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BINARY CURVES AND LADDERS WITH GOOD POSTULATION IN $\mathbb{P}^3$

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Abstract: A genus $g \geq 2$ binary curve is a nodal curve $X = D_1 \cup D_2$ such that $D_1 \cong D_2 \cong \mathbb{P}^1$ and $\sharp(D_1 \cap D_2) = g+1$. A ladder of genus g is a nodal curve $X = D_1 \cup D_2 \cup E_1 \cup \cdots \cup E_{g+1}$ such that $D_j \cong E_i \cong \mathbb{P}^1$ and $\sharp(D_j \cap E_i) = 1$ for all i, j, $D_1 \cap D_2 = E_h \cap E_k = \emptyset$ for all $h \neq k$. Here we prove the existence of many embeddings of a general genus g binary curve (resp. ladder) in $\mathbb{P}^3$ and with good postulation (resp. good postulation and in which each E_i is embedded as a line).

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1. Introduction

Fix an integer $g \geq 2$. We recall that a *binary curve* is a reduced projective curve with 2 irreducible components, say C_1, C_2, such that $C_1 \cong C_2 \cong \mathbb{P}^1$, $\sharp(C_1 \cap C_2) = g+1$ and every point of $C_1 \cap C_2$ is an ordinary node of X (see [5]). Thus $p_a(X) = g$ and X is a stable curve. Let $X = C_1 \cup C_2$ be a genus g binary curve. Every line bundle L on X has a bidegree (d_1, d_2), where $d_i := \deg(L|C_i)$. Obviously, $d_1 + d_2 = \deg(L)$. A key numerical condition for line bundles on stable curves is the so-called Basic Inequality introduced by L. Caporaso (see [4], p. 611). The line bundles satisfying it are called *semibalanced* in [9] and [5]. In the case of a binary curve the Basic Inequality just says

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$$|d_1 - d_2| \le g + 1 . \tag{1}$$

Let $\mu : W \to X$ be the full quasi-stable model of X. Hence X is a connected nodal curve with $g+3$ irreducible components, say $W = D_1 \cup D_2 \cup \bigcup_{P \in \mathrm{Sing}(X)} E_P$, all of them smooth and rational such that $D_1 \cap D_2 = \emptyset$, $E_P \cap E_Q = \emptyset$ for all $P \neq Q$, and $\mu(E_P) = \{P\}$ for all $P \in \mathrm{Sing}(X)$. W is called the *ladder* associated to X. Motivated by [4] we only consider embeddings $j : W \hookrightarrow \mathbb{P}^3$ such that all curves $j(E_P)$ are lines. The pair $(\deg(j(D_1)), \deg(j(D_2)))$ is called the bidegree of $j(W)$ or of the line bundle $j^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Notice that $j^*(\mathcal{O}_{\mathbb{P}^3}(1))$ has degree $d_1 + d_2 + g + 1$. The line bundle $j^*(\mathcal{O}_{\mathbb{P}^3}(1))$ satisfies the Basic Inequality if and only if $d_1 = d_2$. Hence if the hyperplane line bundle satisfies the Basic Innquality, then its degree d satisfies the congruence relation $d \equiv g + 1 \pmod 2$. Let $T \subset \mathbb{P}^3$ be any scheme. For every integer $t \ge 0$ let $\rho_{T,t} : H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(t)) \to H^0(T, \mathcal{O}_T(t))$ denote the restriction map. The scheme T has *maximal rank* if for every integer $t \ge 0$ the linear map $\rho_{T,t}$ has maximal rank, i.e. it is injective or surjective. Now assume $\dim(T) \le 1$ and that $h^1(T, \mathcal{O}_T(x)) = 0$ for every $x \ge 1$. Let d be the degree of the one-dimensional part of T. Obviously, $\rho_{T,0}$ has maximal rank. Fix an integer $t \ge 1$. Since $h^0(T, \mathcal{O}_T(t)) = td + \chi(\mathcal{O}_T)$, $\rho_{T,t}$ has maximal rank if and only if either $h^1(\mathbb{P}^3, I_T(t)) = 0$ (case $td + \chi(\mathcal{O}_T) \le \binom{3+t}{3}$) or $h^0(\mathbb{P}^n, I_T(t)) = 0$ (case $td + \chi(\mathcal{O}_T) \ge \binom{n+t}{n}$). We will use the so-called Horace method introduced in [8] to prove the following results.

Theorem 1. *Fix an integer $g \ge 2$. Set $\alpha(g) := (6g+30)^2/6$. Fix integers $d_i \ge \alpha(g)$, $i = 1, 2$. Then then the image of a general embedding $X \hookrightarrow \mathbb{P}^3$ of bidegree (d_1, d_2) of a general binary curve X of genus g has maximal rank.*

Theorem 2. *Fix an integer $g \ge 2$. Set $\alpha(g) := (6g+30)^2/6$. Fix integers $d_i \ge \alpha(g)$, $i = 1, 2$. Then then the image of a general embedding $j : X \hookrightarrow \mathbb{P}^3$ of bidegree (d_1, d_2) of a general ladder X of genus g has maximal rank.*

Remember that in the statement of Theorem 2 the curve $j(X)$ has degree $d_1 + d_2 + g + 1$ and that for each of the curves $E_1, \dots, E_{g+1}$ of the ladder $D_1 \cup D_2 \cup E_1 \cup \cdots \cup E_{g+1}$ the embedding $j|E_i$ has degree 1, i.e. each $j(E_i)$ is a line meeting at a unique point both $j(D_1)$ and $j(D_2)$ and quasi-transversally.

2. The Proofs

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. For any locally complete intersection curve $C \subset \mathbb{P}^3$ let N_C denote its normal bundle. Hence N_C is a rank 2 vector bundle on C. If C is smooth, then $\deg(N_C) = 4 \cdot \deg(C) + \deg(\omega_C)$.

Remark 1. Let M be a projective scheme and D an effective Cartier divisor of M. For any closed subscheme $Y \subseteq M$ let $\mathrm{Res}_D(Y)$ denote the residual scheme of Y with respect to D, i.e. the closed subscheme of M with $\mathcal{I}_{D,M} : \mathcal{I}_{Y,M}$ as its ideal sheaf. For any $L \in \mathrm{Pic}(M)$ we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Y),M} \otimes L(-D) \to \mathcal{I}_{Y,M} \otimes L \to \mathcal{I}_{Y\cap D,D} \otimes (L|D) \to 0\,. \tag{2}$$

From (2) we get

$$h^i(M, \mathcal{I}_{Y,M} \otimes L) \le h^i(M, \mathcal{I}_{\mathrm{Res}_D(Y),M} \otimes L(-D)) + h^i(D, \mathcal{I}_{Y\cap D,D} \otimes (L|D)) \tag{3}$$

for all integers $i \ge 0$.

Remark 2. Fix an integer $d \ge 3$. Let $C \subset \mathbb{P}^3$ be a general smooth rational curve of degree d. Then N_C is semistable (see [6], [11]), i.e. N_C is a direct sum of two line bundles of degree $2d-1$.

Lemma 1. *Fix integers d, s such that ,d such that $0 < s \le 2d-1$. Let $S \subset \mathbb{P}^3$ be a set such that $\sharp(S) = s$. Let $C \subset \mathbb{P}^3$ be a smooth rational curve such that $S \subset C$ and N_C is semistable. Fix $P \in \mathbb{P}^3\backslash C$ and let $D \subset \mathbb{P}^3$ be a line such that $P \in D$, $\sharp(C \cap D) = 1$, and D intersects quasi-transversally C. Then there are an integral affine curve T, $o \in T$, and a flat family $\{Y_t\}_{t\in T}$ of closed subschemes of $\mathbb{P}^3$ such that $Y_o = C \cup D$, Y_t is a smooth rational curve of degree $d+1$ for all $t \in T\backslash\{o\}$, $S \cup \{P\} \subset Y_t$ for all t, and $h^1(Y_t, N_{Y_t}(-S-P)) = 0$ for all $t \in T$.*

Proof. $N_C(-S)$ is a direct sum of two line bundles of degree $2d-1-s$. Obviously, $N_D(-P) \cong \mathcal{O}_D^{\oplus 2}$. Since $C \cup D$ is a locally complete intersection, $N_{C\cup D}$ is a rank 2 vector bundle. Since $P \notin D$ and $S \cap D = \emptyset$, $N_{C\cup D}(-S-P)|C$ is obtained from $N_C(-S)$ making a positive elementary transformation, while $N_{C\cup D}(-S-\{P\})|D$ is obtained from $N_D(-P)$ making a positive elementary transformation (see [7], §2, [12]). Hence $N_{C\cup D}(-S-P)|C$ is a direct sum of a line bundle of degree $2d-1-s+1$ and a line bundle of degree $2d-1-s$, while $N_{C\cup D}(-S-\{P\})|D$ is a direct sum of a line bundle of degree 1 and a line bundle of degree 0. Consider the Mayer-Vietoris exact sequence

$$\begin{aligned} 0 &\to N_{C\cup D}(-S-P) \to N_{C\cup D}(-S-P)|C \oplus N_{C\cup D}(-S-P)|D \\ &\to N_{C\cup D}(-S-P)|(C\cap D) \to 0\,. \end{aligned} \tag{4}$$

Since $C \cap D$ is a length 1 scheme, $h^1(D, N_{C\cup D}(-S-P-D\cap C)|D) = 0$. We saw that $h^1(C, N_{C\cup D}(-S-P)|C) = 0$. Hence the cohomology exact sequence of (4) gives $h^1(C\cup D, N_{C\cup D}(-S-P)) = 0$. Any vector bundle obtained from $N_C(-S)$

making a negative elementary transformation is a direct sum of a line bundle of degree $2d-1-s$ and a line bundle of degree $2d-2-s$. Since $s \le 2d-1$, it has its h^1 vanishes. To get the existence of the smoothing keeping fixed $S \cup \{P\}$ use [10], Theorem 1.5, and [7], Theorem 4.1 (in which the roles of C and D are reversed). The last assertion for $t \in T\backslash\{o\}$ follows from semicontinuity. $\square$

Lemma 2. *Fix integers $x > g \ge 2$, a smooth quadric Q, and a line T of type $(1,0)$ on Q. Fix a set $S \subset Q\backslash T$ such that $\sharp(S) = g+1$ and no two points of S are contained in the same line of Q. Fix x distinct lines $T_1, \dots, T_x$ of type $(0,1)$ on Q such that $\sharp(T_i \cap S) = 1$ if $1 \le i \le g+1$ and $T_i \cap S = \emptyset$ if $g+2 \le i \le x$. Set $D := T \cup T_1 \dots\dots T_x$. Then D is a limit in $\mathrm{Hilb}(\mathbb{P}^3)$ of curves whose general member is a smooth rational curve B of degree $x+1$ containing S and such that $h^1(B, N_B(-S)) = 0$.*

Proof. Since $S \cap T = \emptyset$, $N_D(-S)|T$ is obtained from N_T (i.e. from a direct sum of two degree 1 line bundles) making x positive elementary transformations. Since $\sharp(T_i \cap S) = 1$ if $i \le g+1$, (resp. $T_i \cap S = \emptyset$ if $i > x$, $N_{T_i}(-S)$ is trivial if $i \le g+1$,. Hence every vector bundle N^- (resp. N^+) obtained from $N_{T_i}(-S)$ making a negativee (resp. positive) elementary transformation satisfies $h^1(T_i, N^-) = 0$ (resp. satisfies $h^1(T_i, N^+) = 0$ and N^+ is spanned). If $g+2 \le i \le x$, then $N_{T_i}(-S)$ is a direct sum of two line bundles on degree 1 and hence the previous statements on N^- and N^+ are trivially true. Apply [10], Theorem 1.5, the proof of [7], Theorem 4.1, taking all vector bundles twisted by $(-S)$ with $C := T$ and $D := \cup_{i=1}^{x} T_i$. $\square$

Lemma 3. *Fix integers g, d, x such that $x > g \ge 2$ and $d \ge \lceil (g+3)/2 \rceil$. Fix a smooth rational curve $C \subset \mathbb{P}^3$ such that $\deg(C) = d$ and N_C is a direct sum of 2 line bundles of degree $2d-1$. Fix a smooth quadric Q intersecting transversally C and a line T of type $(1,0)$ on Q such that $T \cap C = \emptyset$. Assume the existence of x distinct lines $T_1, \dots, T_x$ of type $(0,1)$ on Q such that $\sharp(T_i \cap (Q \cap C)) = 1$ if $1 \le i \le g+1$ and $T_i \cap (Q \cap C) = \emptyset$ if $g+2 \le i \le x$. Set $D := T \cup T_1 \dots\dots T_{g+1}$. Then $D \cup T$ is a limit in $\mathrm{Hilb}(\mathbb{P}^3)$ of curves whose general member is a union $A \cup B$ with $A \cong B \cong \mathbb{P}^1$, $\deg(A) = d$, $\deg(B) = x+1$, $\sharp(A \cap B) = g+1$, and $A \cup B$ with only nodes as singularities.*

Proof. Apply Lemma 2 to the curve D. $\square$

We recall the terminology of [3]. A *tree* of degree d in $\mathbb{P}^3$ is a connected nodal curve R such that $\deg(R) = d$, each irreducible component of R is a line, and $p_a(R) = 0$. Hence we may order the oreder the lines $R_1, \dots, R_d$ of R in such a way that for $2 \le j \le d$ the line R_j intersects a unique line $L_{\tau(j)}$ such that $\tau(j) < j$. The function $\tau : \{2, \dots, d\} \to \{1, \dots, d-1\}$ obtained in this way is called a *type* of R. A line of a tree T is called a *final line* if it intersects at most another irreducible component of T. Every tree is smoothable (see [8], [3]). Fix a reduced projective curve $C \subset \mathbb{P}^3$. A reduced curve $Z \subset \mathbb{P}^3$ is said to be a *useful* curve for C if $C \subseteq Z$, $\deg(Z) = d$, the closure $\overline{Z\backslash C}$ of $Z\backslash C$ in $\mathbb{P}^3$ is a union of s trees $Y_1, \dots, Y_s$ (some integer $s > 0$) such that $Y_i \cap Y_j = \emptyset$ for all $i \neq j$, $\sharp(Y_i \cap C) \leq 1$ for all i, and each Y_i intersecting C intersects it quasi-transversally; a line $L \subset \overline{Z\backslash C}$ is called a *free final line* of Z if it is a final line of some Y_i and it intersects at most another irreducible component of Z. We need the following lemma (see [3], Lemma 2.1).

Lemma 4. *Fix non-negative integers a, b, x, y, u, d, e, f such that $b \geq 4$, $x \leq y$, $x + y \leq b + 1$, $y = 0$ if $a = 0$, $x = 0$ if $a = 1$, $e \leq 2u$, $e \leq b + 1$, $f \leq 2(d - u)$. Fix a non-degenerate curve $C \subset \mathbb{P}^3$, such that $\deg(C) = u$ and a smooth quadric surface $Q \subset \mathbb{P}^3$ intersecting transversally C and not containing secant lines to C. Fix $A \subseteq C \cap Q$ such that $\sharp(A) = e$. Fix a type τ for useful curves of degree d for C. Then there exists (Y, D, D', S, S', A, B) with the following properties:*

(a) Y is a useful curve for C of type τ; Y intersects transversally Q; $B \subset (Y\backslash C) \cap Q)$; $\sharp(B) = f$;

(b) D, D' are lines of type $(1, 0)$ on Q, $S \subset D$, $S' \subset D'$, $\sharp(S) = y$, $\sharp(S') = x$, and $h^0(Q, \mathcal{I}_W(a, b)) = \max\{(a+1)(b+1)-e-f-x-y, 0\}$, where $W := A \cup B \cup S \cup S'$; we may also assume that D' intersects a prescribed irreducible component of Y; if $y \leq b$, we may also assume that D' intersects a prescribed irreducible component of Y;

(c) let $\pi : Q \to D'$ be the projection; we assume $S' \subseteq \pi(S) \subseteq \pi(Y \cap (Q\backslash(D \cup D')))$.

Remark 3. Here we show how to use Lemma 4 to check the postulation of the intersection with a smooth quadric Q of a general genus g binary curve $X \subset \mathbb{P}^3$ with a fixed bidegree. Fix an integer $g \geq 2$ and set $\delta := \lfloor (g+3)/2 \rfloor$ Fix integers $d_1 \geq \delta$ and $d_2 \geq \delta$. We will also prove that we may find X as above such that $X \cap Q$ has some non-general properties. Fix the smooth quadric Q and take a general $S \subset \mathbb{P}^3$ such that $\sharp(S) = g + 1$. We require $S \cap Q = \emptyset$. Let $X(d_1, d_2)$ be the set of all binary curves of bidegree (d_1, d_2) with S as their singular locus. Since S is general, there are smooth rational curves A_1, A_2 such that $\deg(A_1) = \deg(A_2) = \delta$, $S = A_1 \cap A_2$ and $A_1 \cup A_2$ nodal (use Remark 2 and that $2 \cdot \delta \geq \sharp(S)$). Fix integers $s_j > 0$, $j = 1, 2$, integers $d_{j,i} > 0$,

$j = 1, 2$, $1 \le i \le s_j$, such that $\sum_{i=1}^{s_j} = d_j - \delta$ $(j = 1, 2)$ and types $\tau_{j,i}$, $j = 1, 2$, $1 \le i \le s_j$, for trees of degree $d_{j,i}$. Take general trees $B_{j,i}$, $j = 1, 2$, $1 \le i \le s_j$, with the only restriction that each $B_{j,i}$ intersects quasi-transversally A_j and at a unique point. Set $B_j := \cup_i B_{j,i}$. By construction each $A_j \cup B_j$ is connected, nodal, $\deg(A_j \cup B_j) = d_j$ and $p_a(A_j \cup B_j) = 0$. We also know that $A_j \cup B_j$ is a flat limit of a family of smooth rational curves containing S. Let C_j be the general element of this family. Set $X = C_1 \cup C_2$. Notice that $X \in X(d_1, d_2)$. By semicontinuity for general X the postulation of $X \cap Q$ is controlled by the postulation of $(A_1 \cup B_1 \cup A_2 \cup A_2) \cap Q$.

Remark 4. Here we show how to use Lemma 4 to check the postulation of the intersection with a smooth quadric Q of a general genus g ladder embedded in $\mathbb{P}^3$ with a fixed bidegree. Fix an integer $g \ge 2$ and set $\delta := \lfloor (g+3)/2 \rfloor$ Fix integers $d_1 \ge \delta$ and $d_2 \ge \delta$. Fix $g+1$ general lines $E_1, \dots, E_{g+1} \subset \mathbb{P}^3$. Let $X(d_1, d_2)$ be the set of all pairs $(Y_1, Y_2) \subset \mathbb{P}^3 \times \mathbb{P}^3$ such that $Y_1 \cup Y_2 \cup E_1 \cup \cdots \cup E_{g+1}$ is a ladder of bidegree (d_1, d_2). There are smooth rational curves A_1, A_2 such that $\deg(A_1) = \deg(A_2) = \delta$ and $A_1 \cup A_2 \cup E_1 \cup \cdots \cup E_{g+1}$ is a ladder of bidegree (δ, δ). From now on we may copy verbatim Remark 3.

As in [1], p. 214, for all integers $k \ge 0$ and $g \ge 0$ set $r(6k+1, g) = 6k^2 + 8k + 3$, $q(6k+1, g) = g$, $r(6k+2, g) = 6k^2 + 10k + 4$, $q(6k+2, g) = 3k + 1 + g$, $r(6k+3, g) = 6k^2 + 12k + 6$, $q(6k+3, g) = 2k + 1 + g$, $r(6k+4, g) = 6k^2 + 14k + 8$, $q(6k+4, g) = 3k + 2 + g$, $r(6k+5, g) = 6k^2 + 16k + 11$, $q(6k+5, g) = g$, $r(6k+6, g) = 6k^2 + 18k + 13$, and $q(6k+6, g) = 5k + 5 + g$). The integers $r(n, 0)$ and $q(n, 0)$ are uniquely determined by the relations:

$$n \cdot r(n, 0) + 1 = \binom{n+3}{3}, \quad 0 \le q(n, 0) \le n - 1\,. \tag{5}$$

Remark 5. Fix integers $g \ge 2$ and $d \ge 4g + 4$. In the case of ladders assume $d \ge 5g + 5$. Let $c_{g,d}$ be the first positive integer t such that $td + 1 - g \le \binom{t+3}{3}$. Let $X \subset \mathbb{P}^3$ be either a ladder or a binary curve such that $p_a(X) = g$ and $\deg(X) = g$. Notice that $h^1(X, \mathcal{O}_X(c_{g,d})) = 0$. Hence Castelnuovo-Mumford Lemma and Riemann-Roch give that X has maximal rank if and only $h^0(\mathbb{P}^3, \mathcal{I}_X(c_{g,d} - 1)) = 0$ and $h^1(\mathbb{P}^3, \mathcal{I}_X(c_{g,d})) = 0$.

Proof of Theorems 1 and 2. Fix a smooth quadric $Q \subset \mathbb{P}^3$. Fix an integer $g \ge 2$ (the genus). Fix a large integer d. In part (b) of the proof we will discuss how the very rough value $\alpha(g)$ comes . A binary curve or a ladder $X \subset \mathbb{P}^3$ has maximal rank if and only if $h^0(\mathbb{P}^3, \mathcal{I}_X(c_{g,d} - 1)) = 0$ and $h^1(\mathbb{P}^3, \mathcal{I}_X(c_{g,d})) = 0$ (Remark 5). Let n_1 be the first positive integer such that either $n_1 \equiv 1 \pmod 6$, say $n_1 = 6k_1 + 1$ for some $k_1 \in \mathbb{N}$, such that $g \le k_1$ or $n_1 \equiv 5 \pmod 6$, say

$n_1 = 6k_1 + 5$ for some $k_1 \in \mathbb{N}$ and $g \le k_1 + 1$. Notice that $n_1 \equiv 1, 5 \pmod 6$ if and only if $q(n_1, 0) = 0$ and that in both cases we are requiring an inequality which implies $r(n_1+2, 0) - r(n_1, 0) \ge g+2$ (but we made a stronger assumption). Let C be a general smooth rational curve of degree $r(n_1, 0)$. C has maximal rank (see [8]), i.e. $h^i(\mathbb{P}^3, \mathcal{I}_C(n_1)) = 0$, $i = 0, 1$ (here we use that $q(n_1, 0) = 0$). The generality of C and Remark 2 give that $N_C(-2)$ is a direct sum of 2 line bundles of degree -1. Hence $h^i(C, N_C(-2)) = 0$, $i = 0, 1$. Thus for general C we may assume that $C \cap Q$ is a general subset of Q with cardinality $2 \cdot r(n_1, g)$ (see [10], Theorem 1.5). In particular we may assume that each line of Q contains at most one point of $C \cap Q$. Fix a line $T \subset Q$ of type $(1, 0)$ such that $T \cap C = 0$ and $r(n_1+2, 0) - r(n_1, 0) - 1$ lines T_i, $1 \le i \le r(n_1+2, 0) - r(n_1, 0) - 1$, such that $T_i \cap (C \cap Q) \neq \emptyset$ if and only if $1 \le i \le g+1$. Set $A := T \cup \bigcup_{i=1}^{r(n_1+2,0)-r(n_1,0)-1} T_i$. Thus A is connected and nodal, $p_a(A) = 0$, $A \cup C$ is connected and nodal, and $\sharp(A \cap D) = g + 1$. Notice that $C \cup A$ is a genus g ladder. By Lemma 2 we may deform A to a smooth rational curve A' of degree $r(n_1 + 2, 0) - r(n_1, 0)$ keping fixed the points $A \cap C$. Notice that $C \cup A'$ is a genus g binary curve. Fix a line D of type $(0, 1)$ on Q and $S \subset D$ such that $\sharp(S) = q(n_1+2, g) = q(n_1+2, 0)+g$. Our assumptions on k_1 give $q(n_1, 0)+g \le n_1$. Set $B := S \cup (C \cap (Q \backslash A))$. Notice that $\sharp(B) = (n_1 + 2)(n_1 + 4 - r(n_1 + 2, 0) + r(n_1, 0))$.

First Claim: $h^i(Q, \mathcal{I}_B(n_1+2, n_1+3-r(n_1+2, 0)+r(n_1, 0))) = 0$, $i = 0, 1$.

Proof of the First Claim: Since $\sharp(B) = h^0(Q, \mathcal{O}_Q(n_1+1, n_1+3-r(n_1+2, 0)+r(n_1, 0)))$, it is sufficient to prove the case $i = 1$. S gives $\sharp(S)$ independent conditions to $H^0(Q, \mathcal{O}_Q(n_1 + 1, n_1 + 3 - r(n_1 + 2, 0) + r(n_1, 0)))$, because S is on a line of type $(0, 1)$ and $\sharp(S) \le n_1 + 2$. Since $C \cap Q$ is general in Q, the First Claim follows.

Second Claim: $h^i(\mathbb{P}^3, \mathcal{I}_{C \cup A \cup S}(n_1 + 2)) = 0$, $i = 0, 1$.

Proof of the Second Claim: Since $(C \cup A \cup S) \cap Q = B$ and $\mathrm{Res}_Q(C \cup A \cup S)) = C$, the Second Claim follows from the First Claim, the vanishings of $h^i(\mathbb{P}^3, \mathcal{I}_C(n_1))$, $i = 0, 1$, and Remark 1.

Then as in [3] we may continue and get for all integers $m \ge n_1 + 4$ such that $m \equiv n_1 \pmod 2$, i.e. for all odd integers $m \ge n_1 + 4$ a pair (Y_m, A_m), where Y_m is a ladder or a binary curve, $p_a(Y) = g$, $\deg(Y) = r(m, 0)$, $A_m \subset \mathbb{P}^3$, $\sharp(A_m) = q(m, g) = q(m, 0) + g$ and $h^i(\mathbb{P}^3, \mathcal{I}_{Y_m \cup A_m}(m)) = 0$, $i = 0, 1$. Up to now we proved the good postulation of certain space curves with respect to the line bundle $\mathcal{O}_{\mathbb{P}^3}(t)$ when t is large and odd. Now we consider the case t even. For each $P \in \mathbb{P}^3$ let $2P$ denote the first infinitesimal neighborhood of P in $\mathbb{P}^3$, i.e. the closed subscheme of $\mathbb{P}^3$ with $\mathcal{I}_P^2$ as its ideal sheaf. Thus $2P$ has length 4 and $(2P)_{red} = \{P\}$. If $P \in Q$, then $2P \cap Q$ is the first infinitesimal neighborhood of P in Q, i.e. a length 3 scheme, while $\mathrm{Res}_Q(2P) = \{P\}$. For any finite $B \subset \mathbb{P}^3$ set

$2B = \cup_{P\in B} 2P$. Set $k_0 := g+2$ and $n_0 := 6k_0$. Notice that $q(n_0,0) = 5k_0$. Since $k_0 = g+2$, $q(n_0,0) > g$ and $q(n_0+2,g) = q(n_0+2,0)+g = 3k_0+1+g \le q(n_0,0)$. Let C be a general smooth rational curve of degree $r(n_0,0)$. C has maximal rank (see [8]), i.e. $h^1(\mathbb{P}^3,\mathcal{I}_C(n_0)) = 0$, and $h^0(\mathbb{P}^3,\mathcal{I}_C(n_0)) = q(n_0,0) = 5k_0$. The generality of C and Remark 2 give that $N_C(-2)$ is a direct sum of 2 line bundles of degree -1. Hence $h^i(C,N_C(-2)) = 0$, $i = 0,1$. Thus for general C we may assume that $C\cap Q$ is a general subset of Q with cardinality $2\cdot r(n_0,g)$ (see [10], Theorem 1.5). In particular we may assume that each line of Q contains at most one point of $C\cap Q$. Notice that $r(n_0+2,0) - r(n_0,2) = 4k_0+3$ and that $2(4k_0+3-2) \ge 5k_0$. Fix lines T, T' subset Q of type $(1,0)$ such that $T\cap C = T'\cap C = \emptyset$. Fix $4k_0+1$ lines T_i, $1 \le i \le 4k_0+1$ of type $(0,1)$ on Q such that $T_i\cap C \ne \emptyset$ (i.e. $T_i\cap C$ is a point) if and only if $1 \le i \le g+k_0$. Set $A_i := T_i\cap T$ and $A'_i := T_i\cap T'$. Set $A'_1 := \cup_{i=1}^{4k_0}\{P'_i\}$, $A_1 := \cup_{i=2+g}^{k_0+g+1}\{P_i\}$. Thus $\sharp(A_1\cup A'_1) = 5k_0$. For general C we may also get $h^i(\mathbb{P}^3,\mathcal{I}_{C\cup A_1\cup A'_1}(n_0)) = 0$, $i = 0,1$ (as in the proofs in [8]). Fix a general line $T_2 \subset Q$ of type $(0,1)$ and a general $B \subset T_2$ such that $\sharp(B) = 3k+1+g$. Set $Y := C\cup T\cup T'\cup\bigcup_{i=1}^{4k_0+1} T_i\cup 2A'_1\cup 2A_1$. Hence $\mathrm{Res}_Q(Y\cup B) = \mathrm{Res}_Q(Y) = C\cup A_1\cup A'_1$ has no cohomology in degree n_0. The one-dimensional part of $(Y\cup B)\cap Q)$ is an effective divisor of degree $(2,4k_0+1)$. Outside it, $(Y\cup B)\cap Q$ is a union of B and $2\cdot\deg(C) - g - k_0$ points of $C\cap Q$. Since $C\cap Q$ is general in Q we get $h^i(Q,\mathcal{I}_{(Y\cup B)\cap Q}(n_0,n_0+2-4k_0-1)) = 0$, $i = 0,1$. Thus Remark 1 gives $h^i(\mathbb{P}^3,\mathcal{I}_{Y\cup B}(n_0+2)) = 0$, $i = 0,1$. Y may be deformed to a union of C and some lines, in such a way that the lines intersecting C intersects again C (outside Q now) and two deformed lines intersects if and only if the corresponding lines of $Y\cap Q$ meet at a point not in $A'_1\cup A_1$. To get a ladder we take as one of the smooth rational component a smooth rational curve C' which is a deformation of $C\cup\bigcup_{i=2g+2}^{k_0+g+1} T'_i$, the lines of the ladder are deformations of the lines T_i, $1 \le i \le g+1$, while the other smooth rational component is a deformation of the remaing lines of $Y\cap Q$. To get a binary curve we take again C' and as the other components a deformation C'' of the lines in $Y\cap Q$ (opening the intersection points corresponing of a point of $A_1\cup A'_1$ to get a curve with arithmetic genus g) keeping fixed the $g+1$ points of intersection with C. Then we deform again the configuration in such a way that the new curve intersects transversally Q.Then as in [3] we may continue and get for all integers $m \ge n_0+4$ such that $n \equiv n_0 \pmod 2$, i.e. for all even integers $m \ge n_0+4$ a pair (Y_m, A_m), where Y_m is a ladder or a binary curve, $p_a(Y) = g$, $\deg(Y) = r(m,0)$, $A_m \subset \mathbb{P}^3$, $\sharp(A_m) = q(m,g) = q(m,0)+g$ and $h^i(\mathbb{P}^3,\mathcal{I}_{Y_m\cup A_m}(m)) = 0$, $i = 0,1$. For both parities of the critical value $c_{g,d}$ we may also find $Y'_{c(g,d)}$ similar to $Y_{c(g,d)}$, except we do not smooth some of the lines added in the step $c_{g,d}-2 \mapsto c_{g,d}$ (see [8], [1], [2], §V). We know how to add something to $Y_{c_{g,d}} - 1$ or delete some line from $Y_{c_{g,d}}$ to get a binary curve or a ladder $X \subset \mathbb{P}^3$ such that $p_a(X) = g$, $\deg(X) = d$, $h^0(\mathbb{P}^3, \mathcal{I}_X(c_{g,d} - 1)) = 0$ and $h^1(\mathbb{P}^3, \mathcal{I}_X(c_{g,d})) = 0$ (see [2], Proposition V.2).

(a) The main point of the proof is to see how to control the bidegree of the winning curve X. In each inductive step $n \mapsto n+2$ such that $q(n+2, 0) \geq q(n, 0)$ we may link the added lines to the components of the binary curve (or the ladder) we prefer. Indeed, each of these two components have degrees $\geq n$ (if we want $d_1 \geq n$ and $d_2 \geq n$, otherwise we are never forced to add a line linked to a component which has the degree d_i required by the statement of the theoremto be proved. We have $q(n+2, 0) \geq q(n, 0)$ if and only if $n \equiv 1, 2, 4, 5$. Now assume $n \equiv 3 \pmod 6$, say $n = 6k+3$. In this case we add a tree of degree $2k+2$ which must be linked to one of the two degree ≥ 2 components on Y_n and $2k+3$ lines which we are free to link to any of the degree ≥ 2 components of the curve Y_n. Call (a_1, a_2) the bidegree of two degree ≥ 2 components on Y_n. As bidegree for Y_{n+2} we may obtain in this way any pair (b_1, b_2) such that $b_1 \geq a_1$, $b_2 \geq a_2$, $b_1 + b_2 = 4k+5$. Hence even in this case we get no restriction for the possible bidegrees of Y_{n+2}. Now assume $n \equiv 0 \pmod 6$, say $n = 6x$. In this case we add a tree of degree $2x$ which must be linked to on of the two degree ≥ 2 components of Y_n and $2x+3$ lines which we are free to link to any of the degree ≥ 2 components of the curve Y_n. This case is done as above.

(b) Here we discuss why the given values for $\alpha(g)$ and $\beta(g)$ work. We need to apply several times Lemma 4 and each time appears a set $A \subset C \cap Q$ such that $\sharp(A) = e$, but the curve C is not quite general and hence $C \cap Q$ is not general in Q. We may deform C to a union of a binary curve (or a ladder) and some trees of any prescribed types as in Remarks 3 and 4. This troble gives very weak restrictions on $\min\{d_1, d_2\}$. We also took large integers n_1 and n_0 to be able to get $q(n, 0) + g \leq n$ for all $n \geq \max\{n_0, n_1\}$. We certain need $d_1 \geq \max\{r(n_1, 0), r(n_0, 0)\}$, but we also need two further steps to be able to covers degrees $\neq r(n, 0)$ for some n. Notice that in the first steps after we n_0 and n_1 we may increase d_2 only by a small amount. We need at least two full seps $n \mapsto ton + 2$ with $n \equiv 0, 3 \pmod 6$ to add a tree to the minimal degree component of the binary curve or the lower degree non-linear component of the ladder. Notice that $r(n, 0) \leq (n+10)^2/6$. Hence we took $\alpha(g) := (6g+30)^2/6$. □

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