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WEIERSTRASS POINTS OF BINARY CURVES

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Abstract: A binary curve of genus g is a nodal curve X with two irreducible components, say $X = C_1 \cup C_2$, such that $C_1 \cong C_2 \cong \mathbb{P}^1$ and $\sharp(C_1 \cap C_2) = g+1$. Following papers by L. Caporaso here we study Weierstrass points on some binary curves.

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1. Introduction

Recall that a binary curve of genus g is a nodal curve X with two irreducible components, say $X = C_1 \cup C_2$, such that $C_1 \cong C_2 \cong \mathbb{P}^1$ and $\sharp(C_1 \cap C_2) = g+1$ (see [3]). Hence $p_a(X) = g$ and $C_1 \cap C_2 = \mathrm{Sing}(X)$. Fix any $S \subseteq \mathrm{Sing}(X)$. Let $u_S : X_S \to X$ be the quasi-stable curve obtained "blowing-up" S, i.e. X_S is semistable and connected, $u_S|X_S\backslash u^{-1}(S) \to X\backslash S$ and $E_P := u_S^{-1}(P)$ is a smooth rational curve intersecting the other components of X_S at two points for every $P \in S$. For the elementary properties of depth 1 coherent sheaves on reduced curves, see [5], parts VII and VIII. We say that a depth 1 sheaf F on a nodal curve Y has pure rank 1 if its restriction to Y_{reg} is a pure rank 1 vector bundle. Let F be sheaf on Y with pure rank 1 and with depth 1. Set

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$\mathrm{Sing}(F) := \{P \in Y : F$ is not locally free at $P\}$. Hence $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(Y)$. The degree $\deg(F)$ of F may be defined by the Riemann-Roch formula $\chi(F) = \deg(F) + \chi(\mathcal{O}_Y)$. Now take $Y = X$ with $X = C_1 \cup C_2$ a binary curve of genus g. Let F be a depth 1 sheaf on X with pure rank 1. Set $s(F) := \sharp(\mathrm{Sing}(F)$ and $d_i(F) := \deg((F|C_i)/\mathrm{Tors}(F|C_i))$. Since X is nodal, it is well-known that $\deg(F) = d_1(F)+d_2(F)+s(F)$. We will say that $(d_1(F), d_2(F), s(F))$ is the type or the multidegree of F. If F is locally free, then it satisfies the Basic Inequality of L. Caporaso (see [2], p. 611) if and only if $|d_1 - d_2| \le g+1$. Now assume that F is not locally free. Set $S := \mathrm{Sing}(X)$ and $L := u_S^*(F)/\mathrm{Tors}(u_S^*(F))$. $L \in \mathrm{Pic}(X_S)$, $\deg(L) = \deg(F)$ and $\deg(L|E_P) = 1$ for all $P \in S$. The sheaf F on X is said to satisfy the Basic Inequality if the line bundle L on X_S satisfies the Basic Inequality. For any zero-dimensional subscheme Z of a nodal curve Y set $\mathcal{O}_Y(Z) := \mathcal{H}om(\mathcal{I}_Z, \mathcal{O}_Y)$. $\mathcal{O}_Y(Z)$ is a depth 1 sheaf on Y with pure rank 1. Since Y is Gorenstein, any depth 1 coherent sheaf on Y is reflexive. In particular we have $\mathcal{H}om(\mathcal{O}_Y(Z), \mathcal{O}_Y) \cong \mathcal{I}_Z$. Hence $\deg(\mathcal{O}_Y(Z)) = \mathrm{length}(Z)$. Fix $P \in Y$. The point P is said to a *Weierstrass point* of Y if there is a closed subscheme $Z \subset Y$ such that $Z_{red} = \{P\}$, $\mathrm{length}(Z) \le g$ and $h^0(Y, \mathcal{O}_Y(Z)) \ge 2$. Take as above with $\mathrm{length}(Z)$ minimal. In this case $h^0(Y, \mathcal{O}_Y(Z)) = 2$, the sheaf $\mathcal{O}_X(Z)$ is spanned and the integer $d := \mathrm{length}(Z) = \deg(\mathcal{O}_Y(Z))$ is called the *order* of P. Now assume that P is an ordinary node of Y. Let $f : C \to Y$ be the partial normalization of Y in which we normalize only the point P. Set $\{P_1, P_2\} := f^{-1}(P)$ and $L := f^*(\mathcal{O}_Y(Z))/\mathrm{Tors}(f^*(\mathcal{O}_Y(Z)))$. Since $\mathcal{O}_Y(Z)$ is locally free and with rank 1 outside P, $\{P_1, P_2\} \subset C_{reg}$ and L has no torsion, $L \in \mathrm{Pic}(C)$. Since the tensor product is a right exact functor and $\mathcal{O}_Y(Z)$ is spanned, $f^*(\mathcal{O}_Y(Z))$ is spanned. Hence L is spanned. Since $\mathcal{O}_Y(Z)$ has no torsion, the natural map $H^0(Y, \mathcal{O}_Y(Z)) \to H^0(C, L)$ is injective. Hence $h^0(C, L) \ge 2$. From now on we assume that $Y = X = C_1 \cup C_2$ is a binary curve. It is easy to check that no smooth point of a binary curve is a Weierstrass point (see Lemma 1). Here we study Weierstrass points of order 3 (see Examples 2 and 3, Remark 2 and Proposition 1). We also prove the following results.

Theorem 1. *Fix integers g, d_1, d_2, ϵ such that $g \ge 3$, $d_1 > 0$, $d_2 > 0$, $\epsilon \in \{0, 1\}$ and $d_1 + d_2 + \epsilon \le g$. Then there are a binary curve X of genus g and $P \in \mathrm{Sing}(X)$ such that P is a Weierstrass point of X with multidegree (d_1, d_2, ϵ).*

Theorem 2. *Let X be a general binary curve of genus $g \ge 3$. Then X has no Weierstrass point with order at most $g-1$.*

2. Proofs and Other Results

Lemma 1. *Let X be a binary curve of genus g. No point of X_{reg} is a Weierstrass point of X.*

Proof. Assume the existence of a Weierstrass point $P \in X_{reg}$. Let d be the order of P. Thus $d \le g$ and $h^0(X, \mathcal{O}_X(dP)) = 2$. The line bundle $\mathcal{O}_X(dP)$ has multidegree $(d, 0)$ or $(0, d)$. Hence $h^0(X, \mathcal{O}_X(dP)) \le 1$ (see [3], Lemma 10), contradiction. □

For any $P \in \mathrm{Sing}(X)$ let $f_P : Y_P \to X$ be the partial normalization of X in which we normalize only the point P. Hence Y_P is nodal, $p_a(Y) = g - 1$, and Y_P has two irreducible components, say $Y_P = D(P)_1 \cup D(P)_2$ with $D(P)_i \cong \mathbb{P}^1$. Assume that P is a Weierstrass point of X with order d. Notice that [3], Lemma 10, implies $d \ge 2$. Let Z be a degree d zero-dimensional subscheme of X such that $Z_{red} = \{P\}$, $\mathrm{length}(Z) = d$, $h^0(X, \mathcal{O}_X(Z)) = 2$ and $\mathcal{O}_X(Z)$ is spanned. Let (d_1, d_2, ϵ), $\epsilon \in \{0, 1\}$, be the multidegree of $\mathcal{O}_X(Z)$. We have $d = d_1 + d_2 + \epsilon$. We have $d_1 > 0$ and $d_2 > 0$ (use [3], Lemma 10, on X if $\epsilon = 0$ and on Y_P is $\epsilon = 1$). The sheaf $\mathcal{O}_X(Z)$ satisfied the Basic Inequality, i.e. it is semibalanced, if and only if $|d_1 - d_2| \le g + 1 - \epsilon$. Since $d_1 > 0$, $d_2 > 0$, $d \le g$, and $d_1 + d_2 + \epsilon = d$, the Basic Inequality is satisfied by the sheaf associated to any Weierstrass point of any binary curve. Hence from now on we do not need to check or mention the Basic Inequality.

Example 1. Fix $P \in \mathrm{Sing}(X)$ and assume that P is a Weierstrass point of X with order 2. Since $d_1 > 0$, $d_2 > 0$ and $d_1 + d_2 + \epsilon = 2$, P has multidegree $(1, 1, 0)$. Hence X is hyperelliptic (see [3], §3). Every singular point of every hyperelliptic binary curve is a Weierstrass point with order 2.

Example 2. Assume $g \ge 3$. Fix $P \in \mathrm{Sing}(X)$ and assume that P is a Weierstrass point of X with order 3. Hence P has either multidegree $(1, 1, 1)$ or multidegree $(1, 2, 0)$ or multidegree $(1, 2, 0)$. X has multidegree $(1, 1, 1)$ if and only if Y_P is hyperelliptic and $\mathcal{O}_X(Z) = f_{P*}(R)$, where R is the hyperelliptic line bundle on Y_P. Conversely, assume that Y_P is hyperelliptic with hyperelliptic line bundle and set $A := f_*(R)$. Since $\deg(A) = 3$ and $h^0(X, A) = h^0(Y_P, R) = 2$, A is spanned if X is not hyperelliptic, i.e. if $R \ne \mathcal{O}_{Y_P}(f_P^{-1}(P))$. Since in this case $h^0(Y_P(-f^{-1}(P)) = 0$, A has a section vanishing only at P, i.e. $A \cong \mathcal{O}_X(Z)$ with $\mathrm{length}(Z) = 3$ and A spanned. The classification of hyperelliptic binary curves made in [3], §3, shows that the latter condition is also a necessary condition. Now assume that P has multidegree $(1, 2, 0)$ (the case $(2, 1, 0)$ being identical).

Remark 1. Let X be a binary curve of genus $g \geq 3$ such that there is a spanned sheaf F on X with degree 3 and $\mathrm{Sing}(F) \neq 0$. Fix $P \in \mathrm{Sing}(F)$. The degree 2 spanned sheaf $L := f_P^*(F)/\mathrm{Tors}(f_P^*(F))$ on Y_P shows that Y_P is hyperelliptic. Recall that $F \cong f_*(L)$. The study of the hyperelliptic binary curves made in [3], §3, gives that L is locally free. Hence P is the only singular point of F. Conversely, asume Y_P hyperelliptic and call R the hyperelliptic line bundle of Y_P. Then $f_*(R)$ is spanned sheaf on X with degree 3 and $\mathrm{Sing}(F) = \{P\}$.

Remark 2. Let X be a binary curve of genus 3. Thus $\sharp(\mathrm{Sing}(X)) = 4$. If X is hyperelliptic, then each singular point of X is a Weierstrass point of multidegree $(1, 1, 0)$. Now assume that X is not hyperelliptic. Since each binary curve of genus 2 is hyperelliptic, Example 2 shows that each singular point of X is a Weierstrass point of multidegree $(1, 1, 1)$. For some genus 3 binary curve a singular point may also have a structure of Weierstrass point of multidegree $(2, 1, 0)$ or $(1, 2, 0)$.

Example 3. Let $X = C_1 \cup C_2$ be a non-hyperelliptic binary curve of genus 4. Since X is not hyperelliptic, ω_X is very ample (see [3], Proposition 19). We use its canonical embedding to see X as a degree 6 curve $X \subset \mathbb{P}^3$ such that $\mathcal{O}_X(1) \cong \omega_X$. In this embedding each C_i is a rational normal curve. There is a bijection between the non-locally free g_3^1 and the points $P \in \mathrm{Sing}(X)$ such that the partial nomrmalization $f_P : Y_P \to X$ of X at P is hyperelliptic. If P has this property, then there are infinitely many lines of $\mathbb{P}^3$ passing through P and intersecting X at two other distinct points. Hence if X has a non-locally free g_3^1, then the canonical model of X is not contained in a smooth quadric surface. For an arbitrary X Riemann-Roch shows that X is contained in a quadric surface $Q \subset \mathbb{P}^3$. Since C_1 is not contained in a plane, Q is irreducible. First assume that Q is smooth. Hence $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, X is a curve of type $(3, 3)$ on Q and (up to renaiming the two pencils of Q), C_1 is a curve of type $(2, 1)$ on Q and C_2 is a curve of type $(1, 2)$ on Q. X has 2 locally free g_3^1, one of multidegree $(2, 1)$ and the other one of multidegree $(2, 1)$. Now assume that Q is a quadric cone with vertex O. Let $m : F_2 \to Q$ be the blowing-up of the vertex O. F_2 is a smooth surface isomorphic to the Hirzebruch surface F_2. Take $h := m^{-1}(O)$ and a fiber T of the ruling of F_2 as a basis of $\mathrm{Pic}(Y_O)$. Both C_1 and C_2 passes through O. Consider the partial normalization $f_O : Y_O \to X$ in which we normalize only the point O. $Y_O \in |2h + 2T|$ and hence Y_O is a hyperelliptic binary curve. Notice that O is the only point such that there are infinitely many lines of $\mathbb{P}^3$ passing through O and intersecting X at two other distinct points. In summary, if the canonical model of X is contained in a smooth quadric, then X has no Weierstrass point of type $(1,1,1)$, while if the canonical model of X is contained in a quadric cone, then X has a unique Weierstrass point of type $(1,1,1)$.

Proposition 1. *Let X be a binary curve of genus $g \geq 4$. Then X has at most one Weierstrass point of multidegree* $(1,1,1)$.

Proof. If X is hyperelliptic, then X has no Weierstrass point of multidegree $(1,1,1)$. Hence we may assume that X is not hyperelliptic. The case $g=4$ was checked in Example 3. Hence we may assume $g \geq 5$. Assume the existence of two different Weierstrass points P_1, P_2 of X of multidegree $(1,1,1)$. Fix $S \subset \mathrm{Sing}(X)$ such that $\sharp(S) = g-4$ and $\{P_1,P_2\} \cap S = \emptyset$. Let $f_S : Y_S \to X$ be the partial normalization of X in which we only normalize the points of S. Hence Y_S is a binary curve of genus 4. Set $O_i := f_S^{-1}(P_i) \in \mathrm{Sing}(Y_S)$, $i=1,2$. Since a partial normalization of a hyperelliptic binary curve is hyperelliptic (see [3], part (iii) of Lemma 15), the first part of Example 2 shows that O_1 and O_2 are Weierstrass points of Y_S of multidegree $(1,1,1)$. Since $O_1 \neq O_2$ and Y_S has genus 4, the contradiction comes from Example 3. □

Proposition 2. *Every binary curve has at least one Weierstrass point.*

Proof. Fix a genus $g \geq 3$. It is suficient to prove that a general binary curve X of genus g has a Weierstrass point. Since X is general, it has no non-trivial automorphism. Hence $\overline{\mathcal{M}}_g$ has a universal family in a Zariski neighborhood of X. For a general $C \in \mathcal{M}_g$ there is a length g scheme such that $\sharp((Z_C)_{red}) = 1$ and $h^0(C, \mathcal{O}_C(Z_C)) \geq 2$. The properness of the relative Hilbert scheme shows that the flat family $\{Z_C\}$ has a limit $Z \subset X$ such that $\mathrm{length}(Z) = g$ and Z_{red} is a unique point P. The semicontinuity theorem for cohomology gives $h^0(X, \mathcal{O}_X(Z)) \geq 2$. Hence there s a subscheme $Z' \subseteq Z$ such that $h^0(X, \mathcal{O}_X(Z')) = 2$. Since $Z'_{red} = \{P\}$, P is a Weierstrass point of X. □

Proof of Theorem 1. Fix integers $g \geq 3$, $d_1 > 0$ and $d_2 > 0$. We first do the case $\epsilon 00$. Fix 3 copies $C_1, C_2, \mathbb{P}^1$ of $\mathbb{P}^1$, $P_1 \in C_1$, $P_2 \in C_2$ and $Q \in \mathbb{P}^1$. Let V_i, $i=1,2$, be a sufficiently general one-dimensional projective subspace of the complete linear system $|\mathcal{O}_{C_i}(d_i)$ containing the effective divisor d_iP_i. V_i has no base points and defines a degree d_i morphism $f_i : C_i \to \mathbb{P}^1$ such that P_i is a total ramification point of f_i. Up to a choice of projective coordinates we may assume $f_i(P_i) = Q$ for all i. Fix a general $S \subset \mathbb{P}^1$, say $S = \{Q_1, \dots, Q_g\}$, such that $\sharp(S) = g$. The generality of S implies $Q \notin S$ and that no point of $f_i^{-1}(S)$ is a ramification point of f_i, $i=1,2$. For every integer j such that $1 \leq j \leq g$ fix $A_j \in f_1^{-1}(Q_j)$ and $B_j \in f_2^{-1}(Q_j)$. Let X be the binary curve obtained gluing the points $P_1, A_1, \dots, A_g$ of C_1 (in this order) with the points $P_2, B_1, \dots, B_g$ of C_2. Since $f_1(P_1) = f_2(P_2)$ and $f_1(A_j) = f_2(B_j)$ for all j, the morphisms $f_1 : C_1 \to \mathbb{P}^1$ and $f_2 : C_2 \to \mathbb{P}^1$ induce a set-theoretic map $f : X \to \mathbb{P}^1$ such that $f|C_i = f_i$ for all i. Since the target $\mathbb{P}^1$ is a smooth curve, a well-known property of nodal singularities (universal singularity in the sense of [2], p. 387) implies that $f : X \to \mathbb{P}^1$ is a morphism. Call P the points of X obtained gluing together the point $P_1 \in C_1$ and $P_2 \in C_2$. Set $L := f^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $Z := f^{-1}(Q)$ (scheme-theoretic counterimage). L is a spanned line bundle on X such that $\deg(L|C_i) = d_i$, $i = 1, 2$. Hence Z is an effective Cartier divisor of X with length $d_1 + d_2$ and such that $L \cong \mathcal{O}_X(Z)$. Now assume $d_1 + d_2 \leq g$. To conclude the proof for the triple (g, d_1, d_2) it is sufficient to prove that we may choose V_1, V_2, S so that $h^0(X, L) = 2$. Let M_j, $1 \leq j \leq g$, denote the point of X obtained gluing the point $A_j \in C_1$ with the point $B_j \in C_2$. Let $u_j : Y_j \to X$, $0 \leq j \leq g$, be the partial normalization of X in which we normalize only the points M_i with $1 \leq i \leq j$. Set $\phi_j := f \circ u_j$ and $L_j := u_j^*(L)$. Since $p_a(Y_0) = 0$, it is easy to check that $h^0(Y_0, L_0) = d_1 + d_2 + 1$. Notice that Y_0 does not depend from the choice of $Q_1, \dots, Q_g$. For general Q_1 we may chose A_1, B_1 in the fiber of Q_1 of the pencils associated to f_1 and f_2 so that $h^0(Y_1, L_1) = \max\{d_1 + d_2, 2\}$. Then we continue taking general $Q_1, \dots, Q_g$. Now we do the case $\epsilon = 1$. We make the same construction, except we assume $f_1(P_1) \neq f_2(P_2)$. Let X' be the genus $g - 1$ curve obtained gluing each point $A_j \in C_j$ with the point $B_j \in C_2$, $1 \leq i \leq g$. Now $f : X' \to \mathbb{P}^1$ is a degree $d_1 + d_2$ pencil and $L \in \mathrm{Pic}(X')$. Let $v_j : Y_j' \to X'$ be the partial normalization of X' in which we normalize only the points M_i with $1 \leq i \leq j$. Set $\psi_j := f \circ v_j$ and$L_j := u_j^*(L)$. Notice that $Y_0' = C_1 \sqcup C_2$ (disjoint union). Hence $h^0(Y_0', L_0) = d_1 + d_2 + 2$. For general Q_1 we may chose A_1, B_1 in the fiber of Q_1 of the pencils associated to f_1 and f_2 so that $h^0(Y_1', L_1) = \max\{d_1 + d_2 + 1, 2\}$. Then we continue taking general $Q_1, \dots, Q_g$. Since $d_1 + d_2 \leq g - 1$, at the end get $h^0(X', L_g) = 2$. Take as X the curve obtained from X' gluing the point $P_1 \in C_1$ with the point $P_2 \in C_2$. Call $a : X' \to X$ the gluing map. Set $F := a_*(L_g)$. Obviously, $h^0(X, F) = 2$. F has multidegree $(d_1, d_2, 1)$. Since $f_1(P_1) \neq f_2(P_2)$ we get that F is spanned at the point $P := a_1(P_1)$. Since each f_i has a total ramification point at P_i, there is a section of F whose zero-locus Z is supported by P. Hence $F \cong \mathcal{O}_X(Z)$. Since $h^0(X, F) = 2$ and F is spanned at P, Z gives to P a structure of Weierstrass point of multidegree $(d_1, d_2, 1)$. □

Proof of Theorem 2. Since X is not hyperelliptic, its canonical line bundle is very ample (see [3], Proposition 19). Let $X = C_1 \cup C_2 \subset \mathbb{P}^{g-1}$ be the canonical model. Fix $P \in \mathrm{Sing}(X)$ and assume the existence of a zero-dimensional scheme $Z \subset X$ such that $Z_{red} = \{P\}$, $d := \mathrm{length}(Z) \leq g - 1$

and $h^0(X, \mathcal{O}_X(Z)) \geq 2$. By the duality for locally Cohen-Macaulay projective schemes (see [1]) the last inequality is equivalent to $h^0(X, Hom(\mathcal{O}_X(Z), \omega_X)) \geq 1 + g - d$. Since $\mathcal{O}_X$ is Gorenstein, every torsion free sheaf on X is reflexive and in particular $Hom(\mathcal{I}_Z, \mathcal{O}_X) \cong \mathcal{O}_X(Z)$ and $Hom(\mathcal{O}_X(Z), \mathcal{O}_X) \cong \mathcal{I}_Z$. Hence $h^0(X, Hom(\mathcal{O}_X(Z), \omega_X)) = h^0(X, \mathcal{I}_Z \otimes \omega_X)$. Thus the linear span $M := \langle Z \rangle$ of Z has codimension at least $1+g-d$, i.e. $\dim(M) \leq d-2$. Let a_i, $i \in \{1, 2\}$, be the order of contact of M with C_i at P. Hence $a_i \geq 1$ and $a_i = 1$ if and only if M is transversal to C_i. Assume $a_i = 1$. Since C_{2-i} is a rational normal curve, P is not a ramification point of C_{2-i}, i.e. $a_{2-i} \leq \dim(M)$, contradicting the inequality $\dim(M) \leq d - 2$. Since $a_1 \geq 2$ and $a_2 \geq 2$, the tangent line of C_1 at P is contained in the osculating hyperplane of C_2 at P, i.e. the hyperplane of $\mathbb{P}^{g-1}$ spanned by the effective Cartier divisor $(g-1)P$ of C_2. Since X is general, C_2 may be seen as the general rational normal curve of $\mathbb{P}^{g-1}$ intersecting C_1 at $g+1$ general points of C_1. For general C_2 as above there is no $Q \in C_1 \cap C_2$ and $i \in \{1, 2\}$ such that the tangent line to C_i at Q is contained in the osculating hyperplane of C_{2-i} at Q, contradiction. $\square$

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