$$\beta_0(z)=\frac{1}{120960}(-138z-73z^2+126z^3-84z^5-14z^6+12z^7+3z^8),$$
$$\beta_1(z)=\frac{1}{30240}(1866z+2593z^2+630z^3+126z^5+14z^6-18z^7-3z^8),$$

$$\beta_2(z) = \frac{1}{20160}(3630z + 6619z^2 + 3234z^3 - 252z^5 + 14z^6 + 24z^7 + 3z^8),$$
$$\beta_3(z) = \frac{1}{30240}(282z + 1417z^2 + 2142z^3 + 1260z^4 + 210z^5 - 70z^6 - 30z^7 - 3z^8),$$
$$\beta_4(z) = \frac{1}{120960}(6z - 241z^2 - 378z^3 + 252z^5 + 154z^6 + 36z^7 + 3z^8).$$

The following symmetric 4-step LMM is obtained by evaluating (8) at $x = x_{n+4}$.

$$y_{n+4} - 4y_{n+3} + 6y_{n+2} - 4y_{n+1} + y_n = \frac{h^4}{720}\left(-f_n + 124f_{n+1} + 474f_{n+2} + 124f_{n+3} - f_{n+4}\right). \quad (14)$$

In particular, to start the initial value problem for $n = 0$, we obtain the following equations from (13):

$$h\delta_0 = -\frac{11}{6}y_0 + 3y_1 - \frac{3}{2}y_2 + \frac{1}{3}y_3 + \frac{h^4}{20160}\left(-193f_0 - 3620f_1 - 1254f_2 + 28f_3 - f_4\right), \quad (15)$$

$$h^2\gamma_0 = 2y_0 - 5y_1 + 4y_2 - y_3 + \frac{h^4}{60480}\left(4463f_0 + 42124f_1 + 7962f_2 + 1132f_3 - 241f_4\right), \quad (16)$$

$$h^3\sigma_0 = -y_0 + 3y_1 - 3y_2 + y_3 + \frac{h^4}{160}\left(-53f_0 - 184f_1 + 10f_2 - 16f_3 + 3f_4\right). \quad (17)$$

On the other hand, the methods for BVPs are developed by invoking the following remark.

Remark 3.1. *The functions $\delta(x)$, $\gamma(x)$, and $\eta(x)$ are piecewise continuous on the sub-intervals $[x_0, x_k], \ldots, [x_{N-k}, x_N]$ of $[a, b]$ with matching points at $x_{n+k}, n = 0, k, 2k, \ldots, N - 2k$.*

From Remark 3.1, for $k = 4$, we impose that $\delta(x)$, $\gamma(x)$ and $\eta(x)$ are continuous at $x = x_{n+4}$. Hence, the following three additional methods

$$11y_n - 42y_{n+1} + 57y_{n+2} - 26y_{n+3} - 11y_{n+4} + 18y_{n+5} - 9y_{n+6} + 2y_{n+7} = \frac{h^4}{10080}\left(-151f_n + 19012f_{n+1} + 76758f_{n+2} + 29956f_{n+3} + 1004f_{n+4} + 10860f_{n+5} + 3762f_{n+6} - 84f_{n+7} + 3f_{n+8}\right), \quad (18)$$

$$\begin{aligned}
&-2y_n + 7y_{n+1} - 8y_{n+2} + 3y_{n+3} - 2y_{n+4} + 5y_{n+5} - 4y_{n+6} + y_{n+7} \\
&\quad = \frac{h^4}{60480}(409f_n - 21964f_{n+1} - 87594f_{n+2} - 62956f_{n+3} + 168f_{n+4} \\
&\qquad + 42124f_{n+5} + 7962f_{n+6} + 1132f_{n+7} - 241\,, f_{n+8}) \qquad (19)
\end{aligned}$$

$$\begin{aligned}
&y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3} - y_{n+4} + 3y_{n+5} - 3y_{n+6} + y_{n+7} \\
&\quad = \frac{h^4}{1440}(-29f_n + 392f_{n+1} + 858f_{n+2} + 1904f_{n+3} + 952f_{n+4} + 1656f_{n+5} \\
&\qquad - 90f_{n+6} + 144f_{n+7} - 27f_{n+8})\,, \qquad (20)
\end{aligned}$$

are obtained from (12) with continuity equations imposed at $x = x_{n+4}$ as follows:

$$\lim_{x\to x_{n+4}^-} \delta(x) = \lim_{x\to x_{n+4}^+} \delta(x), \quad \lim_{x\to x_{n+4}^-} \gamma(x) = \lim_{x\to x_{n+4}^+} \gamma(x),$$
$$\lim_{x\to x_{n+4}^-} \eta(x) = \lim_{x\to x_{n+4}^+} \eta(x)\,,$$

where

$$\delta(x) = \begin{cases} \frac{1}{h}(-\frac{11}{6}y_n + 7y_{n+1} - \frac{19}{2}y_{n+2} + \frac{13}{3}y_{n+3} + \frac{h^4}{60480}(-151f_n + 19012f_{n+1} \\ +76758f_{n+2} + 29956f_{n+3} + 425f_{n+4})), \quad x_n \le x \le x_{n+4} \\ \frac{1}{h}(-\frac{11}{6}y_{n+4} + 3y_{n+5} - \frac{3}{2}y_{n+6} + \frac{1}{3}y_{n+7} + \frac{h^4}{20160}(-193f_{n+4} \\ -3620f_{n+5} - 1254f_{n+6} + 28f_{n+7} - f_{n+8}))\,, \end{cases} \qquad (21)$$

$$\gamma(x) = \begin{cases} \frac{1}{h^2}(-2y_n + 7y_{n+1} - 8y_{n+2} + 3y_{n+3} + \frac{h^4}{60480}(-409f_n + 21964f_{n+1} \\ +87594f_{n+2} + 62956f_{n+3} + 4295f_{n+4})), \quad x_n \le x \le x_{n+4}\,, \\ \frac{1}{h^2}(2y_{n+4} - 5y_{n+5} + 4y_{n+6} - y_{n+7} + \frac{h^4}{60480}(4463f_{n+4} + 42124f_{n+5} \\ +7962f_{n+6} + 1132f_{n+7} - 241f_{n+8})), \quad x_{n+4} \le x \le x_{n+8}\,, \end{cases} \qquad (22)$$

$$\eta(x) = \begin{cases} \frac{1}{h^3}(-y_n + 3y_{n+1} - 3y_{n+2} + y_{n+3} + \frac{h^4}{1440}(-29f_n + 392f_{n+1} \\ +858f_{n+2} + 1904f_{n+3} + 475f_{n+4})), \quad x_n \le x \le x_{n+4}\,, \\ \frac{1}{h^3}(-y_{n+4} + 3y_{n+5} - 3y_{n+6} + y_{n+7} + \frac{h^4}{160}(-53f_{n+4} - 184f_{n+5} \\ +10f_{n+6} - 16f_{n+7} + 3f_{n+8})), \quad x_{n+4} \le x \le x_{n+8}\,. \end{cases} \qquad (23)$$

It is worth noting that the derivatives are provided by $\delta(x_{n+\tau}) = \delta_{n+\tau}$, $\gamma(x_{n+\tau}) = \gamma_{n+\tau}, \eta(x_{n+\tau}) = \eta_{n+\tau}, \tau = 1, \ldots, 4$ as follows:

$h\delta_{n+1} = -\frac{1}{3}y_n - \frac{1}{2}y_{n+1} + y_{n+2} - \frac{1}{6}y_{n+3} + \frac{h^4}{60480}(-31f_n + 2908f_{n+1} + 2382f_{n+2}$

$$
\begin{aligned}
&\qquad\qquad -260f_{n+3}+41f_{n+4}),\\
h\delta_{n+2} &= \tfrac{1}{6}y_n - y_{n+1} + \tfrac{1}{2}y_{n+2} + \tfrac{1}{3}y_{n+3} + \tfrac{h^4}{60480}(55f_n - 1972f_{n+1} - 3318f_{n+2}\\
&\qquad +236f_{n+3} - 41f_{n+4}),\\
h\delta_{n+3} &= -\tfrac{1}{3}y_n + \tfrac{3}{2}y_{n+1} - 3y_{n+2} + \tfrac{11}{6}y_{n+3} + \tfrac{h^4}{20160}(-23f_n + 1244f_{n+1}\\
&\qquad +3630f_{n+2} + 188f_{n+3} + f_{n+4}),\\
h\delta_{n+4} &= -\tfrac{11}{6}y_n + 7y_{n+1} - \tfrac{19}{2}y_{n+2} + \tfrac{13}{3}y_{n+3} + \tfrac{h^4}{60480}(-151f_n + 19012f_{n+1}\\
&\qquad +76758f_{n+2} + 29956f_{n+3} + 425f_{n+4}),\\
h^2\gamma_{n+1} &= y_n - 2y_{n+1} + y_{n+2} + \tfrac{h^4}{60480}(-157f_n - 4748f_{n+1} - 102f_{n+2} - 44f_{n+3}\\
&\qquad +11f_{n+4}),\\
h^2\gamma_{n+2} &= y_{n+1} - 2y_{n+2} + y_{n+3} + \tfrac{h^4}{60480}(11f_n - 212f_{n+1} - 4638f_{n+2} - 212f_{n+3}\\
&\qquad +11f_{n+4}),\\
h^2\gamma_{n+3} &= -y_n + 4y_{n+1} - 5y_{n+2} + 2y_{n+3} + \tfrac{h^4}{60480}(-73f_n + 10372f_{n+1}\\
&\qquad +39714f_{n+2} + 5668f_{n+3} - 241f_{n+4}),\\
h^2\gamma_{n+4} &= -2y_n + 7y_{n+1} - 8y_{n+2} + 3y_{n+3} + \tfrac{h^4}{60480}(-409f_n + 21964f_{n+1}\\
&\qquad +87594f_{n+2} + 62956f_{n+3} + 4295f_{n+4}),\\
h^3\eta_{n+1} &= -y_n + 3y_{n+1} - 3y_{n+2} + y_{n+3} + \tfrac{h^4}{1440}(25f_n - 364f_{n+1} - 438f_{n+2}\\
&\qquad +68f_{n+3} - 11f_{n+4}),\\
h^3\eta_{n+2} &= -y_n + 3y_{n+1} - 3y_{n+2} + y_{n+3} + \tfrac{h^4}{1440}(-13f_n + 328f_{n+1} + 474f_{n+2}\\
&\qquad -80f_{n+3} + 11f_{n+4}),\\
h^3\eta_{n+3} &= -y_n + 3y_{n+1} - 3y_{n+2} + y_{n+3} + \tfrac{h^4}{160}(f_n + 20f_{n+1} + 154f_{n+2} + 68f_{n+3}\\
&\qquad -3f_{n+4}),\\
h^3\eta_{n+4} &= -y_n + 3y_{n+1} - 3y_{n+2} + y_{n+3} + \tfrac{h^4}{1440}(-29f_n + 392f_{n+1} + 858f_{n+2}\\
&\qquad +1904f_{n+3} + 475f_{n+4}).
\end{aligned}
$$

4. Analysis and Implementation of the Methods

The methods specified in Section 3 can be conventionally represented as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^4 \sum_{j=0}^{k} \beta_j f_{n+j}\,, \tag{24}$$

or can be compactly written in the form

$$\rho(E)y_n = h^4\sigma(E)f_n\,, \tag{25}$$

where $\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j$ and $\sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j$ are the characteristic polynomials, $\zeta \epsilon \mathbb{C}$, and $E^j y_n = y_{n+j}$ is a shift operator.

Following Fatunla [5] and Lambert [16] we define the local truncation error associated with (24) to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k} \{\alpha_j y(x+jh) - h^4 \beta_j y^{(4)}(x+jh)\}\,. \tag{26}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (26) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 h y' + \ldots, + C_q h^q y^q(x) + \ldots, \tag{27}$$

where the constant coefficients C_q, $q = 0, 1, \ldots$ are given as follows:

$C_0 = \sum_{j=0}^{k} \alpha_j$,

$C_1 = \sum_{j=1}^{k} j\alpha_j$,

$C_2 = \frac{1}{2!} \sum_{j=1}^{k} j^2 \alpha_j$,

$C_3 = \frac{1}{3!} \sum_{j=1}^{k} j^3 \alpha_j$,

$\vdots$

$C_q = \frac{1}{q!} \left(\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2)(q-3) \sum_{j=1}^{k} j^{q-4} \beta_j \right)$.

In the sense of Henrici [9], we say that the method (18) has order p and error constant C_{p+4} if

$$C_0 = C_1 = \ldots = C_p = C_{p+3} = 0, \quad C_{p+4} \neq 0\,.$$

We also note that $C_{p+4} h^{p+4} y^{(p+4)}(x_n)$ is the principal local truncation error at the point x_n. Our calculations reveal that the NFDMs have high order and relatively small error constants as shown in Table 1.

We emphasized that the methods (14)-(17) are used for IVPs and in order to analyze them for zero-stability, they are normalized and written as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(RA^0 - A^1) = R^3(R-1)\,. \tag{28}$$

A^0 is an identity matrix of dimension 4 and A^1 is a matrix of dimension 4 given by

$$A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, the methods (14) and (18) to (20) are applicable to BVPs and in order to analyze them for zero-stability, they are normalized and written as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(RA^0 - A^1) = (R-1)^4\,, \qquad (29)$$

where A^0 is an identity matrix of dimension k, A^1 is a matrix of dimension k and is given by

$$A^1 = \begin{pmatrix} -1 & 4 & -6 & 4 \\ -4 & 15 & -20 & 10 \\ -10 & 36 & -45 & 20 \\ -20 & 70 & -84 & 35 \end{pmatrix}.$$

Following Fatunla [5], the block methods obtained by combining the methods (14) to (17) and the methods (14) and (18) to (20) are zero-stable, since from (28) and (29), $\rho(R) = 0$ satisfy $|R_j| \le 1$, $j = 1, \ldots, k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 4. The methods are consistent as they have order $p > 1$. According to Henrici [9], we can safely assert the convergence of the block methods.

Method	Order p	Error constant C_{p+4}
(14)	6	1/3024
(15)	5	−1/3600
(16)	5	23/6048
(17)	5	−883/60480
(18)	6	−59/45360
(19)	5	23/3024
(20)	6	−151/3024

Table 1: Orders and error constants for NFDMs

Our method is implemented efficiently by combining the NFDMs as simultaneous numerical integrators for fourth order IVPs and BVPs. For instance, the methods (14) to (17) are combined as simultaneous numerical integrators for IVPs without looking for any other methods to provide the starting values by explicitly obtaining initial conditions at $x_{n+4}, n = 0, 4, \ldots, N-4$ using the computed values $Y(x_{n+4}) = y_{n+4}$, $\delta(x_{n+4}) = \delta_{n+4}$, $\gamma(x_{n+4}) = \gamma_{n+4}$, and $\eta(x_{n+4}) = \eta_{n+4}$ over sub-intervals $[x_0, x_4], \ldots, [x_{N-4}, x_N]$. On the other hand, the methods (14) and (18) to (20) are combined to give a single matrix of finite difference equations which simultaneously solves BVPs. In particular, for linear problems, we can solve (1) directly from the start with Gaussian elimination using partial pivoting, and for nonlinear problems, we can use a modified Newton-Raphson method.

5. Numerical Examples

In this section, we have tested the performance of our methods on one IVP and two BVPs. For each example we find absolute errors of the approximate solution in π_N, where N is chosen to be divisible by k. All computations were carried out using our written *Mathematica* code in *Mathematica 6.0*.

Example 5.1. We consider the linear IVP

$$y^{(iv)} = y''' + y'' + y' + 2y,\ y(0) = 0,\ y'(0) = y''(0) = 0,\ y'''(0) = 30,\quad 0 \leq x \leq 2\,,$$

$$\text{Exact} : y(x) = 2e^{2x} - 5e^{-x} + 3\cos(x) - 9\sin(x)\,.$$

The theoretical solution at $x = 2$ is $y(2) \simeq 99.08750629903285$. The errors in the solution were obtained at $x = 2$ using NFDMs for fixed stepsizes $h = 2/20, 2/40, 2/60, 2/80, 2/100$ corresponding to the number of steps $N = 20, 40, 60, 80, 100$ as shown in Table 2.

Steps	y	Errors
20	99.08738068697005	1.25612×10^{-4}
40	99.08750439151542	1.90752×10^{-6}
60	99.08750613251559	1.66517×10^{-7}
80	99.08750626939175	2.96411×10^{-8}
100	99.08750629038168	8.65117×10^{-9}

Table 2: Absolute errors, $|y(x) - y|$, at $x = 2$ for Example 5.1

Example 5.2. We consider the given linear BVP that involves a cantilever beam of length L with both ends fixed, distributed load, $\kappa(x)$, modulus of elasticity E and the moment of inertia I. The problem is solved for $\kappa(x) = x$, $L = 1$, and $EI = 1$ (see [7]).

$$EI\frac{d^4y}{dx^4}(x) = \kappa(x),\ y(0) = 0,\ y'(0) = 0,\ y''(L) = 0,\ y'''(L) = 0,\quad 0 \leq x \leq L\,,$$

$$\text{Exact} : y(x) = \frac{1}{120}(20x^2 - 10x^3 + x^5)\,.$$

The boundary conditions indicate that there are no bending moment and no shear at $x = 1$. The approximate solutions of this problem at the grid points were compared with the theoretical solution. The details of the numerical results are given in Table 3.

$x \times 8$	$y(x)$	y	$Error$
0	0.000000000000000000	0.000000000000000000	0.0000000000000
1	0.002441660563151041	0.002441660563151065	2.38524×10^{-17}
2	0.009122721354166667	0.009122721354166701	3.46945×10^{-17}
3	0.019104766845703124	0.019104766845703235	1.11022×10^{-16}
4	0.031510416666666666	0.031510416666666895	2.28983×10^{-16}
5	0.045553843180338540	0.045553843180338850	3.12250×10^{-16}
6	0.060571289062500000	0.060571289062500366	3.67761×10^{-16}
7	0.076051584879557300	0.076051584879557740	4.44089×10^{-16}
8	0.091666666666666660	0.091666666666667170	5.13478×10^{-16}

Table 3: Absolute errors, $|y(x) - y|$, $h = 1/8$ for Example 5.2

Example 5.3. We consider the following linear BVP for $x\epsilon\,[0,1]$ that was also solved by Conte [4].

$$y^{(iv)} - 3601y'' + 3600y = -1 + 1800x^2,\; y(0) = y(1),$$
$$y'(0) = 1, y(1) = 3/2 + \sinh(1), y'(1) = 1 + \cosh(1)\,,$$
$$\text{Exact} : y(x) = 1 + x^2/2 + \sinh(x)\,.$$

$x \times 8$	$y(x)$	y	$Error$
0	1.0000000000000000	1.0000000000000000	0.0000000000000
1	1.1331382752411154	1.1331382752420516	9.36140×10^{-13}
2	1.2838623168081682	1.2838623168084253	2.57128×10^{-13}
3	1.4541635679136147	1.4541635679138223	2.07612×10^{-13}
4	1.6460953054937475	1.6460953054919225	1.82498×10^{-12}
5	1.8618047644566160	1.8618047644536126	3.00338×10^{-12}
6	2.1035667319358300	2.1035667319337548	2.07523×10^{-12}
7	2.3738191371442947	2.3738191371424930	1.80167×10^{-12}
8	2.6752011936438014	2.6752011936438014	0.0000000000000

Table 4: Absolute errors, $|y(x) - y|$, $h = 1/8$ for Example 5.3

Our method is compared with the orthogonalisation method discussed in Conte [4] and it is observed that the maximum absolute error (3.00×10^{-12}, $h = 0.125$) for our method is smaller than the maximum absolute error (1.00×10^{-7},

$h = 0.01$) for the method given in [4]. Hence, for this example, our method is more acurate and clearly superior. The details of the numerical results are given in Table 4.

6. Conclusions

We have proposed NFDMs which are obtained from the same k-step LMM with continuous coefficients and applied as simultaneous numerical integrators to solve $y^{(iv)} = f(x, y, y', y'', y''')$ without first adapting the ODE to an equivalent first order system. An essential ingredient in the methods involves the way in which they are applied. For instance, for IVPs, we proceed by explicitly obtaining initial conditions at $x_{n+4}, n = 0, 4, \ldots, N-4$ using the computed values $Y(x_{n+4}) = y_{n+4}$, $\delta(x_{n+4}) = \delta_{n+4}$, $\gamma(x_{n+4}) = \gamma_{n+4}$, and $\eta(x_{n+4}) = \eta_{n+4}$ over sub-intervals $[x_0, x_4], \ldots, [x_{N-4}, x_N]$. On the other hand, BVPs are solved by combining the NFDMs (14) and (18) to (20) as simultaneous numerical integrators into a single matrix equation. Our future research will be focused on developing a global error estimation strategy for the methods and an automatic code for their implementation.

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