International Journal of Pure and Applied Mathematics
Volume 48 No. 2 2008, 81-102

STABILITY OF FUNCTIONAL EQUATION IN QUASI-BANACH SPACES

K. Ravi[1 §], R. Kodandan[2], N. Jammuna[3]

[1]Department of Mathematics
Sacred Heart College
Tamil Nadu, Tirupattur, 635 601, INDIA
e-mail: shckravi@yahoo.co.in

[2]Department of Mathematics
Srinivasa Institute of Technology and Management Studies
Chittoor, Andhra Pradesh, 517 127, INDIA
e-mail: rkodandan1979@rediff.co.in

[3]Department of Mathematics
Magna College of Engineering
Red Hills, Tiruvallur High Road, Chennai, 600 055, INDIA

Abstract: In this paper, we introduce and investigate the general solution of a new functional equation

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+f(4x)$$

and discuss its Hyers-Ulam-Rassias stability in quasi-Banach spaces.

AMS Subject Classification: 39B52, 39B82, 39B72, 46B03, 47Jxx
Key Words: Hyers-Ulam-Rassias stability, quadratic function, additive function, quasi-Banach space, $p-$Banach spaces

1. Introduction

The stability problem of functional equations originates from the fundamental

Received: August 19, 2008

§Correspondence author

question: when is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?

In connection with the above questions, in 1940, S.M. Ulam [24] raised a question concerning the stability of homomorphisms. Let G be a group and let G' be a metric group with metric d(.,.). Given $\epsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there is a homomorphism $H : G \to G'$ with $d(f(x), H(x)) < \epsilon$ for all $x \in G$?

The first partial solution to Ulam's question was given by D.H. Hyers [8], he considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies Hyers inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$, it was shown that the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exist for all $x \in E$ and that $a : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \leq \epsilon\,.$$

Moreover, it was proved that if $f(tx)$ is continuous in t for each fixed $x \in E$, then a is linear.

In 1978, Th.M. Rassias [18] provided a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. He proved the following theorem by using a direct method and we state the theorem without proof.

Theorem 1.1. (see [18]) *If a function $f : E \to E'$ between Banach spaces satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\,[||x||^p + ||y||^p]$$

for some $\theta \geq 0$, $0 \leq p < 1$ and for all $x, y \in E$. Then there exists a unique additive function $a : E \to E'$ such that

$$\|f(x) - a(x)\| \leq \frac{2\theta}{2-2^p}||x||^p$$

for all $x \in E$. Moreover, it $f(tx)$ is continuous in t for each fixed $x \in E$ then a is linear.

The theorem of Rassias was later extended to all $p \neq 1$ and generalized by many mathematicians (see [6, 7, 14, 18]).

The quadratic function $f(x) = kx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Hence the functional equation (1.1) is called the quadratic functional equation or the Euler-Lagrange functional equation and every solution of the functional equation (1.1) is called a quadratic function. The functional equation (1.1) is a familiar equation and this equation was dealt by many authors F. Skof [23], P.W. Cholewa [4], S. Czerwik [5] and J.M. Rassias [16].

S.M. Jung [11] investigated the Hyers-Ulam stability of the quadratic functional equation of Pexider type

$$f_1(x+y) + f_2(x-y) = 2f_3(x) + 2f_4(y). \tag{1.2}$$

The generalized Hyers-Ulam-Rassias stability of a quadratic equation

$$f(x+y+z) + f(x-y) + f(y-z) + f(z-x) = 3f(x) + 3f(y) + 3f(z) \tag{1.3}$$

was discussed by J.H. Bae and K.W. Jun [2]. J.M. Rassias [17] established Hyers-Ulam-Rassias stability of a quadratic functional equations on several variables.

In 2005, K.W. Jun and H.M. Kim [12] obtained the general solution of a generalized quadratic and additive type functional equation of the form

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y)$$

for any integer a with $a \neq -1, 0, 1$.

A. Najati and M.B. Moghimi [15] dealt the functional equation

$$f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 2f(2x) - 2f(x) \tag{1.4}$$

which is derived from quadratic and additive functions and established the general solution of equation (1.4) and investigated the Hyers-Ulam-Rassias stability for equation (1.4).

In this paper, we introduce and investigate the general solution of a new functional equation

$$f(3x+y) + f(3x-y) = f(x+y) + f(x-y) + f(4x) \tag{1.5}$$

and discuss its Hyers-Ulam-Rassias stability of this equation in quasi-Banach spaces. It may be noted that $f(x) = ax^2 + bx$ is a solution of the functional equation (1.5) and which contain quadratic and additive terms and not the constant term which appeared in [15], we also found that the results we obtained here are more simple than the one appeared in [15].

Before giving the main results, we will present here some basic facts concerning quasi-Banach spaces.

Definition 1.2. (see [3], [22]) Let X be a linear space. A quasi-norm $\| \cdot \|$ is real - valued function on X satisfying the following:

(i) $\| x \| \geq 0$ for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.

(ii) $\| \lambda x \| = | \lambda | \| x \|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\| x + y \| \leq K(\| x \| + \| y \|)$ for all $x, y \in X$.

From (iii) we have

$$\left\| \sum_{i=1}^{2n} x_i \right\| \leq K^n \sum_{i=1}^{2n} \|x_i\| \quad \text{and} \quad \left\| \sum_{i=1}^{2n+1} x_i \right\| \leq K^{n+1} \sum_{i=1}^{2n+1} \|x_i\|$$

for all integers $n \geq 1$ and all $x_1, x_2, \ldots, x_{2n+1} \in X$. The pair $(X, \| \cdot \|)$ is called quasi-normed space if $\| \cdot \|$ is a quasi-norm on X. The smallest possible K is called the modulus of concavity of $\| \cdot \|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\| \cdot \|$ is called p-norm $(0 < p \leq 1)$ if $\| x+y \|^p \leq \| x \|^p + \| y \|^p$ for all $x, y \in X$. In this case, a quasi-Banach space is called p-Banach space.

According to Aoki-Rolewicz Theorem [22] each quasi-norm is equivalent to some p-norm, it is easier to work in p-norms than quasi-norm and hence we focus our attention on p-norms in some of our results which occur in this paper.

2. Solution of Equation (1.5)

In this section, let E_1 and E_2 denote real vectors spaces, we will prove the following two lemmas, which will be useful to prove our main theorems.

Lemma 2.1. *If $f : E_1 \to E_2$ is an even function with $f(0) = 0$ satisfying (1.5) for all $x, y \in E_1$ then f is quadratic.*

Proof. The function f is even and therefore $f(-x) = f(x)$ for all $x \in E_1$. Replacing y by $3x$ in (1.5) and using evenness, we obtain

$$f(6x) = 2f(4x) + f(2x), \quad \forall x \in E_1. \tag{2.1}$$

Replace x by $\frac{x}{2}$ in (2.1), we obtain

$$f(3x) = 2f(2x) + f(x), \quad \forall x \in E_1. \tag{2.2}$$

Again replacing y by $5x$ in (1.5), we arrive that

$$f(8x) + f(2x) = f(6x) + 2f(4x), \quad \forall x \in E_1. \tag{2.3}$$

Use (2.1) in (2.3), we obtain

$$f(8x) = 4f(4x), \quad \forall x \in E_1. \tag{2.4}$$

Replacing x by $\frac{x}{2}$ in (2.4), we obtain

$$f(4x) = 4f(2x), \quad \forall x \in E_1. \tag{2.5}$$

Again replacing x by $\frac{x}{2}$ in (2.5), we obtain

$$f(2x) = 4f(2x), \quad \forall x \in E_1. \tag{2.6}$$

Now substitute (2.6) in (2.5), we obtain

$$f(4x) = 16f(x), \quad \forall x \in E_1. \tag{2.7}$$

Replace y by $x - 2y$ in (1.5) and use (2.6), we arrive

$$4f(2x + y) + 4f(x + y) = 4f(x - y) + 4f(y) + f(4x), \quad \forall x \in E_1. \tag{2.8}$$

Replacing y by $-y$ in (2.8), we obtain

$$4f(2x - y) + 4f(x - y) = 4f(x + y) + 4f(y) + f(4x), \quad \forall x, y \in E_1. \tag{2.9}$$

Adding (2.8) and (2.9), we obtain

$$f(2x + y) + f(2x - y) = 8f(x) + 2f(y), \quad \forall x, y \in E_1. \tag{2.10}$$

Replacing y by x in (2.10), we obtain

$$f(3x) = 9f(x), \quad \forall x \in E_1. \tag{2.11}$$

Again, replacing y by $3y$ in (1.5) and using (2.11), we obtain

$$9f(x + y) + 9f(x - y) = f(x + 3y) + f(x - 3y) + f(4x), \quad \forall x, y \in E_1. \tag{2.12}$$

Interchanging x and y in (1.5), and using evenness of f, we obtain

$$f(x + 3y) + f(x - 3y) = f(x + y) + f(x - y) + f(4y), \quad \forall x, y \in E_1. \tag{2.13}$$

Using (2.13) in (2.12), we obtain

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad \forall x, y \in E_1.$$

Therefore $f : E_1 \to E_2$ is quadratic. □

Lemma 2.2. *If $f : E_1 \to E_2$ be an odd function, satisfying (1.5) for all $x, y \in E_1$. Then f is additive.*

Proof. Replace y by $2x$ in (1.5) and using oddness of f, we obtain

$$f(5x) = f(3x) + f(4x) - 2f(x), \quad \forall x \in E_1. \tag{2.14}$$

Replacing y by $x - y$ in (1.5), we obtain

$$f(4x - y) + f(2x + y) = f(2x - y) + f(y) + f(4x), \quad \forall x, y \in E_1. \tag{2.15}$$

Again, replacing y by $-y$ in (2.15) and using oddness of f, we obtain

$$f(4x + y) + f(2x - y) = f(2x + y) - f(y) + f(4x), \quad \forall x, y \in E_1. \tag{2.16}$$

Now adding (2.15) and (2.16), we arrive

$$f(4x+y)+f(4x-y)=2f(4x), \quad \forall x,y\in E_1. \tag{2.17}$$

Replacing y by $4x$ in (2.17), we obtain

$$f(8x)=2f(4x), \quad \forall x\in E_1. \tag{2.18}$$

Replacing x by $\frac{x}{2}$ in (2.18), we obtain

$$f(4x)=2f(2x), \quad \forall x\in E_1. \tag{2.19}$$

Again replacing x by $\frac{x}{2}$ in (2.19), we obtain

$$f(8x)=2f(4x), \quad \forall x\in E_1. \tag{2.20}$$

Changing y by x in (2.17), we get

$$f(5x)+f(3x)=2f(4x), \quad \forall x\in E_1. \tag{2.21}$$

Using (2.14) in (2.21), we obtain

$$2f(3x)-2f(x)=f(4x), \quad \forall x\in E_1. \tag{2.22}$$

Use (2.19) in (2.22), we obtain

$$f(3x)-f(x)=f(2x), \quad \forall x\in E_1. \tag{2.23}$$

Again using (2.20) in (2.23), we obtain

$$f(3x)=3f(x), \quad \forall x\in E_1.$$

Using (2.19) in (2.20), we obtain

$$f(4x)=4f(x), \quad \forall x\in E_1.$$

Therefore in general $f(nx)=n\ f(x), \quad \forall x\in E_1$. Now replace x by $\frac{x}{3}$ and multiplying (1.5) by 3 and using oddness of f, we obtain

$$f(x+3y)-f(3y-x)=3f(x+y)-3f(x-y)-f(4x), \quad \forall x,y\in E_1. \tag{2.24}$$

Interchanging x and y in (1.5), and using oddness of f we obtain

$$f(3y+x)+f(3y-x)=f(x+y)-f(x-y)+f(4y), \quad \forall x,y\in E_1. \tag{2.25}$$

Adding (2.24) and (2.25), we get

$$f(x+3y)=4f(x+y)+2f(x-y)-f(4x)+f(4y), \quad \forall x,y\in E_1. \tag{2.26}$$

Replacing y by $-y$ in (2.26), we obtain

$$f(x-3y)=4f(x-y)+2f(x+y)-f(4x)-f(4y), \quad \forall x,y\in E_1. \tag{2.27}$$

Again replacing x by $x-2y$ in (2.26), we obtain

$$2f(x-3y)=4f(x-y)+2f(x-3y)-f(x-2y)+f(4y), \quad \forall x,y\in E_1. \tag{2.28}$$

Now replacing (2.27) and (2.28), we obtain

$$2f(x-y)=f(x-2y)+f(x), \quad \forall x,y\in E_1. \tag{2.29}$$

Replace now, x by $x+y$ in (2.29), we get

$$2f(x) = f(x-y) + f(x+y), \quad \forall x, y \in E_1. \tag{2.30}$$

Interchange x and y in (2.30), we obtain

$$2f(y) = f(y-x) + f(x+y), \quad \forall x, y \in E_1. \tag{2.31}$$

From (2.30) and (2.31), we obtain

$$f(x+y) = f(x) + f(y), \quad \forall x, y \in E_1.$$

Therefore the mapping $f : E_1 \to E_2$ is additive. □

Theorem 2.3. *A function $f : E_1 \to E_2$ satisfies (1.5) for all $x, y \in E_1$, if and only if there exists a symmetric bi-additive function $B : E_1 \times E_1 \to E_2$ and an additive function $A : E_1 \to E_2$ such that*

$$f(x) = B(x,x) + A(x), \quad x \in E_1 .$$

Proof. Suppose there exists a symmetric bi-additive function $B : E_1 \times E_1 \to E_2$ and an additive function $A : E_1 \to E_2$ such that

$$f(x) = B(x,x) + A(x), \quad x \in E_1, \tag{2.32}$$

then using (2.32), we obtain

$$f(3x+y) = B(3x+y, 3x+y) + A(3x+y) \tag{2.33}$$

$$f(3x-y) = B(3x-y, 3x-y) + A(3x-y) \tag{2.34}$$

for all $x, y \in E_1$. From (2.33) and (2.34), we obtain

$$\begin{aligned} f(3x+y) + f(3x-y) = B(3x+y, 3x+y) + B(3x-y, 3x-y) \\ + A(3x+y) + A(3x-y) \end{aligned} \tag{2.35}$$

for all $x, y \in E_1$. Using properties of symmetric bi-additive function in (2.35), we arrive

$$f(3x+y) + f(3x-y) = f(x+t) + f(x-y) + f(4x), \quad x, y \in E_1 .$$

Hence the function satisfies (1.5).

Conversely, we decompose f into even part and the odd part by letting $f_e(x) = \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in E_1$. Replacing x by $-x$, y by $-y$ in (1.5) and adding, subtracting the resultant equation with (1.5), we find that $f_e(x), f_o(x)$ satisfies (1.5). Hence by Lemma 2.1 and Lemma 2.3, we obtain that the functions $f_e(x)$ and $f_o(x)$ are quadratic and additive respectively. It shows that there exists a symmetric bi-additive function $B : E_1 \times E_1 \to E_2$ such that $f_e(x) = B(x,x)$ and an additive function $A : E_1 \to E_2$ such that $A(x) = f_o(x)$ and $f(x) = B(x,x) + A(x), \quad x \in E_1$. □

3. Hyers-Ulam-Rassias Stability of Equation (1.5)

In this section, we assume that E_1 is a quasi-normed space with quasi-norm $||\cdot||_{E_1}$ and that E_2 is a p-Banach space with p-norm $||\cdot||_{E_2}$. Let k be the modulus of concavity of $||\cdot||_{E_2}$. We use the following notation:

$$D\ f(x,y) = f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-f(4x), \quad \forall x,y \in E_1, \tag{3.1}$$

and we state the following Lemma 3.1 [15] without proof, it will be useful in proving our theorems.

Lemma 3.1. (see [15]) *Let $0 \le p \le 1$ and let $x_1, x_2, \ldots, x_n$ be non negative real numbers then*

$$\left(\sum_{i=1}^{n} x_i\right)^p \quad \le \quad \sum_{i=1}^{n} x_i^p. \tag{3.2}$$

Theorem 3.2. *Let $\phi : E_1 \times E_1 \to [0,\infty)$ be a function such that for all $x, y \in E_1$*

$$\lim_{n\to\infty} 9^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = 0 \tag{3.3}$$

and

$$\sum_{i=1}^{\infty} 9^{ip}\ \phi^p\left(\frac{x}{3^i}, \frac{y}{3^i}\right) < \infty \tag{3.4}$$

for all $x \in E_1$ and for all $y \in \{-x, 3x\}$. Suppose that an even function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality

$$\|D\ f(x,y)\|_{E_2} \le \phi(x,y), \qquad \forall x,y \in E_1. \tag{3.5}$$

Then the limit

$$Q(x) = \lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right) \tag{3.6}$$

exists for all $x \in E_1$ and $Q : E_1 \to E_2$ is a unique quadratic function satisfying

$$\|f(x) - Q(x)\|_{E_2} \le \frac{k^2}{9}\left[\tilde{\Psi}_e(x)\right]^{\frac{1}{p}}, \quad \forall x \in E_1, \tag{3.7}$$

where

$$\widetilde{\Psi}_e(x) = \sum_{i=1}^{\infty} 9^{ip}\left\{\frac{1}{2^p}\ \phi^p\left(\frac{x}{2\cdot 3^i}, \frac{-x}{2\cdot 3^i}\right) + \ \phi^p\left(\frac{x}{2\cdot 3^i}, \frac{3x}{2\cdot 3^i}\right)\right\}. \tag{3.8}$$

Proof. Replacing y by $3x$ in (3.5), we obtain

$$\|f(6x) - f(2x) - 2f(4x)\|_{E_2} \le \phi(x,3x), \forall x,y \in E_1. \tag{3.9}$$

Replace x by $x+y$ in (3.5), we obtain

$$\|f(4x+y)+f(2x+y)-f(2x-y)-f(y)-f(4x)\|_{E_2} \leq \phi(x,x+y), \quad \forall x,y \in E_1. \tag{3.10}$$

Again replacing y by $-y$ in (3.10) and using evenness, we obtain

$$\|f(4x-y)+f(2x+y)-f(2x-y)-f(y)-f(4x)\|_{E_2} \leq \phi(x,x-y), \tag{3.11}$$

then from (3.9), (3.10) and (3.11), we obtain

$$\|f(4x+y)+f(4x-y)-2f(y)-4f(4x)+f(6x)-f(2x)\|_{E_2} \leq k^2\left[\phi(x,3x)+\phi(x,x+y)+\phi(x,x-y)\right], \quad \forall x,y \in E_1. \tag{3.12}$$

In equation (3.12), replace y by $2x$, we get

$$\|f(6x)-9f(2x)\|_{E_2} \leq \frac{k^2}{2}\left[\phi(x,-x)+2\phi(x,3x)\right], \quad \forall x \in E_1. \tag{3.13}$$

If we replace x by $\frac{x}{2}$ in (3.13), we arrive

$$\|f(3x)-9f(x)\|_{E_2} \leq \frac{k^2}{2}\left[\phi\left(\frac{x}{2},\frac{-x}{2}\right)+2\phi\left(\frac{x}{2},\frac{3x}{2}\right)\right], \quad \forall\, x \in E_1,$$

which can be written as

$$\|f(3x)-9f(x)\|_{E_2} \leq k^2\ \psi(x), \quad \forall x \in E_1, \tag{3.14}$$

and

$$\psi(x)=\frac{1}{2}\left[\phi\left(\frac{x}{2},\frac{-x}{2}\right)+2\phi\left(\frac{x}{2},\frac{3x}{2}\right)\right], \quad \forall\, x \in E_1. \tag{3.15}$$

In equation (3.14), replace x by $\frac{x}{3^{n+1}}$ and multiplying both sides by 9^n, we have

$$\left\|9^{n+1}f\left(\frac{x}{3^{n+1}}\right)-9^n f\left(\frac{x}{3^n}\right)\right\|_{E_2} \leq k^2\ 9^n\ \psi\left(\frac{x}{3^n}\right)(x,y), \quad \forall\, x \in E_1, \tag{3.16}$$

for all non-negative integers n. Since E_2 is a p-Banach space and using (3.16), we obtain

$$\left\|9^{n+1}f\left(\frac{x}{3^{n+1}}\right)-9^n f\left(\frac{x}{3^n}\right)\right\|_{E_2}^p \leq \sum_{i=m}^{n}\left\|9^{i+1}f\left(\frac{x}{3^{i+1}}\right)-9^i f\left(\frac{x}{3^i}\right)\right\|_{E_2}^p \leq k^{2p}\ \sum_{i=m}^{n} 9^{ip}\ \psi^p\left(\frac{x}{3^i}\right) \tag{3.17}$$

for all non-negative integers m and n with $n \geq m$ and all $x \in E_1$. Now $0<p\leq 1$ and with the help of Lemma 3.1, the equation (3.15) can be written as

$$\psi^p(x)=\frac{1}{2^p}\left[\phi^p\left(\frac{x}{2},\frac{-x}{2}\right)+2^p\phi\left(\frac{x}{2},\frac{3x}{2}\right)\right], \quad \forall\, x \in E_1. \tag{3.18}$$

Therefore it follows from (3.4) and (3.18) that

$$\sum_{i=1}^{\infty} 9^{ip}\ \psi^p\left(\frac{x}{3^i}\right) < \infty, \quad \forall\ x, y\ \in E_1. \tag{3.19}$$

Therefore, we conclude from (3.17) and (3.19) that the sequence $\{9^n\ f\left(\frac{x}{3^n}\right)\}$ is a Cauchy sequence for all $x \in E_1$, since E_2 is complete, the sequence $\{9^n\ f\left(\frac{x}{3^n}\right)\}$ converges for all $x \in E_1$. Now we define the mapping $Q : E_1 \to E_2$ by (3.6) for all $x \in E_1$. Allowing $n \to \infty$ in (3.17), we get

$$\begin{aligned} \|f(x) - Q(x)\|_{E_2}^p &\leq k^{2p}\ \sum_{i=0}^{\infty} 9^{ip}\ \psi^p\left(\frac{x}{3^{i+1}}\right) \\ &= \frac{k^{2p}}{9^p}\ \sum_{i=0}^{\infty} 9^{ip}\ \psi^p\left(\frac{x}{3^i}\right), \quad \forall x \in E_1. \end{aligned} \tag{3.20}$$

Use (3.15) in the equation (3.20), we arrive the result (3.7). Now, we show that Q is a quadratic it follows from (3.3), (3.5) and (3.6),

$$\|D\ Q(x,y)\|_{E_2} = \lim_{n\to\infty} 9^n \left\|D\ f\left(\frac{x}{3^n}, \frac{y}{3^n}\right)\right\|_{E_2} \leq 9^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}\right), \quad \forall\ x, y\ \in E_1.$$

Therefore the mapping $Q : E_1 \to E_2$ satisfies (1.5). Since $Q(x) = 0$, then by Lemma 2.1, we obtain that the mapping $Q : E_1 \to E_2$ is quadratic. To prove the uniqueness of Q. Let $T : E_1 \to E_2$ be another quadratic mapping satisfying (3.7). Since

$$\lim_{n\to\infty} 9^n \sum_{i=1}^{\infty} 9^{ip}\phi^p\left(\frac{x}{3^{n+i}}, \frac{y}{3^{n+i}}\right) = \lim_{n\to\infty} 9^{ip}\phi^p\left(\frac{x}{3^i}, \frac{y}{3^i}\right) = 0, \quad \forall\ x\ \in E_1\,,$$

and for all $y \in \{-x, 3x\}$ then

$$\lim_{n\to\infty} 9^{np}\ \tilde{\psi}_e\ \left(\frac{x}{3^n}\right) = 0, \quad \forall\ x\ \in E_1. \tag{3.21}$$

It follows from (3.7) and (3.21),

$$\|Q(x) - f(x)\|_{E_2}^p = \lim_{n\to\infty} 9^n \left\|f\left(\frac{x}{3^n}\right) - T\left(\frac{y}{3^n}\right)\right\|_{E_2}^p \leq \lim_{n\to\infty} 9^{np}\ \tilde{\psi}_e\ \left(\frac{x}{3^n}\right) = 0$$

for all $x, y\ \in E_1$, so $Q = T$. □

Theorem 3.3. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{9^n}\phi(3^n x, 3^n y) = 0, \quad \forall\ x, y\ \in E_1, \tag{3.22}$$

and

$$\sum_{i=0}^{\infty} \frac{1}{9^n}\phi^p(3^i x, 3^i y) < \infty, \quad \forall\ x \in E_1, \tag{3.23}$$

and for all $\{-x, 3x\}$. Suppose that an even function $f : E_1 \to E_2$ with $f(0) = 0$

satisfies the inequality (3.5) for all $x, y \in E_1$, the limit

$$Q(x) = \lim_{n\to\infty} \frac{1}{9^n} f(3^n x) \tag{3.24}$$

exists for all $x \in E_1$ and $Q : E_1 \to E_2$ is a unique quadratic function

$$\|f(x) - Q(x)\|_{E_2} \le \frac{k^2}{9} \left[\tilde{\psi}_e(x)\right]^{\frac{1}{p}}, \quad \forall\ x \ \in E_1, \tag{3.25}$$

where

$$\widetilde{\psi}_e(x) = \sum_{i=0}^{\infty} \frac{1}{9^{ip}} \left\{ \frac{1}{2^p}\ \phi^p \left(\frac{3^i x}{2}, \frac{-3^i x}{2} \right) +\ \phi^p \left(\frac{3^i x}{2}, \frac{3^{i+1} x}{2} \right) \right\}. \tag{3.26}$$

Proof. If we replacing x by $3^n x$ in (3.14) and dividing by 9^{n+1} on both sides of (3.14), we obtain

$$\left\| \frac{1}{9^{n+1}} f(3^{n+1}x) - \frac{1}{9^n} f(3^n x) \right\|_{E_2} \le \frac{k^2}{9^n + 1}\ \psi(3^n x) \tag{3.27}$$

for all $x \ \in E_1$ and for all non-negative integers n. Since E_2 is a p-Banach space, using (3.27), we obtain

$$\begin{aligned} \left\| \frac{1}{9^{n+1}} f(3^{n+1}x) - \frac{1}{9^m} f(3^m x) \right\|_{E_2}^p &\le \sum_{i=m}^{n} \left\| \frac{1}{9^{i+1}} f(3^{i+1}x) - \frac{1}{9^i} f(3^i x) \right\|_{E_2}^p \\ &\le \frac{k^{2p}}{9^p} \sum_{i=m}^{n} \frac{1}{9^{ip}}\ \psi^p(3^i x) \end{aligned} \tag{3.28}$$

for all non-negative integers m and n with $n \ge m$ and all $x \in E_1$. Since $\sum_{i=0}^{\infty} \frac{1}{9^{ip}}\ \psi^p(3^i x) < \infty$ for all $x \in E_1$ then (3.28) implies that the sequence $\{\frac{1}{9^n} f(3^n x)\}$ is a Cauchy sequence for all $x \in E_1$. Since E_2 is complete, the sequence $\{\frac{1}{9^n} f(3^n x)\}$ converges for all $x \in E_1$. Now we define the mapping $Q : E_1 \to E_2$ by (3.24) for all $x \in E_1$. Letting $m = 0$ and $n \to \infty$ allowing in (3.28), we get

$$\|f(x) - Q(x)\|_{E_2}^p \le \frac{k^{2p}}{p^p}\ \sum_{i=0}^{\infty} \frac{1}{p^{ip}} \psi^p(3^i x), \quad \forall\ x \ \in E_1\,. \tag{3.29}$$

Use (3.15) in the equation (3.29), we arrive the result (3.25).

Now using (3.29), (3.22) in the equation (3.4), we obtain

$$\|D\ Q(x,y)\|_{E_2} \le \lim_{n\to\infty} \frac{1}{9^n} \phi(3^n x, 3^n y) = 0, \quad \forall\ x, y \ \in E_1.$$

Therefore the mapping $Q : E_1 \to E_2$ satisfies (1.5). Since $Q(0) = 0$ then by Lemma 2.1, we obtain that the mapping Q is quadratic. Uniqueness is proved in similar manner, as in the proof of Theorem 3.2. □

Corollary 3.4. *Let θ, r, s be non negative real numbers such that $r, s \neq 2$, suppose that an even function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality*

$$\|Df(x,y)\|_{E_2} \le \theta \left[\|x\|^r_{E_1} + \|y\|^s_{E_1}\right], \quad \forall x, y \in E_1. \tag{3.30}$$

Then there exists a unique quadratic function $Q : E_1 \to E_2$ satisfies

$$\|f(x) - Q(x)\|_{E_2} \le \frac{\theta\ k^2}{2}\left[\frac{2^p+1}{2^r p}\ \frac{1}{|3^{rp}-9^p|}\ \|x\|^{rp}_{E_1} + \frac{1+2^p\ 3^{sp}}{2^s p}\ \frac{1}{|3^{sp}-9^p|}\ \|x\|^{sp}_{E_1}\right]^{\frac{1}{p}}.$$

If $r = s = t$. Then

$$\|f(x) - Q(x)\|_{E_2} \le \frac{\theta\ k^2}{2^{t+1}}\left[\left\{2 + 2^p\left(1 + 3^{tp}\right)\right\}\ \frac{1}{|3^{tp}-9^t|}\right]^{\frac{1}{p}}\ \|x\|^t_{E_1}.$$

Proof. In Theorem 3.2, take $\phi(x,y) = \theta\left[\|x\|^r_{E_1} + \|y\|^s_{E_1}\right]$ for all $x, y \in E_1$. Then using the equations (3.8) in (3.7), it becomes

$$\begin{aligned}
&\|f(x) - Q(x)\|_{E_2} \\
&\le \frac{\theta\ k^2}{2}\left[\sum_{i=1}^{\infty}\left(\frac{\frac{1}{2^p}+1}{2^{rp}\ 3^{irp}}\right)\|x\|^{rp}_{E_1} + \frac{\frac{1}{2^p}+3^{sp}}{2^{sp}\ 9^p}\ \left(\sum_{i=1}^{\infty}\frac{9^{ip}}{3^{isp}}\right)\|x\|^{sp}_{E_1}\right]^{\frac{1}{p}} \\
&\le \frac{\theta\ k^2}{2}\left[\frac{2^p+1}{2^{p(r+1)}}\ \frac{1}{3^{rp}}\frac{1}{|1-\frac{9^p}{3^{rp}}|}\ \|x\|^{rp}_{E_1} + \frac{\frac{1}{2^p}+3^{sp}}{2^{sp}\ 3^{sp}}\ \frac{1}{|1-\frac{9^p}{3^{sp}}|}\ \|x\|^{sp}_{E_1}\right]^{\frac{1}{p}} \\
&\le \frac{\theta\ k^2}{2}\left[\frac{2^p+1}{2^r p}\ \frac{1}{|3^{rp}-9^p|}\ \|x\|^{rp}_{E_1} + \frac{1+2^p\ 3^{sp}}{2^s p}\ \frac{1}{|3^{sp}-9^p|}\ \|x\|^{sp}_{E_1}\right]^{\frac{1}{p}}.
\end{aligned}$$

Take $r = s = t$, then the above inequality reduces to

$$\|f(x) - Q(x)\|_{E_2} \le \frac{\theta\ k^2}{2^{t+1}}\left[\left\{2 + 2^p\left(1 + 3^{tp}\right)\right\}\ \frac{1}{|3^{tp}-9^t|}\right]^{\frac{1}{p}}\ \|x\|^t_{E_1}. \qquad \square$$

Corollary 3.5. *Let θ, r, s be non negative real numbers such that $\lambda = r + s \neq 2$. Suppose that an even function $f : E_1 \to E_2$ with $f(0)$ satisfying the inequality*

$$\|Df(x,y)\|_{E_2} \le \theta\,\|x\|^r_{E_1}\cdot\|y\|^s_{E_1}, \quad \forall\ x, y \in E_1. \tag{3.31}$$

Then there exists a unique quadratic function $Q : E_1 \to E_2$ satisfies

$$\|f(x) - Q(x)\|_{E_2} \le \frac{\theta\ k^2}{2^{\lambda+1}}\left[\left\{1 + 2^p\ 3^{sp}\right\}\ \frac{1}{|3^{\lambda\ p}-9^p|}\right]^{\frac{1}{p}}\ \|x\|^{\lambda}_{E_1}.$$

Proof. Taking $\phi(x,y) = \theta\,\|x\|^r_{E_1}\cdot\|y\|^s_{E_1}$ in Theorem 3.2, the proof is similar to that of proof of Corollary 3.4. $\square$

Theorem 3.6. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function such that for all $x, y \in E_1$*

$$\lim_{n\to\infty} 3^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = 0, \quad \forall\, x, y \ \in E_1\,, \tag{3.32}$$

and

$$\sum_{i=1}^{\infty} 3^{ip}\ \phi^p\left(\frac{x}{3^i}, \frac{y}{3^i}\right) < \infty, \quad \forall\, x, y \ \in E_1\,, \tag{3.33}$$

and for all $y \in \{-x, 3x\}$. Suppose that an odd function $f : E_1 \to E_2$ satisfies the inequality (3.5) for all $x, y \in E_1$, then the limit

$$A(x) = \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) \tag{3.34}$$

exists for all $x \in E_1$ and $A : E_1 \to E_2$ is a unique quadratic function satisfying

$$\|f(x) - A(x)\|_{E_2} \leq \frac{k^2}{9}\left[\tilde{\varphi}_o(x)\right]^{\frac{1}{p}}, \quad \forall\, x \ \in E_1, \tag{3.35}$$

where

$$\widetilde{\varphi}_o(x) = \sum_{i=1}^{\infty} 3^{ip}\left\{\frac{1}{2^p}\ \phi^p\left(\frac{x}{2\cdot 3^i}, \frac{-x}{2\cdot 3^i}\right) +\ \phi^p\left(\frac{x}{2\cdot 3^i}, \frac{3x}{2\cdot 3^i}\right)\right\}. \tag{3.36}$$

Proof. Replacing y by $3x$ and x by $x - y$ in (3.5), we obtain

$$\|f(6x) - f(2x) - 2f(4x)\|_{E_2} \leq \phi(x, 3x), \tag{3.37}$$

$$\|f(4x - y) + f(2x + y) - f(2x - y) - f(y) - f(4x)\|_{E_2} \leq \phi(x, x - y), \tag{3.38}$$

for all $x, y \ \in E_1$. Again replacing y by $-y$ in (3.38) and using oddness, we obtain

$$\|f(4x + y) + f(2x - y) - f(2x + y) + f(y) - f(4x)\|_{E_2} \leq \phi(x, x + y), \tag{3.39}$$

for all $x, y \ \in E_1$. Using (3.37), (3.38) and (3.39), and the definition of quasi-norm, we obtain

$$\begin{aligned}\|f(4x + y) &+ f(4x - y) - 2f(4x) + f(6x) + f(2x) - 2f(4x)\|_{E_2} \\ &\leq k^2\left[\phi(x, 3x) + \phi(x, x + y) + \phi(x, x - y)\right], \quad \forall\, x, y \ \in E_1.\end{aligned} \tag{3.40}$$

Again replacing y by $2x$ in (3.40), we obtain

$$\|f(6x) - 3f(2x)\|_{E_2} \leq \frac{k^2}{2}\left[\phi(x, -x) + 2\phi(x, 3x)\right], \quad \forall\, x \ \in E_1, \tag{3.41}$$

In (3.41), replace x by $\frac{x}{2}$ in, we arrive

$$\|f(3x) - 3f(x)\|_{E_2} \leq \frac{k^2}{2}\left[\phi\left(\frac{x}{2}, \frac{-x}{2}\right) + 2\phi\left(\frac{x}{2}, \frac{3x}{2}\right)\right], \quad \forall\, x \ \in E_1. \tag{3.42}$$

Now we use

$$\psi(x)=\frac{1}{2}\left[\phi\left(\frac{x}{2},\frac{-x}{2}\right)+2\phi\left(\frac{x}{2},\frac{3x}{2}\right)\right],\quad \forall\, x\ \in E_1.$$

If we replace x by $\frac{x}{3^{n+1}}$ in (3.42)and multiply both sides of (3.42) by 3^n, we get

$$\left\|3^{n+1}f\left(\frac{x}{3^{n+1}}\right)-3^n f\left(\frac{x}{3^n}\right)\right\|_{E_2}\leq k^2\ 3^n\ \psi\left(\frac{x}{3^{n+1}}\right),\quad \forall\, x\ \in E_1\,, \tag{3.43}$$

and for all non-negative integers n. Since E_2 is a p-Banach space and using (3.43), we obtain

$$\begin{aligned}\left\|3^{n+1}f\left(\frac{x}{3^{n+1}}\right)-3^m f\left(\frac{x}{3^m}\right)\right\|_{E_2}^p &\leq \sum_{i=m}^{n}\left\|3^{i+1}f\left(\frac{x}{3^{i+1}}\right)-3^i f\left(\frac{x}{3^i}\right)\right\|_{E_2}^p\\ &\leq k^{2p}\ \sum_{i=m}^{n}3^{ip}\ \psi^p\left(\frac{x}{3^{i+1}}\right)\end{aligned} \tag{3.44}$$

for all non-negative integers m and n with $n\geq m$ and all $x\in E_1$. But

$$\psi^p(x)=\frac{1}{2^p}\left[\phi^p\left(\frac{x}{2},\frac{-x}{2}\right)+2^p\phi\left(\frac{x}{2},\frac{3x}{2}\right)\right],\quad \forall\, x\ \in E_1. \tag{3.45}$$

Therefore it follows from (3.44) and (3.45) that

$$\sum_{i=1}^{\infty}3^{ip}\ \psi^p\left(\frac{x}{3^i}\right)<\infty,\quad \forall\, x\ \in E_1\,. \tag{3.46}$$

Hence we conclude from (3.44) and (3.45) that the sequence $\{3^n\ f\left(\frac{x}{3^n}\right)\}$ is a Cauchy sequence for all $x\in E_1$, since E_2 is complete, the sequence $\{3^n\ f\left(\frac{x}{3^n}\right)\}$ converges for all $x\in E_1$. Now we define the mapping $A:E_1\to E_2$ by (3.44) for all $x\in E_1$ and letting $m=0$ and allowing $n\to\infty$ in (3.14), we get

$$\begin{aligned}\|f(x)-A(x)\|_{E_2}^p &\leq k^{2p}\ \sum_{i=0}^{\infty}3^{ip}\ \psi^p\left(\frac{x}{3^{i+1}}\right)\\ &=\frac{k^{2p}}{3^p}\ \sum_{i=0}^{\infty}3^{ip}\ \psi^p\left(\frac{x}{3^i}\right),\quad \forall x\in E_1.\end{aligned} \tag{3.47}$$

Therefore (3.36) follows from (3.45) and (3.47). We will now show that A is additive.

Use the equations (3.5),(3.32) and (3.39), we obtain

$$\|D\ A(x,y)\|_{E_2}=\lim_{n\to\infty}3^n\left\|D\ f\left(\frac{x}{3^n},\frac{y}{3^n}\right)\right\|_{E_2}\leq 3^n\phi\left(\frac{x}{3^n},\frac{y}{3^n}\right)=0$$

for all $x,y\ \in E_1$, therefore the mapping $A:E_1\to E_2$ satisfies (1.5). But f is a off function, therefore (3.34) implies that the mapping $A:E_1\to E_2$ is odd. Hence by Lemma 2.2, we get that the mapping $A:E_1\to E_2$ is quadratic. To prove the uniqueness of Q, let $T : E_1 \to E_2$ be additive.

To prove uniqueness of A: Consider $T : E_1 \to E_2$ is another additive mapping satisfying (3.35). But

$$\lim_{n\to\infty} 3^{np} \sum_{i=1}^{\infty} 3^{ip} \phi^p \left(\frac{x}{3^{n+i}}, \frac{y}{3^{n+i}} \right) = \lim_{n\to\infty} \sum_{i=1}^{\infty} 3^{ip} \phi^p \left(\frac{x}{3^i}, \frac{y}{3^i} \right) = 0, \quad \forall x \in E_1 \tag{3.48}$$

for all $y \in \{-x, 3x\}$ and then apply (3.48) in equation (3.36), we get

$$\lim_{n\to\infty} 3^{np}\ \tilde{\phi}_o \ \left(\frac{x}{3^n} \right) = 0 \quad \forall\ x \ \in E_1. \tag{3.49}$$

Hence it follows from (3.35) and (3.49),

$$\|A(x) - f(x)\|^p_{E_2}$$
$$= \lim_{n\to\infty} 3^{np} \left\| f\left(\frac{x}{3^n} \right) - T\left(\frac{x}{3^n} \right) \right\|^p_{E_2} \le \frac{k^{2p}}{9^p} \lim_{n\to\infty} 3^{np}\ \tilde{\phi}_o \ \left(\frac{x}{3^n} \right) = 0$$

for all $x \ \in E_1$, so $A = T$. Thus the proof of the theorem is complete. □

Theorem 3.7. *Let $\phi : E_1 \times E_1 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{3^n} \phi(3^n x, 3^n y) = 0, \qquad \forall\ x, y \ \in E_1, \tag{3.50}$$

and

$$\sum_{i=0}^{\infty} \frac{1}{3^i p} \phi^p(3^i x, 3^i y) < \infty, \qquad \forall\ x \in E_1, \tag{3.51}$$

and for all $\{-x, 3x\}$. Suppose that an even function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality (3.5) for all $x, y \in E_1$, the limit

$$A(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \tag{3.52}$$

exists for all $x \in E_1$ and $A : E_1 \to E_2$ is a unique quadratic function

$$\|f(x) - A(x)\|_{E_2} \le \frac{k^2}{3} \left[\tilde{\varphi}_o(x)\right]^{\frac{1}{p}}, \qquad \forall\ x \ \in E_1\,, \tag{3.53}$$

where

$$\tilde{\varphi}_o(x) = \sum_{i=1}^{\infty} \frac{1}{3^{ip}} \left\{ \frac{1}{2^p}\ \phi^p \left(\frac{3^i x}{2}, \frac{-3^i x}{2} \right) + \ \phi^p \left(\frac{3^i x}{2}, \frac{3^{i+1} x}{2} \right) \right\}. \tag{3.54}$$

Proof. If we replace x by $3^n x$ in (3.42) and divide by 9^{n+1} on both sides, we obtain

$$\left\| \frac{1}{3^{n+1}} f(3^{n+1} x) - \frac{1}{3^n} f(3^n x) \right\|_{E_2} \le \frac{k^2}{3^{n+1}}\ \varphi(3^n x), \qquad \forall\ x \ \in E_1 \tag{3.55}$$

and for all non-negative integers n. Since E_2 is a p-Banach space, using (3.55), we obtain

$$\left\|\frac{1}{3^{n+1}} f(3^{n+1}x) - \frac{1}{3^n} f(3^m x)\right\|_{E_2}^p \leq \sum_{i=m}^{n} \left\|\frac{1}{3^{i+1}} f(3^{i+1}x) - \frac{1}{3^i} f(3^i x)\right\|_{E_2}^p$$
$$\leq \frac{k^{2p}}{3^p} \sum_{i=m}^{n} \frac{1}{3^{ip}}\ \psi^p(3^i x) \quad (3.56)$$

for all non-negative integers m and n with $n \geq m$ and all $x \in E_1$. Since

$$\sum_{i=0}^{\infty} \frac{1}{3^{ip}}\ \psi^p(3^i x) < \infty, \quad \forall\ \ x \in E_1\,,$$

then (3.56) implies that the sequence $\{\frac{1}{3^n} f(3^n x)\}$ is a Cauchy sequence for all $x \in E_1$. Since E_2 is complete, the sequence $\{\frac{1}{3^n} f(3^n x)\}$ converges for all $x \in E_1$. Now we define the mapping $A : E_1 \to E_2$ by (3.52) for all $x \in E_1$. The rest of the proof is similar to the proof of Theorem 3.6. $\square$

Corollary 3.8. *Let θ, be non negative real numbers and r, s be real numbers such that $r, s \neq 1$. Suppose that an odd function $f : E_1 \to E_2$ satisfies the inequality (3.30) for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ satisfies*

$$\|f(x) - A(x)\|_{E_2}$$
$$\leq \frac{\theta\ k^2}{2}\left[\frac{1+2^p}{2^r p}\ \frac{1}{|3^{rp} - 3^p|}\ \|x\|_{E_1}^{rp} + \frac{1+2^p\ 3^{sp}}{2^s p}\ \frac{1}{|3^{sp} - 3^p|}\ \|x\|_{E_1}^{sp}\right]^{\frac{1}{p}}. \quad (3.57)$$

If $r = s = t$. Then

$$\|f(x) - Q(x)\|_{E_2} \leq \frac{\theta\ k^2}{3 \cdot 2^{t+1}}\left[\left\{2 + 2^p\left(1 + 3^{tp}\right)\right\}\ \frac{1}{|3^{tp} - 3^t|}\right]^{\frac{1}{p}}\ \ \|x\|_{E_1}^t. \quad (3.58)$$

Proof. In Theorem 3.6, using the equation (3.35), we obtain

$$\|f(x) - A(x)\|_{E_2} \leq \frac{\theta\ k^2}{9}$$
$$\times\left[\sum_{i=1}^{\infty} 3^{ip}\left\{\frac{1}{2^p}\left(\left\|\frac{x}{2 \cdot 3^i}\right\|_{E_1}^{rp} + \left\|\frac{-x}{2 \cdot 3^i}\right\|_{E_1}^{rp}\right)\right\} + \left\{\left\|\frac{x}{2 \cdot 3^i}\right\|_{E_1}^{rp} + \left\|\frac{3x}{2 \cdot 3^i}\right\|_{E_1}^{rp}\right\}\right]^{\frac{1}{p}}$$
$$\leq \frac{\theta\ k^2}{9}\left[\frac{\frac{1}{2^p}+1}{2^{pr}}\ \sum_{i=1}^{\infty}\frac{3^{ip}}{3^{irp}}\ \|x\|_{E_1}^{rp} + \frac{\frac{1}{2^p}+3^{sp}}{2^{sp}}\ \sum_{i=1}^{\infty}\frac{3^{ip}}{3^{isp}}\ \|x\|_{E_1}^{sp}\right]^{\frac{1}{p}}, \quad (3.59)$$

which give raise to our result (3.57). Again substituting $r = s = t$ in (3.57), we find that

$$\|f(x)-A(x)\|_{E_2} \leq \frac{\theta\ k^2}{6\cdot 2^t}\left[\{(2+2^p)+\left(1+2^p\ 3^{tp}\right)\}\ \frac{1}{|3^{tp}-3^t|}\right]^{\frac{1}{p}}\ \ \|x\|^t_{E_1}.$$

this gives our result (3.58). □

Corollary 3.9. *Let θ, be non negative real numbers and r,s be any real numbers such that $\lambda = r,s \neq 1$. Suppose that an odd function $f : E_1 \to E_2$ satisfies the inequality (3.31) for all $x,y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ satisfies*

$$\|f(x)-A(x)\|_{E_2} \leq \frac{\theta\ k^2}{3\cdot 2^{\lambda+1}}\left[(1+2^p\ 3^{sp})\ \ \frac{1}{|3^{tp}-3^p|}\right]^{\frac{1}{p}}\ \ \|x\|^{\lambda}_{E_1}. \tag{3.60}$$

Proof. In Theorem 3.6, using the equation (3.35), we obtain

$$\|f(x)-A(x)\|_{E_2} \leq \frac{\theta\ k^2}{9}$$

$$\times\left[\sum_{i=1}^{\infty} 3^{ip}\left\{\frac{1}{2^p}\left(\left\|\frac{x}{2\cdot 3^i}\right\|^{rp}_{E_1}+\left\|\frac{-x}{2\cdot 3^i}\right\|^{rp}_{E_1}\right)\right\}+\left\{\left\|\frac{x}{2\cdot 3^i}\right\|^{rp}_{E_1}+\left\|\frac{3x}{2\cdot 3^i}\right\|^{rp}_{E_1}\right\}\right]^{\frac{1}{p}}$$

$$\leq \frac{\theta\ k^2}{9}\left[\frac{1}{2^p}\cdot\frac{1}{3^p}\ \frac{1}{|3^{\lambda p}-3^t|}+\frac{3^{\lambda}p}{3^p}\ \frac{1}{|3^{\lambda p}-3^t|}\right]^{\frac{1}{p}}\|x\|^{\lambda}_{E_1}$$

which give raise to our result (3.60). □

Theorem 3.10. *Let $\phi : E_1 \times E_1 \to [0,\infty)$ be a function satisfies (3.3) and (3.4) for all $x,y \in E_1$ and for all $y \in \{x,-3x\}$. Suppose that a function $f : E_1 \to E_2$ satisfies the inequality (3.5) with $f(0)=0$ for all $x,y \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ and a unique additive function $A : E_1 \to E_2$ satisfies (1.5) and*

$$\|f(x)-Q(x)-A(x)\|_{E_2} \leq \frac{k^4}{18}\left[\left\{\tilde{\psi}_e(x)+\tilde{\psi}_e(-x)\right\}^{\frac{1}{p}}+\{\tilde{\varphi}_o(x)+\tilde{\varphi}_o(-x)\}^{\frac{1}{p}}\right] \tag{3.61}$$

for all $x \in E_1$ where $\tilde{\psi}_e(x)$ and $\tilde{\varphi}_o(x)$ has been defined in (3.8) and (3.36) respectively, for all $x \in E_1$.

Proof. We have $f_e(x) = \frac{f(x)+f(-x)}{2}$ for all $x \in E_1$. Therefore $f_e(0)=0$, $f_e(-x) = f_e(x)$ and $||D\ f_e(x,y)|| \leq \frac{k}{2}[\phi(x,y)+\phi(-x,-y)]$ for all $x,y \in E_1$.

Let

$$\Phi(x,y) \leq \frac{k}{2}\left[\phi(x,y)+\phi(-x,-y)\right], \quad \forall\ x,y \in E_1. \tag{3.62}$$

Then using Lemma 3.1, we obtain

$$\Phi^p(x,y) \leq \frac{k^p}{2^p}\left[\phi^p(x,y)+\phi^p(-x,-y)\right], \quad \forall x,y \in E_1.$$

Then therefore $\sum_{i=1}^{\infty} 9^{ip}\, \Psi^p\left(\frac{x}{3^i},\frac{y}{3^i}\right) \leq \infty$ for all $x \in E_1$ and for all $y \in \{x,-3x\}$. Hence, in view of Theorem 3.2, there exists a unique quadratic function $Q : E_1 \to E_2$ satisfies

$$\|f_e(x)-Q(x)\|_{E_2} \leq \frac{k^2}{9}\left[\tilde{\Psi}_e(x)\right]^{\frac{1}{p}}, \quad \forall\ x \in E_1\,, \tag{3.63}$$

where

$$\tilde{\Psi}_e(x) = \sum_{i=1}^{\infty} 9^{ip}\left\{\frac{1}{2^p}\ \Phi^p\left(\frac{x}{2\cdot 3^i},\frac{-x}{2\cdot 3^i}\right) +\ \Phi^p\left(\frac{x}{2\cdot 3^i},\frac{3x}{2\cdot 3^i}\right)\right\}. \tag{3.64}$$

Applying (3.62) in (3.64) and using Lemma 3.1, we obtain

$$\|f(x)-Q(x)\|_{E_2} \leq \frac{k^p}{2^p}\left[\tilde{\psi}_e(x)+\tilde{\psi}_e(-x)\right], \quad \forall x\ \in E_1. \tag{3.65}$$

Therefore it follows from (3.62),

$$\|f(x)-Q(x)\|_{E_2} \leq \frac{k^3}{18}\left[\tilde{\psi}_e(x)+\tilde{\psi}_e(-x)\right]^{\frac{1}{p}}, \quad \forall x\ \in E_1. \tag{3.66}$$

Also we have $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x\ \in E_1$. Therefore $f_o(0) = 0$, $f_p(-x) = -f_o(x)$ and $||D\ f_o(x,y)|| \leq \Phi(x,y)$ for all $x,y\ \in E_1$. From Theorem 3.6, it follows that there exists a unique additive function $A : E_1 \to E_2$ satisfies

$$\|f_o(x)-A(x)\|_{E_2} \leq \frac{k^2}{9}\left[\tilde{\varphi'}_o(x)\right]^{\frac{1}{p}}, \quad \forall\ x \in E_1 \tag{3.67}$$

where

$$\tilde{\varphi'}_o(x) = \sum_{i=1}^{\infty} 3^{ip}\left\{\frac{1}{2^p}\ \Phi^p\left(\frac{x}{2\cdot 3^i},\frac{-x}{2\cdot 3^i}\right) +\ \Phi^p\left(\frac{x}{2\cdot 3^i},\frac{3x}{2\cdot 3^i}\right)\right\}. \tag{3.68}$$

Using the above ideas as given in (3.65), we arrive

$$\tilde{\varphi'}_o(x) = \frac{k^p}{2^p}\left[\tilde{\varphi}_o(x)+\tilde{\varphi}_o(-x)\right], \quad \forall x \quad \in E_1. \tag{3.69}$$

Again using (3.69) in (3.67), we obtain

$$\|f(x)-A(x)\|_{E_2} \leq \frac{k^p}{2^p}\left[\tilde{\varphi}_o(x)+\tilde{\varphi}_o(-x)\right], \quad \forall x \quad \in E_1. \tag{3.70}$$

Then the result (3.61) follows from (3.66) and (3.70). □

Theorem 3.11. *Let $\phi : E_1 \times E_1 \to [0,\infty)$ be a function satisfies (3.50) and (3.51) for all $x,y \in E_1$ and for all $y \in \{x,-3x\}$. Suppose that a function $f : E_1 \to E_2$ satisfies the inequality (3.5) for all $x,y \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ and a unique additive function*

$A : E_1 \to E_2$ satisfies (1.5) and

$$\|f(x) - Q(x) - A(x)\|_{E_2} \leq \frac{k^4}{18}\left[\left\{\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)\right\}^{\frac{1}{p}} + \{\tilde{\varphi}_o(x) + \tilde{\varphi}_o(-x)\}^{\frac{1}{p}}\right] \quad (3.71)$$

for all $x \in E_1$ where $\tilde{\psi}_e(x)$ and $\tilde{\varphi}_o(x)$ has been defined in (3.26) and (3.54) respectively, for all $x \in E_1$.

Proof. The proof of this theorem follows from Theorem 3.3 and Theorem 3.7 and it is very similar to the Theorem 3.10 and so the proof is omitted here. □

Corollary 3.12. *Let θ, r, s be non negative real numbers and such that $r, s \neq 1, 2$. Suppose that an function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality (3.30) for all $x, y \in E_1$. Then there exists a unique quadratic function $Q : E_1 \to E_2$ and a unique additive function $A : E_1 \to E_2$ satisfying (1.5)*

$$\|f(x) - Q(x) - A(x)\|_{E_2} \leq \frac{\theta\ k^2}{6}\left[\begin{array}{l} 3^p\left\{\frac{2^p+1}{2^{rp}}\ \frac{1}{|3^{rp}-3^p|}\|x\|_{E_1}^{rp} + \frac{1+2^p\ 3^{sp}}{2^s p}\ \frac{1}{|3^{sp}-3^p|}\|x\|_{E_1}^{sp}\right\} \\ +\frac{2^p+1}{2^{rp}}\ \frac{1}{|3^{rp}-3^p|}\|x\|_{E_1}^{rp} + \frac{1+2^p\ 3^{sp}}{2^s p}\ \frac{1}{|3^{sp}-3^p|}\|x\|_{E_1}^{sp}\end{array}\right]^{\frac{1}{p}}. \quad (3.72)$$

If $r = s = t$. Then

$$\|f(x) - Q(x) - A(x)\|_{E_2} \leq \frac{\theta\ k^4}{2^2 \cdot 6}\left[\begin{array}{l} 3^p\left\{2 + 2^p(1 + 3^{tp})\right\}\ \frac{1}{|3^{tp}-3^p|} \\ +\left\{2 + 2^p(1 + 3^{tp})\right\}\ \frac{1}{|3^{sp}-3^p|}\end{array}\right]^{\frac{1}{p}}\|x\|_{E_1}^t. \quad (3.73)$$

Proof. Define the function $\phi : E_1 \times E_1 \to [0, \infty)$ by

$$\phi(x, y) = \theta\left[\|x\|_{E_1}^r + \|y\|_{E_1}^s\right].$$

Using Corollary 3.4, we obtain

$$\|f(x) - Q(x)\|_{E_2} \leq \frac{\theta\ k^2}{6} \times \left[3^p\ \left\{\frac{2^p+1}{2^{rp}}\ \frac{1}{|3^{rp}-3^p|}\|x\|_{E_1}^{rp} + \frac{1+2^p\ 3^{sp}}{2^s p}\ \frac{1}{|3^{sp}-3^p|}\|x\|_{E_1}^{sp}\right\}\right]^{\frac{1}{p}} \quad (3.74)$$

again using Corollary 3.8, we obtain

$$\|f(x) - A(x)\|_{E_2}$$

$$\leq \frac{\theta\ k^2}{6}\left[\frac{2^p+1}{2^{rp}}\ \frac{1}{|3^{rp}-3^p|}\ \|x\|_{E_1}^{rp} + \frac{1+2^p\ 3^{sp}}{2^s p}\ \frac{1}{|3^{sp}-3^p|}\ \|x\|_{E_1}^{sp}\right]^{\frac{1}{p}}. \quad (3.75)$$

Adding the equations (3.74) and (3.75), we obtain (3.72). By taking $r = s = t$, equation (3.72) becomes (3.73). □

Corollary 3.13. *Let θ, r, s be non negative real numbers and such that $r, s \neq 1, 2$. Suppose that an function $f : E_1 \to E_2$ with $f(0) = 0$ satisfies the inequality for all $x, y \in E_1$. Then there exists a unique quadratic function $Q : E_1 \to E_2$ and a unique additive function $A : E_1 \to E_2$ satisfying (1.5)*

$$\|f(x) - Q(x) - A(x)\|_{E_2} \leq \frac{\theta\ k^4}{3 \cdot 2^{\lambda+1}} \times \left[(1+2^p\ 3^{sp})\ \left\{\frac{3^p}{|3^{\lambda p}-9^p|} \cdot \frac{1}{|3^{\lambda p}-3^p|}\right\}\right]^{\frac{1}{p}} \|x\|_{E_1}^{\lambda}. \quad (3.76)$$

Proof. Using Corollary 3.5, we obtain

$$\|f_e(x) - Q(x)\|_{E_2} \leq \frac{\theta\ k^4}{2^{\lambda+1}}\left[(1+2^p\ 3^{sp})\ \frac{1}{|3^{\lambda p}-9^p|}\right]^{\frac{1}{p}} \|x\|_{E_1}^{\lambda} \quad (3.77)$$

again using Corollary 3.9, we obtain

$$\|f_o(x) - A(x)\|_{E_2} \leq \frac{\theta\ k^4}{3 \cdot 2^{\lambda+1}}\left[(1+2^p\ 3^{sp})\ \frac{1}{|3^{\lambda p}-3^p|}\right]^{\frac{1}{p}} \|x\|_{E_1}^{\lambda}. \quad (3.78)$$

Adding the equations (3.77) and (3.78), we obtain (3.76). □

References

[1] J. Aczel, J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press (1989).

[2] J.H. Bae, K.W. Jun, On the generalized Hyers-Ulam-Rassias stability of quadratic functional equation, *Bull. Korean. Math. Soc.*, **38** (2001), 325-336.

[3] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, **1**, Colloq. Publ., Volume 48, Amer. Math. Soc., Providence, RI (2000).

[4] P.W. Cholewa, Remarks on the stability of functional equations, *Equations Math.*, **27** (1984), 76-86.

[5] S. Czerwik, On the stability of the quadratic mappings in normed spaces, *Abh. Math. Sem. Univ. Hamburg*, **62** (1992), 59-64.

[6] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431-436.

[7] A. Grabiec, The generalized Hyers-Ulam stability of a class of functional equations, *Publ. Math. Debrecen*, **48** (1996), 217-235.

[8] D.H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.*, **27** (1941), 222-224.

[9] D.H. Hyers, G. ISac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel (1998).

[10] D.H. Hyers, Th.M. Rassias, Approximate homomorphisms, *Equationes Math.*, **44** (1992), 125-153.

[11] S.M. Jung, Quadratic functional equations of pexider type, *Int. J. Math. and Math. Sci.*, **24**, No. 5 (2000), 351-359.

[12] K.W. Jun, H.M. Kim, On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive functional equation, *Bull. Korean Math. Soc.*, **42**, No. 1 (2005), 133-148.

[13] Pl. Kannappan, Quadratic functional equation and inner product spaces, *Results Math.*, **27** (1995), 368-372.

[14] Y.W. Lee, The stability of derivatives on Banach algebras, *Bull. Inst. Math. Acad. Sinicia*, **28** (2000), 113-116.

[15] A. Najati, M.B. Moghimi, Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces, *J. Math. Anal. Appl.* (2007).

[16] J.M. Rassias, On the stability of Euler-Lagrange functional equation, *Chinese J. Math.*, **20** (1992), 185-190.

[17] J.M. Rassias, Hyers-Ulam stability for a quadratic functional equation in several variables, *J. Indian. Math. Soc.*, **68** (2001), 65-73.

[18] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

[19] Th.M. Rassias, On the stability of the quadratic functional equation and its applications, *Studia. Univ. Babes-Bolyai*, **XLIII** (1998), 89-124.

[20] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta. Appl. Math.*, **62** (2000), 23-170.

[21] Th.M. Rassias, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.*, **251** (2000), 264-284.

[22] S. Rolewicz, *Metric Linear Spaces*, PWN-Polish Sci. Publ., Warszawa, Reidel, Dordrecht (1984).

[23] F. Skof, Propricta locali e approssimazione di operatori, *Semi, Mat. Fis.*, Milano, **53** (1983), 113-129.

[24] S.M. Ulam, *Problems in Modern Mathematics*, Science Edition, Wiley, New York (1964).