International Journal of Pure and Applied Mathematics
Volume 48 No. 2 2008, 103-116

OSCILLATION CRITERIA FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS AND A DAMPING TERM

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Abstract: We study the second order differential equation $[r(x)\psi(y(x))(y(x)+c(x)y(d(x)))']' + \int_{\alpha}^{\beta} a(x,\theta)f(y(b(x,\theta)))d\theta = 0$ and establish some criteria for the solutions of this equation to be oscillatory that extend the existing results.

AMS Subject Classification: 34C10, 34K11, 34K40
Key Words: oscillatory, second order non-linear differential equations, distributed arguments, damping term

1. Introduction

In recent years, there have been many results about the oscillation criteria of non-linear second-order differential equations. For example, Jan Seman [1] considers the oscillation criteria of the non-linear differential equation $(a(x)y')' + b(x)g(y) = c(x)$, and S.H. Saker, J.V. Manojlovic and et al [3], [4] study the

Received: August 20, 2008
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oscillation criteria of the damped non-linear differential equation

$$(y(x)+c(x)y(x-x_0))''+a(x)f(y(b(x)))=0. \tag{1.1}$$

Zhiting Xu [5], [6] generalizes equation (1.1) to

$$[r(x)\psi(y(x))(y(x)+c(x)y(x-\tau))']'+\int_\alpha^\beta a(x,\theta)f(y(b(x,\theta)))d\theta=0 \tag{1.2}$$

and establishes some oscillation criteria for this equation. In most of the papers that study the oscillation criteria of differential equations, the authors require a function $H(x,y)$ that satisfies

$$H(x,x)=0, H(x,y)>0 \text{ and } h(x,y)=-\frac{\partial H(x,y)}{\partial y}$$
$$\text{for } (x,y)\in D\{(x,y)|x\geq y\geq x_0\}, \tag{H1}$$

and then use the integral average and inequalities to obtain their criteria. Using this method, the non-negative and monotonous properties of $H(x,y)$ and $H(x,x)$ make the analysis easier and the results obtained look very pretty. For example, Manojlovic [7] obtained the following criteria

$$\limsup_{x\to\infty}\int_{x_0}^x[\frac{H(x,s)}{H(x,x_0)}a(s)-\frac{\beta c(s)h^{\alpha+1}(x,s)}{H(x,x_0)H^\alpha(x,s)}]ds=\infty. \tag{H2}$$

The factor $\frac{H(x,s)}{H(x,x_0)}$ in (H2), as a weight applied to $a(x)$, weakens the property of $a(x)$ as $x\to\infty$ because it gets closer and closer to zero as s approaches the upper limit of the integral. Therefore, it reduces the possibility for (H2) to be satisfied. For example, we choose $c(x)=1$, $h(y)=1,\alpha=1$, $a(x)=\frac{a}{x^2}$ and $f(y)=y$ in equation (1.2). Following the notations given by Manojlovic [8], (H2) becomes

$$\limsup_{x\to\infty}\frac{1}{H(x,x_0)}\int_{x_0}^x[\frac{aH(x,s)}{s^2}-\frac{4h^{\alpha+1}(x,s)}{H^\alpha(x,s)}]ds$$
$$\leq\limsup_{x\to\infty}\frac{1}{H(x,x_0)}\int_{x_0}^x\frac{aH(x,s)}{s^2}ds<\infty$$

for some $H(x,y)$ that satisfies (H2). Hence, criterion (H2) cannot be used to determine whether the equation $y''(x)+\frac{a}{x^2}y(x)=0$ has oscillating solutions. Baculikova [8] studies the oscillation criteria by using a different method. Although this equation is a special case of (1.1) or (1.2), the result is much better since $\frac{H(x,s)}{H(x,x_0)}$ is changed to x.

In this paper, we consider the damped non-linear differential equation

$$(r(x)\psi(y(x))(y(x)+c(x)y(d(x)))')'+a(x)f(y(b(x)))=0 \tag{1.3}$$

and the damped non-linear differential equation with distributed deviating arguments

$$(r(x)\psi(y(x))(y(x)+c(x)y(d(x)))')' + \int_{\alpha}^{\beta} a(x,\theta)f(y(b(x,\theta)))d\theta = 0 \qquad (1.4)$$

and establish some oscillation criteria. It is clear that (1.3) is a generalization of (1.1) and (1.4) a generalization of (1.2). It is also clear that equation (1.4) degenerates into (1.3) if we choose $a(x,\theta) = a(x), b(x,\theta) = b(x), \alpha = 0$ and $\beta = 1$. Therefore, it suffices to study (1.4) only. Our criteria are straight forward, easier to analyze and cover more cases. We will use several examples in Section 3 to show this. In the following discussion, we will need the following conditions.

Condition 1. *$a(x,\theta)$, $b(x,\theta) \in C(I\times[\alpha,\beta], R^+)$ and $c(x), d(x), r(x) \in C^1(I,R^+)$ for $I=[x_0,\infty)$; $b(x,\theta)$ is non-decreasing about θ; $0 < b(x,\theta) < x$ and $b_x(x,\theta) > 0$ for $\theta \in [\alpha,\beta]$; $0 < d(x) < x$; $\lim_{x\to\infty} b(x,\theta) = \lim_{x\to\infty} d(x) = \lim_{x\to\infty} R(x) = \infty$ for $R(x) = \int_{x_0}^{x} \frac{1}{r(s)}ds$.*

Condition 2. *$\psi(y), f(y) \in C^1(-\infty,\infty); 0 < \psi(y) \le M < \infty; yf(y) > 0$ for $y \ne 0; 0 < m_f \le |\frac{f'(y)y}{f(y)}| \le M_f < \infty$; $f(y)$ is increasing; for any $D > 0$, there exists a constant $k_D > 0$ such that $f'(y) \ge k_D$ for $y \in (-\infty,-D)\cup(D,\infty)$.*

Our conditions about $y(x)$ are much general than those given by Zhiting Xu and et al [5], [6]. They give two conditions $S_1 : f'(y) \ge k_1 > 0$ for $y \ne 0$ and $S_3 : \frac{f(y)}{y} \ge k_3 > 0$ for $y \ne 0$. It is clear that $f(y) = y^3$ does not satisfy S_1 and S_3, but it does satisfy our Condition 2.

2. Main Results and Proof

Lemma 1. *If equation (1.4) satisfies the Conditions 1 and 2, $y = y(x)$ is a non-oscillatory solution of (1.4) and $z(x) = y(x) + c(x)y(d(x))$, then, there exists constant $X > x_0$ such that, for $x \ge X$,*

$$z(x)z'(x) \ge 0; z(x)(r(x)\psi(y(x))z'(x))' < 0. \qquad (2.1)$$

Proof. Assume that $y = y(x)$ is a non-oscillating solution of (1.4). Then, there exists $X > x_0$ such that $y = y(x)$ is monotonous and $y(x) \ne 0$ for all $x \in [X,\infty)$. Without loss of generality, we assume that $y(x) > 0$ for $x \in [X,\infty)$. It follows that $z(x) > 0$ and $f(z(x)) > 0$ for $x \in [X,\infty)$. From (1.4), one can get

$$\frac{d}{dx}(\psi(y(x))\frac{dz}{dR}) = -\int_{\alpha}^{\beta} a(x,\theta)f(y(b(x,\theta)))d\theta < 0.$$

Hence, $\psi(y(x))\frac{dz}{dR} = \psi(y(x))r(x)z'(x)$ is decreasing and the second inequality is true. Now, we need to prove that $z'(x) \geq 0$ for $x > X$. If this is not true, we have $z'(x) < 0$ for large x by the property we obtained for $\psi(y(x))r(x)z'(x)$ and

$$\frac{dz}{dx}(x) \leq \frac{\psi(y(X))}{r\psi(y(x))}\frac{dz}{dR}(X) \leq \frac{\psi(y(X))}{rM}\frac{dz}{dR}(X) < 0, \text{ for large } X. \tag{2.2}$$

Integrating (2.2) from X to x, one gets

$$z(x) \leq z(X) + \frac{\psi(y(X))}{M}\frac{dz}{dR}(X)(R(x) - R(X)) \to -\infty \text{ as } x \to \infty.$$

This is clearly a contradiction to the fact $z(x) > 0$ and finishes the proof of the lemma. □

Noticing that equation (1.4) is not changed except that $\psi(y)$ is changed to $\psi(-y)$, we only need to consider the case for $y > 0$ in the proof of the following theorems.

Theorem 3. *If equation (1.4) satisfies Conditions 1 and 2, $c'(x) \leq 0$ and there exists non-negative monotonous function $p(x) \in C^1(0,\infty)$ such that*

$$\limsup_{x\to\infty} \int_{x_0}^{x} [A_{\alpha,\beta}(s)p(s) - \frac{Mr(b(s,\alpha))(1+c(b(s,\alpha)))}{4k_D p(s)b'(s,\alpha)}B_1^2(s,\alpha)]ds = \infty\,, \tag{2.3}$$

where

$$\begin{aligned} A_{\alpha,\beta}(x) &= \int_{\alpha}^{\beta} a(x,\theta)d\theta, \\ B_1(x,\alpha) &= \max\{p'(x) + \frac{m_f p(x)b'(x,\alpha)c'(b(x,\alpha))}{1+c(b(x,\alpha))}, 0\}, \end{aligned}$$

then, the solutions of (1.4) are oscillating.

Proof. Suppose that $y = y(x)$ is a non-oscillatory solution of (1.4). Then, there exists a constant $X_1 > x_0$ such that $y = y(x)$ is monotonous and $y(x) \neq 0$ for $x \in [X_1, \infty)$. Since $\lim_{x\to\infty} b(x,\theta) = \lim_{x\to\infty} d(x) = \infty$ for $\theta \in [\alpha,\beta]$, there exists constant $X \geq X_1 \geq x_0$ such that $b(x,\alpha) \geq X_1$ and $d(x) \geq X_1$ for $x \geq X$. Therefore, we have $b(x,\theta) \geq X_1$ for $x \geq X$ because $b(x,\theta)$ is a non-decreasing function of θ. Now, we can assume that $y(x)$, $y(b(x,\theta))$ and $y(d(x))$ are not zero and monotonous on $[X,\infty)$. Without loss of generality, we may assume that $y(x) > 0$ for $x \geq X$. It follows form Lemma 1 and the condition $c'(x) \leq 0$ that $y'(x) \geq 0$ and $y = y(x)$ is increasing over (X,∞), and hence, $y(x) \geq y(b(x,\theta))$,

$y(x) \geq y(d(x))$ and $z(x) \geq y(x) \geq \frac{z(x)}{1+c(x)}$ for $x \geq X$. Since $f(y) > 0$ is increasing and $b(x,\theta)$ is non-decreasing, we can use (1.4) to get

$$(r(x)\psi(y(x))z'(x))' + A_{\alpha,\beta}(x)f(y(b(x,\alpha))) \leq (r(x)\psi(y(x))z'(x))' + A_{\alpha,\beta}(x)f(\frac{z(b(x,\alpha))}{1+c(b(x,\alpha))}) \leq 0. \quad (2.4)$$

Now, we introduce the Riccati transformation

$$W(x) = \frac{p(x)r(x)\psi(y(x))z'(x)}{f(u(b(x,\alpha)))} \text{ for } x \geq X \text{ and } u(x) = \frac{z(x)}{1+c(x)}. \quad (2.5)$$

Differentiating (2.5) and applying inequality (2.4) and Condition 2 to it, we get

$$W'(x) \leq -A_{\alpha,\beta}(x)p(x) + (p'(x) + \frac{m_f p(x)b'(x,\alpha)c'(b(x,\alpha))}{1+c(b(x,\alpha))})t - \frac{p(x)b'(x,\alpha)f'(u(b(x,\alpha)))z'(b(x,\alpha))}{f(u(b(x,\alpha)))(1+c(b(x,\alpha)))}t, \quad (2.6)$$

where $t = \frac{r(x)\psi(y(x))z'(x)}{f(u(b(x,\alpha)))}$. Applying the previous lemma to (2.6), one gets

$$r(b(x,\alpha))\psi(y(b(x,\alpha)))z'(b(x,\alpha)) \geq r(x)\psi(y(x))z'(x).$$

Hence,

$$W'(x) \leq -A_{\alpha,\beta}(x)p(x) + B_1(x,\alpha)t - \frac{k_D p(x)b'(x,\alpha)t^2}{Mr(b(x,\alpha))(1+c(b(x,\alpha))},$$

where $D = \frac{z(b(X,\alpha))}{1+c(b(X,\alpha))}$. Now we use Conditions 1 and 2 and get

$$W'(x) \leq -A_{\alpha,\beta}(x)p(x) + \frac{Mr(b(x,\alpha))(1+c(b(x,\alpha)))}{4k_D p(x)b'(x,\alpha)}B_1^2(x,\alpha). \quad (2.7)$$

Integrating both sides of (2.7) from x_0 to x, one gets

$$\int_{x_0}^{x} [A_{\alpha,\beta}(s)p(s) - \frac{Mr(b(s,\alpha))(1+c(b(s,\alpha)))}{4k_D p(s)b'(s,\alpha)}B_1^2(s,\alpha)]ds \leq W(x_0) - W(x) < W(x_0).$$

This is a contradiction and finishes the proof of the theorem. □

Now, we give one more condition on $f(y)$.

Condition 4. *There exist constants k_1, k_2 and α such that $0 \leq \alpha \leq 1$ and $k_1|f(y)|^\alpha \geq f'(y) \geq k_2|f(y)|^\alpha$.*

Theorem 5. *If equation (1.4) satisfies Conditions 1, 2, and 3, $c'(x) \leq 0$ and there exists a non-negative monotonous function $p(x) \in C^1(0,\infty)$ such that*

$$\limsup_{x\to\infty} \int_{x_0}^{x} [A_{\alpha,\beta}(s)p(s) - \frac{Mr(b(s,\alpha))(1+c(b(s,\alpha)))}{4k_D p(s)b'(s,\alpha)}B_2^2(s,\alpha)]ds = \infty, \quad (2.8)$$

where

$$B_2(x,\alpha) = \max\{p'(x) + \frac{k_2 p(x) b'(x,\alpha) c'(b(x,\alpha))}{(1-\alpha)k_1(1+c(b(x,\alpha)))}, 0\},$$

then, the solutions of (1.4) are oscillatory.

The proof of this theorem is similar to the one for the previous theorem except changing inequality (2.6) to the following one

$$W'(x) \le -A_{\alpha,\beta}(x)p(x) + (p'(x) + \frac{k_2 p(x) b'(x,\alpha) c'(b(x,\alpha))}{(1-\alpha)k_1(1+c(b\,(x,\alpha))))} t \\ - \frac{k_D p(x) b'(x,\alpha) t^2}{Mr(b(x,\alpha))(1+c(b(x,\alpha)))}. \quad (2.9)$$

Theorem 2 is not too much different form theorem 1 except that condition (2.8) can cover more broad equations that (2.3) when $m_f < \frac{k_2}{(1-\alpha)k_1}$.

Theorem 6. *If equation (1.4) satisfies Condition 3, $c'(x) \ge 0$, $0 \le c(x) \le 1$, and there exists non-negative monotonous function $p(x) \in C_1(0,\infty)$ such that*

$$\limsup_{x\to\infty} \int_{x_0}^{x} [A_{\alpha,\beta}(s)p(s) - \frac{Mr(b(s,\alpha))B_2^2(s,\alpha)}{4k_D p(s) b'(s,\alpha)(1-c(b(s,\alpha)))}]ds = \infty, \quad (2.10)$$

where $B_3(x,\alpha) = p'(x) + \frac{M_f p(x) b'(x,\alpha) c'(b(x,\alpha))}{1-c(b(x,\alpha))}$, then, the solutions of (1.4) are oscillatory.

We sketch the proof of this theorem because it is similar to the proof of theorem 1. First, noticing that $y(x) \le z(x)$ and $z(x) \le y(x) + c(x)z(d(x)) \le y(x) + c(x)z(x)$, we get $y(x) \ge (1-c(x))z(x)$. Thus, $y(x) \ge (1-c(x))z(x)$ for $x > X$. Since $f(y)$ is increasing, we can get from equation (1.4)

$$(r(x)\psi(y(x))z'(x))' + A_{\alpha,\beta}(x) f((1-c(b(x,\alpha)))z(b(x,\alpha))) \le 0. \quad (2.11)$$

Now, we introduce the Riccati transformation again

$$W(x) = \frac{p(x)r(x)\psi(y(x))z'(x)}{f(u(b(x,\alpha))}, \text{ for } x \ge X \text{ and } u(x) = (1-c(x))z(x). \quad (2.12)$$

Clearly, we have $W(x) > 0$ and

$$W'(x) \le -A_{\alpha,\beta}(x)p(x) + (p'(x) + \frac{p(x)b'(x,\alpha)c'(b(x,\alpha))f'(u(b(x,\alpha)))u(b(x,\alpha))}{f(u(b(x,\alpha)))(1-c(b(x,\alpha)))})t \\ - p(x)b'(x,\alpha)\frac{f'(u(b(x,\alpha))(1-c(b(x,\alpha)))z'(b(x,\alpha))}{f(u(b(x,\alpha)))}t, \quad (2.13)$$

where $t = \frac{r(x)\psi(y(x))z'(x)}{f(u(b(x,\alpha))}$. By Condition 2, we can get

$$W'(x) \le -A_{\alpha,\beta}(x)p(x) + B_3(x,\alpha)t - \frac{k_D p(x)b'(x,\alpha)(1-c(b(x,\alpha)))}{Mr(b(x,\alpha))}t^2$$
$$\le -A_{\alpha,\beta}(x)p(x) + \frac{Mr(b(x,\alpha))B_3^2(x,\alpha)}{4k_D p(x)b'(x,\alpha)(1-c(b(x,\alpha)))}. \quad (2.14)$$

Integrating (2.14) from x_0 to x, one gets

$$\int_{x_0}^{x}[A_{\alpha,\beta}(s)p(s) - \frac{Mr(b(s,\alpha))B_3^2(s,\alpha)}{4k_D p(s)b'(s,\alpha)(1-c(b(s,\alpha)))}]ds \le W(x_0) - W(x) < W(x_0).$$

This is a contradiction to condition (2.10).

Similarly we can also prove the following theorems.

Theorem 7. *If equation (1.4) satisfies Condition 3, $c'(x) \ge 0$, $0 \le c(x) < 1$, and there exists non-negative monotonous function $p(x)$ such that*

$$\limsup_{x\to\infty} \int_{x_0}^{x}[A_{\alpha,\beta}(s)p(s) - \frac{Mr(b(s,\alpha))B_4^2(s,\alpha)}{4k_D p(s)b'(s,\alpha)(1-c(b(s,\alpha)))}]ds = \infty, \quad (2.15)$$

where

$$B_4(x,\alpha) = p'(x) + \frac{k_1 p(x)b'(x,\alpha)c'(b(x,\alpha))}{(1-\alpha)k_2(1-c(b(x,\alpha)))},$$

then, the solutions of equation (1.4) are oscillatory.

Theorem 8. *If equation (1.4) satisfies Condition 3, $0 \le c(x) < 1$, $\pm f(\pm uv) \ge M_f f(u)f(v)$ for any u and v with $uv > 0$, and there exists a non-negative monotonous function $p(x) \in C^1(0,\infty)$ such that*

$$\limsup_{x\to\infty} \int_{x_0}^{x}[M_f A_{\alpha,\beta}(s)f(1-c(b(s,\alpha))p(s)) - \frac{Mr(b(s,\alpha))p'^2(s)}{4k_D p(s)b'(s,\alpha)}]ds = \infty, \quad (2.16)$$

or

$$\limsup_{x\to\infty} \int_{x_0}^{x}[M_f \widetilde{A}_{\alpha,\beta}(s)p(s) - \frac{Mr(b(s,\alpha))p'^2(s)}{4k_D p(s)b'(s,\alpha)}]ds = \infty \quad (2.17)$$

where $\widetilde{A}_{\alpha,\beta}(x) = \int_\alpha^\beta a(x,\theta)f(1-c(b(x,\theta)))d\theta$, then, the solutions of (1.4) are oscillatory.

To simplify the conditions in Theorems 1-5, we have the following corollary.

Corollary 9. *If we change conditions (2.3), (2.8), (2.10), (2.15), (2.16) and (2.17) in Theorems 1-5 to conditions (2.18)-(2.23) respectively and keep the other conditions, then, the solutions of (1.4) are oscillatory.*

$$\limsup_{x\to\infty} \int_{x_0}^{x} A_{\alpha,\beta}(s)p(s)ds = \infty, \quad (2.18)$$

$$\int_{x_0}^{x} \frac{Mr(b(s,\alpha))(1+c(b(s,\alpha)))}{4k_D p(s) b'(s,\alpha)} B_1^2(s,\alpha) ds < \infty;$$

$$\limsup_{x\to\infty} \int_{x_0}^{x} A_{\alpha,\beta}(s) p(s) ds = \infty, \tag{2.19}$$

$$\int_{x_0}^{x} \frac{Mr(b(s,\alpha))(1+c(b(s,\alpha)))}{4k_D p(s) b'(s,\alpha)} B_2^2(s,\alpha) ds < \infty$$

$$\limsup_{x\to\infty} \int_{x_0}^{x} A_{\alpha,\beta}(s) p(s) ds = \infty, \tag{2.20}$$

$$\int_{x_0}^{x} \frac{Mr(b(s,\alpha)) B_2^2(s,\alpha)}{4k_D p(s) b'(s,\alpha)(1-c(b(s,\alpha)))} ds < \infty;$$

$$\limsup_{x\to\infty} \int_{x_0}^{x} A_{\alpha,\beta}(s) p(s) ds = \infty, \tag{2.21}$$

$$\int_{x_0}^{x} \frac{Mr(b(s,\alpha)) B_4^2(s,\alpha)}{4k_D p(s) b'(s,\alpha)(1-c(b(s,\alpha)))} ds < \infty;$$

$$\limsup_{x\to\infty} \int_{x_0}^{x} M_f A_{\alpha,\beta}(s) f(1-c(b(s,\alpha))p(s)) ds = \infty, \tag{2.22}$$

$$\int_{x_0}^{x} \frac{Mr(b(s,\alpha)) p'^2(s)}{4k_D p(s) b'(s,\alpha)} ds < \infty;$$

$$\limsup_{x\to\infty} \int_{x_0}^{x} M_f \widetilde{A}_{\alpha,\beta}(s) p(s) ds = \infty, \tag{2.23}$$

$$\int_{x_0}^{x} \frac{Mr(b(s,\alpha)) p'^2(s)}{4k_D p(s) b'(s,\alpha)} ds < \infty.$$

Remark 1. In the paper by S.H. Saker and J.V. Manojlovic [4], they required $0 \le c(x) < 1$, $d(x) = x - x_0$, $r(x) = 1$ and $\psi(y) = 1$. Comparing their results with our Theorem 3, it is easy to find that their results are complicated than ours and all their examples satisfy the conditions of Theorem 3.

Remark 2. If $c(x)$ is monotonous and $\lim_{x\to\infty}(1 - c(b(x))) = 0$, we can find that an equation that satisfies the conditions of Theorem 5 certainly satisfies one of the conditions in Theorem 1-4.

We first use a simple example to show that our theorems can cover many equations although they look simple. In equation (1.4), we let $a(x) = \frac{a}{x^2}$, $b(x) = x$, $r(x) = 1$, $c(x) = 0$, $f(y) = y$, and $p(x) = x^\mu$ for $\mu \ge 1$. Then, condition (2.3) in Theorem 1 becomes

$$\limsup_{x\to\infty} \int_{x_0}^{x} (a - \frac{\mu}{4}) s^{\mu-2} ds = \infty.$$

If we choose $\mu = 1$, then, $a > \frac{1}{4}$. This is actually the necessary and sufficient condition for the solutions of (1.4) to be oscillatory.

3. Examples

For the sake of simplicity, we choose $\psi(y) = 1$ in Examples 1-3.

Example 1. If $0 < \kappa < 1, 0 < \nu \leq 1$, $\beta \geq 1$, $\gamma \leq 1, 0 < b \leq 1, 0 < 0 \leq 1, c > 0, c_1 > 0, \nu_1 \leq 0$, and $r > 0$ when $\gamma \geq 0$, then, the solutions of the following equation

$$\begin{aligned} &((1 + rx^{\gamma})(y + (c + c_1 x^{\nu_1})y(d(x-\sigma)^{\nu}))')' \\ &\qquad + \frac{2+\cos x}{x^{\lambda}}|y(\eta)|^{\beta} \operatorname{sgn} y(\eta) = 0, \\ &\eta = b(x-\tau)^{\kappa},\ x \geq \sigma, \tau \end{aligned} \tag{3.1}$$

are oscillatory when $\lambda \leq 1$, or $1 < \lambda < 1 + \kappa$ and $\gamma \leq 0$, or $1 < \lambda < 1 + \kappa - \kappa\gamma$ and $0 < \gamma \leq 1$.

Noticing that $f(y) = |y|^{\beta} \operatorname{sgn} y$, and $f'(y) = \beta |f(y)|^{1-\frac{1}{\beta}}$, we try to apply Theorem 2 to this example. For the sake of simplicity, we use the corollary of Theorem 2. We choose $p(x) = x^{\mu}$ with $\mu = \max\{\lambda - 1, 0\}$. Then,

$$\limsup_{x\to\infty} \int_{x_0}^{x} p(s)a(s)ds = \int_{x_0}^{x} (2 + \cos s)s^{\mu-\lambda} ds = \infty.$$

If $\lambda \leq 1$, then, $B_2(x) = 0$. Hence,

$$\int_{x_0}^{\infty} \frac{r(b(x))(1 + c(b(x)))}{p(x)b'(x)} B_2^2(x)dx = 0 < \infty.$$

If $\lambda > 1$, then, $B_2(x) = \max\{(\lambda-1)x^{\lambda-2} + \frac{\beta\kappa c_1 \nu_1 b^{\nu_1} x^{\lambda-1}(x-\tau)^{\kappa(\nu_1-2)}}{1+c+c_1 b^{\nu_1}(x-\tau)^{\kappa\nu_1}}, 0\} \approx (\lambda - 1)x^{\lambda-2}$. Thus, condition (2.18) becomes

$$\begin{aligned} &\int_{x_0}^{\infty} \frac{r(b(x))(1 + c(b(x)))}{p(x)b'(x)} B_2^2(x)dx \\ &\qquad \approx \frac{(1+c)(\lambda-1)^2}{b\kappa} \int_{x_0}^{\infty} (1 + rb^{\gamma}(s-\tau)^{\kappa\gamma})(1 - \frac{\tau}{s})^{1-\kappa} s^{\lambda-2-\kappa} ds. \end{aligned}$$

Hence, when $\lambda < 1 + \kappa$ and $\gamma \leq 0$ or $\lambda < 1 + \kappa - \kappa\gamma$ and $0 < \gamma \leq 1$,

$$\int_{x_0}^{\infty} \frac{r(b(x))(1 + c(b(x)))}{p(x)b'(x)} B_2^2(x)dx < \infty$$

and the solutions of equation (3.1) are oscillatory. When $\beta > 1$, $c = \frac{1}{2}$, $\lambda = \frac{3}{2}$, $d = \kappa = \nu = 1$, $b = \frac{1}{2}$, $\tau = r = 0$, and $\sigma = \frac{\pi}{2}$, the conditions in Example 1 are satisfied and then the solutions of (3.1) are oscillatory. This is Example 2.1 given by S.H. Saker and J.V. Manojlovic [4].

Example 2. If $0 < \kappa \leq 1$, $0 < \nu \leq 1$, $\beta \geq 1$, $0 < b \leq 1$, $0 < d \leq 1$, $c \geq 0$, $\nu_1 < 0$, $\gamma \leq 1$ and $r > 0$ when $\gamma \geq 0$, then, the solutions of the equation

$$((1+rx^{\gamma})(y+(c+c_1x^{\nu_1})y(d(x-\sigma)^{\nu}))')' + \frac{a(x-\tau)^{\beta_1}y(\eta)^{\beta}}{x^{\beta_2}(x-\delta)^{\beta_3}} sgny(\eta) = 0, \quad (3.2)$$

where $\eta = b(x-\tau)^{\kappa}$, $x \geq \delta$, $x \geq \sigma$, and $x \geq \tau$, are oscillatory when $-\beta_1 + \beta_2 + \beta_3 \leq 1$, or $\gamma \leq 0$ and $1 < -\beta_1 + \beta_2 + \beta_3 \leq 1 + \kappa$ or $0 < \gamma \leq 1$ and $1 < -\beta_1 + \beta_2 + \beta_3 \leq 1 + \kappa - \kappa\gamma$.

Using very similar discussion to the foregoing one, we can show that equation (3.2) satisfies condition (2.18) and therefore has oscillatory solutions.

When $\nu_1 = -2, c = 0, c_1 = 1, d = \kappa = \nu = b = \tau = 1$, $\sigma = 3, \beta_1 = 6, \beta_2 = 4, \beta_3 = 3$ and $r = 0$, equation (3.2) becomes

$$(y(x) + x^{-2}y(x-3))'' + \frac{a(x-1)^6|y(x-1)|^{\beta}}{x^4(x-\delta)^3} signy(x-1) = 0. \quad (3.3)$$

Hence, the solutions of (3.3) are oscillatory. Actually, equation (3.3) is discussed in Example 2.2 by S.H. Saker and J.V. Manojlovic [4]. Unfortunately, their verification is incorrect because they used the condition

$$\limsup_{x\to\infty} \ln^{-2}\frac{x}{2}\int_2^x \frac{1}{s^2}\ln^2\frac{x}{s}ds = \infty$$

which clearly is not correct.

Example 3. If $q(x) \in C^1[0,\infty)$, $\lim_{x\to\infty} q(x) = 1$, and $q(x) > 0$ is nondecreasing, the solutions of

$$y'(x) + \frac{a}{x^{\lambda}}q(|y(b(x-\tau))|)y(b(x-\tau)) = 0, \quad x \geq \tau\,, \quad (3.4)$$

are oscillatory.

Equation (3.4) is a special case of equation (1.4) with $c(x) = 0$, $\psi(y) = 1$, $a(x) = \frac{a}{x^2}$, $b(x) = b(x-\tau), 0 < b \leq 1$, $f(y) = q(|y|)y$, and $r(x) = 1$. Since $\lim_{x\to\infty} q(x) = 1$, for any $\epsilon > 0$, there exists constant $X > 0$ such that $f'(y) = q'(|y|)|y| + q(|y|) \geq q(|y|) > 1-\epsilon$ as $|y| > X$. Let $p(x) = x$, condition (2.3) in Theorem 1 becomes

$$\limsup_{x\to\infty}\int_{x_0}^x (4ab(1-\epsilon)s^{1-\lambda} - \mu^2 s^{-1})ds = \infty. \quad (3.5)$$

When $\lambda < 2$ or $\lambda = 2$ and $4ab(1-\epsilon) > 1$, condition (3.5) is satisfied and therefore, the solutions of (3.4) are oscillatory. Since $\epsilon > 0$ is arbitrary, condition

$4ab(1-\epsilon)>0$ can be changed to $4ab>0$. If we let $q(y)=\frac{y^\beta}{y^\beta+1}$, $\beta\geq 0$ and $\lambda=2$, equation (3.4) becomes

$$y''(x)+\frac{a\left|y(b(x-\tau))\right|^\beta y(b(x-\tau))}{x^2(\left|y(b(x-\tau))\right|^\beta+1)}=0. \tag{3.6}$$

Thus, the solutions of equation (3.6) are oscillatory when $a>\frac{1}{4b}$. Numerical calculation shows that this is also the necessary condition for equation (3.6) to have oscillatory solutions. If we choose $\beta=1, a=4$ and $b=\frac{1}{2}$, (3.6) becomes Example 1 of Baculikova [5]. If we choose $\beta=0$ and $b=\frac{1}{2}$, equation (3.6) covers Example 3 of Baculikova.

Example 4.

$$((r_1+r_2x^\gamma)\frac{1+r_3|y(x)|}{1+r_4y^2(x)}z'(x))' + \int_0^1\frac{(a(x+\theta-\delta)+a_1)|y(b(x+\theta-\tau))|^\beta}{x^{\beta_1}(x+\theta-\delta)^{\beta_2}sgny(b(x+\theta-\delta))}d\theta=0\,, \tag{3.7}$$

where $x\geq 1$, $x\geq\delta$, $x\geq\sigma$, $x\geq\tau$, $\beta\geq 1$, $0<b\leq 1$, $0<d\leq 1$, $v_1<0$, $\gamma\leq 1$, $r_1^2+r_2^2>0$, $r_i\geq 0$ for $i=1,2$, and 3, $c_i\geq 0$ for $i=0,1$, and 2, and $z(x)=y(x)+\frac{c_0+c_1x^{\nu_1}}{1+c_2x}y(d(x-\sigma))$.

If $c_2\neq 0$, Theorems 3 and 4 cannot be applied to this equation because $c(x)=\frac{c_0+c_1x^{\nu_1}}{1+c_2x}<1$ and is decreasing for large x in this case. We assume that $c_2=0$ and $c_0<1$ and apply Theorem 5 to this equation.

Assume that $r_4\neq 0$ and $\psi(y)=\frac{1+r_3|y|}{1+r_4y^2}\leq M(r_3,r_4)$. We choose $p(x)=x^\mu$ with $\mu=\begin{cases}\max\{\beta_1+\beta_2-2,\,0\}, \text{ if } a\neq 0\,,\\ \max\{\beta_1+\beta_2-1,\,0\}, \text{ if } a=0, a_1\neq 0.\end{cases}$ In this case, we have $M_f=1$ and $k_D=\beta D^{\beta-1}$. Following it, one can get

$$A_{\alpha,\beta}(x)=\int_0^1\frac{a(x+\theta-\delta\,)+a_1}{x^{\beta_1}(x+\theta-\delta\,)^{\beta_2}}d\theta\approx\frac{a(x-\delta)^{1-\beta_1}}{x^{\beta_1}}+\frac{a_1(x-\delta)^{-\beta_2}}{x^{\beta_1}}.$$

Hence,

$$\limsup_{x\to\infty}\int_{x_0}^x[M_fA_{\alpha,\beta}(s)f(1-c(b(s,0)))p(s)-\frac{Mr(b(s,0))p'^2(s)}{4k_Dp(s)b'(s,0)}]ds$$

$$\approx\limsup_{x\to\infty}\int_{x_0}^x[(1-\widetilde{c})^\beta(a(s-\delta)+a_1)s^{\mu-\beta_1-\beta_2}-\frac{\mu^2M(r_{3,r_4)(r_1+r_2b^\gamma(s-\tau)^\gamma)s^{\mu-2}}}{4D^{\beta-1}b}]ds,$$

where $\widetilde{c}=\begin{cases}0,\ c_2\neq 0\\ c_0,\ c_2=0.\end{cases}$ Now, there are four possible cases to deal with. If one of the conditions

$$\gamma \le 0,\ \beta_1+\beta_2<3,\ a\ne 0 \ \text{ or } \ \gamma>0,\ \gamma+\beta_1+\beta_2<3,\ a\ne 0, \tag{3.8}$$

and

$$\gamma \le 0,\ \beta_1+\beta_2<2,\ a=0\ a_1\ne 0 \ \text{ or } \ \gamma>0,\ \gamma+\beta_1+\beta_2<2,\ a\ne 0,\ a_1\ne 0 \tag{3.9}$$

is true, condition (2.16) of Theorem 5 is satisfied.

If one of the conditions

$$a(1-\widetilde{c})-(1-\gamma)^2\frac{M(r_3,r_4)r_2b^{\gamma-1}}{4\beta D^{\beta-1}}>0,\ \gamma+\beta_1+\beta_2=3,\ \gamma>0,\ a\ne 0, \tag{3.10}$$

$$\text{or } a(1-\widetilde{c})-\frac{M(r_3,r_4)r_1}{4\beta D^{\beta-1}b}>0,\ \beta_1+\beta_2=3,\ \gamma<0,\ a\ne 0,$$

and

$$a_1(1-\widetilde{c})-(1-\gamma)^2\frac{M(r_3,r_4)r_2b^{\gamma-1}}{4\beta D^{\beta-1}}>0, \tag{3.11}$$

$$\gamma+\beta_1+\beta_2=2,\ \gamma>0,\ a=0,\ a_1\ne 0$$

$$\textit{or } a_1(1-\widetilde{c})-\frac{M(r_3,r_4)r_1}{4\beta D^{\beta-1}b}>0,\ \beta_1+\beta_2=2,\ \gamma<0,\ a=0,\ a_1\ne 0,$$

one has

$$\limsup_{x\to\infty}\int_{x_0}^{x}[M_fA_{\alpha,\beta}(s)f(1-c(b(s,0)))p(s)-\frac{Mr(b(s,0))p'^2(s)}{4k_Dp(s)b'(s,0)}]ds$$

$$=\begin{cases}\lim\limits_{x\to\infty}\int_{x_0}^{x}[a(1-\widetilde{c})-u\frac{M(r_3,r_4)r_2b^{\gamma-1}}{4\beta D^{\beta-1}}]s^{-1}ds=\infty, & \text{if } \gamma>0 \text{ and } a\ne 0,\\ \lim\limits_{x\to\infty}\int_{x_0}^{x}[a(1-\widetilde{c})-u\frac{M(r_3,r_4)r_1}{4\beta D^{\beta-1}b}]s^{-1}ds=\infty, & \text{if } \gamma<0 \text{ and } a\ne 0,\\ \lim\limits_{x\to\infty}\int_{x_0}^{x}[a_1(1-\widetilde{c})-u\frac{M(r_3,r_4)r_2b^{\gamma-1}}{4\beta D^{\beta-1}}]s^{-1}ds=\infty, & \text{if } \gamma>0,\ a=0 \text{ and } a_1\ne 0,\\ \lim\limits_{x\to\infty}\int_{x_0}^{x}[a_1(1-\widetilde{c})-u\frac{M(r_3,r_4)r_1}{4\beta D^{\beta-1}b}]s^{-1}ds=\infty, & \text{if } \gamma<0,\ a=0 \text{ and } a_1\ne 0,\end{cases}$$

where $(u=\beta_1+\beta_2-2)^2$. Therefore, the solutions of equation (3.7) are oscillatory if one of the foregoing conditions is satisfied.

If $\beta>1$, equation (3.7) does not satisfy the conditions given by Zhiting Xu [5, 6], and thus, the criteria there cannot be applied. Even in the case $\beta=1$, it is very complicated to use the criteria there. Actually, our result can be applied easily to deal with this case. Let $\beta=1$, $r_1=1$, $r_2=0$, $r_3=0$, $r_4=1$, $\gamma=b=d=1$, $c_0=1$, $c_1=0$, $c_2=1$, $\sigma=1$, $\delta=\tau=0$, $\beta_1=2$, $\beta_2=1$, and $a_1=a$. Then, (3.10) becomes $a>\frac{1}{4}$. This is the result obtain by Zhiting Xu [6] in Example 4.1. Let $\beta=1$, $r_1=0$, $r_2=1$, $r_3=0$, $r_4=1$, $\gamma=-2$, $b=d=$

1, $c_0 = \frac{1}{2}$, $c_1 = 0$, $c_2 = 0$, $\sigma = 1$, $\delta = \tau = 0$, $\beta_1 = 0$, $\beta_2 = 2$, and $a_1 = 0$. Then, (3.8) gives that $a > 0$. This is the case given by Zhiting Xu [6] in Example 4.3.

Example 5.

$$((r_1 + r_2e^{-kx})\frac{1 + r_3|y(x)|}{1 + r_4y^2(x)}z'(x))' + \int_0^1 ax^{\beta_1}(x + \theta - \delta)^{\beta_2}e^{-k_2(x+\theta-\delta)}$$
$$- \frac{|y(b(x + \theta - \tau))|^\beta}{sgny(b(x + \theta - \delta))}d\theta = 0\,, \quad (3.12)$$

where $x \geq 1$, $x \geq \delta$, $x \geq \sigma$, $x \geq \tau$, $\beta \geq 1$, $0 < b \leq 1$, $0 < d \leq 1$, $v_1 < 0$, $\gamma \leq 1$, $r_1^2 + r_2^2 > 0$, $r_i \geq 0$ for $i = 1, 2, 3$ *and* $4, c_i \geq 0$ for $i = 0, 1,$ and 2, and $z(x) = y(x) + \frac{c_0 - c_1e^{-k_1x}}{1+c_2x}y(d(x - \sigma)^\nu)$.

Similar to our discussion in the previous example, we apply Theorem 5 to this equation. Let $p(x) = x^\mu e^{k_2x}$ with $\mu = \max\{-\beta_1 - \beta_2 - 1, 0\}$. Then, we have

$$A_{\alpha,\beta}(x) = a\int_0^1 x^{\beta_1}(x + \theta - \delta)^{\beta_2}e^{k_2(x+\theta-\delta)}d\theta = ax^{\beta_1}(x + \widetilde{\theta} - \delta)^{\beta_2}e^{-k_2((x+\widetilde{\theta}-\delta)}\,,$$

where $\widetilde{\theta} \in (0, 1)$. It follows that

$$\limsup_{x\to\infty}\int_{x_0}^x [M_f A_{\alpha,\beta}(s)f(1 - c(b(s, 0)))p(s) - \frac{Mr(b(s, 0))p'^2(s)}{4k_Dp(s)b'(s, 0)}]ds$$
$$\approx \limsup_{x\to\infty}\int_{x_0}^x [a(1 - \widetilde{c})e^{-k_2(\widetilde{\theta}-\delta)}s^{\beta_1+\beta_2+\mu} - \frac{\mu^2M(r_3, r_4)r_1}{4\beta D^{\beta-1}b}s^{\mu-2}]ds\,, \quad (3.13)$$

where $\widetilde{c} = \begin{cases} 0, & \text{if } c_2 \neq 0 \\ c_0, & \text{if } c_2 = 0. \end{cases}$ If $\beta_1+\beta_2 = -2$ and $a(1-\widetilde{c})e^{-k_2(1-\delta)} - \frac{\mu^2M(r_3,r_4)r_1}{4\beta D^{\beta-1}b} > 0$ or $\beta_1 + \beta_2 > -2$, (3.8) is evaluated to ∞ and the solutions of (3.7) are oscillatory. Example 4.3 given by Zhiting Xu [6] is a special case of this example when $\beta = 1$, $r_1 = 0$, $r_2 = 1$, $k = 1$, $r_3 = 0$, $r_4 = 1$, $a = b = d = 1$, $c_0 = 0$, $c_1 = \frac{1}{2}$, $c_2 = 0$, $k_1 = 1$, $\sigma = 1$, $\delta = \tau = 0$, $\beta_1 = \beta_2 = 0$ and $k_2 = \frac{1}{2}$.

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