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BINARY CURVES WITH MAXIMAL RANK IN $\mathbb{P}^n$, $n \geq 4$

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Abstract: A genus $g \geq 2$ binary curve is a nodal curve $X = D_1 \cup D_2$ such that $D_1 \cong D_2 \cong \mathbb{P}^1$ and $\sharp(D_1 \cap D_2) = g + 1$. Here we prove the existence of many embeddings of a general genus g binary curve in $\mathbb{P}^n$, $n \geq 4$, with semibalanced bidegrees in the sense of L. Caporaso and with good postulation.

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1. Introduction

Fix an integer $g \geq 2$. We recall that a *binary curve* is a reduced projective curve with 2 irreducible components, say C_1, C_2, such that $C_1 \cong C_2 \cong \mathbb{P}^1$, $\sharp(C_1 \cap C_2) = g+1$ and every point of $C_1 \cap C_2$ is an ordinary node of X (see [4]). Thus $p_a(X) = g$ and X is a stable curve. Let $X = C_1 \cup C_2$ be a genus g binary curve. Every line bundle L on X has a bidegree (d_1, d_2), where $d_i := \deg(L|C_i)$. Obviously, $d_1 + d_2 = \deg(L)$. A key numerical condition for line bundles on stable curves is the so-called Basic Inequality introduced by L. Caporaso (see [3], p. 611). The line bundles satisfying it are called *semibalanced* in [8] and [4]. In the case of a binary curve the Basic Inequality just says

$$|d_1 - d_2| \leq g + 1\,. \tag{1}$$

Let $T \subset \mathbb{P}^n$ be any scheme. For every integer $t \geq 0$ let $\rho_{T,t} : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(T, \mathcal{O}_T(t))$ denote the restriction map. The scheme T has *maximal rank* if

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for every integer $t \geq 0$ the linear map $\rho_{T,t}$ has maximal rank, i.e. it is injective or surjective. Now assume $\dim(T) \leq 1$ and that $h^1(T, \mathcal{O}_T(x)) = 0$ for every $x \geq 1$. Let d be the degree of the one.dimensional part of T. Obviously, $\rho_{T,0}$ has maximal rank. Fix an integer $t \geq 1$. Since $h^0(T, \mathcal{O}_T(t)) = td + \chi(\mathcal{O}_T)$, $\rho_{T,t}$ has maximal rank if and only if either $h^1(\mathbb{P}^n, I_T(t)) = 0$ (case $td + \chi(\mathcal{O}_T) \leq \binom{n+t}{n}$) or $h^0(\mathbb{P}^n, I_T(t)) = 0$ (case $td+\chi(\mathcal{O}_T) \geq \binom{n+t}{n}$). Here we will prove the following asymptotic result. Up to a certain step, we could get a non-asymptotic result (see the proof of Lemma 5).

Theorem 1. *Fix integers $n \geq 4$ and $g \geq 3$. There is an integer $\gamma(n, g) \geq n+g$ with the following property. Fix an integer $d \geq \gamma(n, g)$. Then there exists a binary curve $X \subset \mathbb{P}^n$ with maximal rank such that $p_a(X) = g$, $\deg(X) = d$ and $\mathcal{O}_X(1)$ satisfies the Basic Inequality.*

We may also prove the following small improvement of Theorem 1.

Theorem 2. *Fix integers $n \geq 4$ and $g \geq 3$.There is an integer $\gamma(n, g) \geq n + g$ with the following property. Fix an integer $d \geq \gamma(n, g)$. Then there exist an integer $\delta_{n,g,d} > 0$ such that $\lim_{d\to+\infty} \delta_{n,g,d} = +\infty$ and the following sentence is true. For every integer δ such that $0 \leq \delta \leq \delta_{n,d,g}$ and $\delta \equiv d \pmod 2$ there a binary curve $X \subset \mathbb{P}^n$ with maximal rank such that $p_a(X) = g$, $\deg(X) = d$ and with bidegrees (d_1, d_2) such that $|d_1 - d_2| = \delta$.*

We will use the so-called Horace method introduced in [7].

2. The Proofs

For all integers g, t, n, such that $n \geq 3$, $t \geq 1$, $g \geq 0$, define the integers $a_{n,t,g}$ and $b_{n,t,g}$, by the following relation:

$$t \cdot a_{n,t,g} + 1 - g + b_{n,t,g} = \binom{n+t}{n}, \; 0 \leq b_{n,t,g} \leq t - 1\,. \tag{2}$$

For every integer $d \geq g+n$ let $c_{n,g,d}$ be the minimal integer t such that $d \leq a_{n,t,g}$. The integer $c_{n,g,d}$ is called the *critical value* of the triple (n, d, g).

Lemma 1. *Fix integers $n \geq 4$, $g \geq 0$ and $k \geq \max\{n, g\}$. Then $a_{n,g,k} \geq 2g + 1$.*

Proof. It is sufficient to check the inequality

$$k \cdot (2g + 1) + 1 - g \leq \binom{n+k}{n}. \tag{3}$$

Since $n \geq 4$, (3) if true if $k \geq \max\{n, g\}$. □

Lemma 2. *Fix integers $n \geq 4$, $g \geq 0$ and $k \geq \max\{n, g\}$. Then $a_{n,0,k-1} - k \geq a_{n,g,k}/2$.*

Proof. It is sufficient to prove the following inequality:

$$2k \cdot a_{n,0,k-1} + 2 - 2g \geq 2k^2 + \binom{n+k}{n}. \tag{4}$$

Since $a_{n,0,k-1} \geq \binom{n+k-1}{n}/(k-1) - 1$, it is sufficient to prove the following inequality:

$$2k \cdot \binom{n+k-1}{n} - 2k(k-1) + (2-2g)(k-1) \geq 2k^2(k-1) + (k-1) \cdot \binom{n+k}{n}. \tag{5}$$

Since $\binom{n+k}{n} = \binom{n+k-1}{n} \cdot (n+k)/k$, (5) is true if $k \geq \max\{n, g\}$. □

Remark 1. Let $C \subset \mathbb{P}^n$ be a reduced and connected curve such that $p_a(C) = g$, $\deg(C) = d$ and $h^1(C, \mathcal{O}_C(1)) = 0$. Hence $h^1(C, \mathcal{O}_C(t)) = 0$ for every $t \geq 1$. The curve C has maximal rank if and only $h^1(\mathbb{P}^n, \mathcal{I}_C(c_{n,g,d})) = 0$ and $h^0(\mathbb{P}^n, \mathcal{I}_C(c_{n,g,d})) = 0$.

Remark 2. Let M be a projective scheme and D an effective Cartier divisor of M. For any closed subscheme $Y \subseteq M$ let $\mathrm{Res}_D(Y)$ denote the residual scheme of Y with respect to D, i.e. the closed subscheme of M with $\mathcal{I}_{D,M} : \mathcal{I}_{Y,M}$ as its ideal sheaf. For any $L \in \mathrm{Pic}(M)$ we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Y),M} \otimes L(-D) \to \mathcal{I}_{Y,M} \otimes L \to \mathcal{I}_{Y \cap D,D} \otimes (L|D) \to 0\,. \tag{6}$$

From (6) we get

$$h^i(M, \mathcal{I}_{Y,M} \otimes L) \leq h^i(M, \mathcal{I}_{\mathrm{Res}_D(Y),M} \otimes L(-D)) + h^i(D, \mathcal{I}_{Y \cap D,D} \otimes (L|D)) \tag{7}$$

for all integers $i \geq 0$.

Remark 3. Fix integers n, x, y such that $n \geq 3$, $x \geq n$, $y \geq 0$. Let $C \subset \mathbb{P}^n$ be a general union of a general degree x smooth rational curve and y lines. C has maximal rank (see [1] if $n = 3$, [2] if $n \geq 4$).

Lemma 3. *Fix integers $n \geq 3$, $g \geq 2$, and $k_2 \geq k_1 \geq n + g$ such that $2k_2 \geq (n-1)g + 2$. Fix a hyperplane $H \subset \mathbb{P}^n$. Let Φ be the set of all binary curves $Y \subset \mathbb{P}^n$ of genus g and bidegree (k_1, k_2) intersecting transversally H. The map $Y \mapsto Y \cap H$ induces a morphism β from Φ into the set $E(k_1 + k_2)$ of all subset of H with cardinality $k_1 + k_2$. Then β is dominant.*

Proof. Fix an integer $k \ge k_1$. Fix a general $B \subset \mathbb{P}^n$ such that $\sharp(B) = g+1$. In particular we assume $B \cap H = \emptyset$. Let $A(k)$ be the set of all degree k smooth rational curves of $\mathbb{P}^n$. Set $A(k, B) := \{C \in A(k) : B \subset C\}$. $A(k)$ is a non-empty irreducible variety. Fix a general $C \in A(k)$ and let N_C its normal bundle. N_C is a rigid vector bundle on $C \ge \mathbb{P}^1$, i.e. its splitting type $a_1 \ge \cdots \ge a_{n-1}$ satisfies $a_{n-1} \ge a_1 - 1$. Since $\mathrm{rank}(N_C) = k-1$ and $\deg(N_C) = (n+1)k-2$, we get $a_{n-1} = \lfloor((n+1)k-2)/(n-1))\rfloor$. Since $h^1(\mathbb{P}^1, L) = 0$ for any line bundle L on $\mathbb{P}^1$ with degree ≥ -1, there is $T \in A(k)$ passing through a general $A \subset \mathbb{P}^n$ such that $\sharp(A) = a_{n-1}+1$ (see [9]). Since $a_{n-1} = \lfloor((n+1)k-2)/(n-1))\rfloor \ge k$ and $g \le k$, we get $A(k, B) \ne \emptyset$. Fix any $F \in A(k, B)$. Fix a general $E \subset H$ such that $\sharp(E)$. Since $g+1 \le \lfloor(2k-2)/(n-1)\rfloor + 1$ and E may be seen as a generalization of $F \cap H$, there is $T \in A(B, k)$ such that $T \cap H = E$. Fix general $E_i \subset H$, $i = 1, 2$, such that $\sharp(E_i) = k_i$. We just prove the existence of $T_i \in A(k_i, B)$ such that $T_i \cap H = E_i$. For general E_1, E_2 we may also find T_1, T_2 such that $T_1 \cap T_2 = B$ and $X := T_1 \cup T_2$ is nodal, i.e. it is a binary curve with genus g. □

Remark 4. Fix integers $m \ge 3$, $k \ge e+2 \ge 2$ and $u \ge 2e$ such that $(k+1)u - e \le \binom{m+k}{m}$. Let $C \subset \mathbb{P}^m$ be a general union of e reducible conics and $u - 2e$ lines. It is easy to modify [5] to get $h^1(\mathbb{P}^m, \mathcal{I}_C(k)) = 0$.

Lemma 4. *Fix integers n, x, y such that $n \ge 3$, $y \ge 0$ and $x \ge n+y$. Let $S \subset \mathbb{P}^n$ be a subset such that $\sharp(S) = y$. Let $T \subset \mathbb{P}^n$ be a smooth rational curve such that $S \subset T$ and $\deg(T) = x$. Fix a line $D \subset \mathbb{P}^n$ such that $\sharp(D \cap T) = 1$, $D \cap S = \emptyset$ and $T \cup D$ is nodal. Then there is a flat family of curves in $\mathbb{P}^n$ whose special fiber is $T \cup D$ and whose general fiber is a degree $x+1$ smooth rational curve containing S.*

Proof. Apply [6], Proposition 3.3 and Theorem 4.1, using $N_T(-S)$ instead of N_T and then use [9], Theorem 1.5. □

Remark 5. Fix integers $t \ge m \ge 3$ and a general $S \subset \mathbb{P}^m$ such that $\sharp(S) = t + \lfloor 2t/(m-1)\rfloor + 1$. Let $D \subset \mathbb{P}^m$ be a general smooth rational curve of degree t. Let N_D be its normal bundle. We have $\deg(N_D) = (m+1)t$ and $\mathrm{rank}(N_D) = m-1$. N_D is a rigid vector bundle on $D \cong \mathbb{P}^1$ (see [10]), i.e. its splitting type $a_1 \ge \cdots \ge a_{m-1}$ satifies $a_{m-1} \ge a_1 - 1$. Thus $a_{m-1} = t + \lfloor 2t/(m-1)\rfloor$. Since S is general, there is a smooth rational curve $C \subset \mathbb{P}^m$ such that $\deg(C) = t$ and $S \subset C$ (see [9], Theorem 1.5).

Lemma 5. *Fix the integers $n \ge 4$, $g \ge 0$ and $k \ge$. Then there exists a genus g binary curve $X \subset \mathbb{P}^n$ of bidegree (d_1, d_2) such that $d_1 + d_2 = a_{n,g,k}$, $|d_1 - d_2| \le k^2$, $h^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ and $h^0(\mathbb{P}^n, \mathcal{I}_X(k)) = b_{n,g,k}$.*

Proof. Let $H \subset \mathbb{P}^n$ be a hyperplane.

(a) Since $k \geq g$, we have $a_{n,0,k} \leq a_{n,g,k} \leq a_{n,0,k} + 1$.

Lemma 2 implies $a_{n,0,k-1} - k \geq a_{n,g,k}/2$. There are uniquely determined integers x, y such that $(k-1)x + ky = \binom{n+k-1}{n}$, $y \geq 0$, and $\lceil a_{n,g,k}/2 \rceil \leq x \leq \lceil a_{n,g,k}/2 \rceil + k(k-1) - 1$. Since $x \geq \lceil a_{n,g,2}/2 \rceil$, we have $y \leq (\binom{n+k-1}{n} - (k-1) \cdot \lfloor a_{n,g,k}/2 \rfloor)/k$. Hence $y \leq (\binom{n+k-1}{n} - (k-1) \cdot a_{n,0,k}/2)/k + 1$ (part (a)).

Let $Y \subset \mathbb{P}^n$ be a general union of a general degree x smooth rational curve and y lines. Y has maximal rank (see [2]). Since $(k-1)x + ky = \binom{n+k-1}{n}$ and Y has maximal rank, $h^i(\mathbb{P}^n, \mathcal{I}_Y(k-1)) = 0$, $i = 0, 1$. Let D be the degree x connected component of Y and R_i, $1 \leq i \leq y$, the lines of Y. The generality of Y implies that $Y \cap H$ may be considered a general union of $x + y$ points. Lemma 1 gives $a_{n,g,k} \geq 2g + 1$. Hence $x \geq g + 1$. Fix $S \subseteq D \cap H$ such that $\sharp(S) = g + 1$. Let u, v be the only integers satisfying the following relations:

$$ku + 1 + v + x - g - 1 = \binom{n+k-1}{n-1}, \quad 0 \leq v \leq k - 1\,. \tag{8}$$

There is a union $C \subset H$ of a degree u smooth rational curve E and v lines E_i, $1 \leq i \leq v$, such that C has maximal rank as a curve in H. Hence $h^1(H, \mathcal{I}_C(k)) = 0$ and $h^0(H, \mathcal{I}_C(k)) = x - g - 1$. Hence $h^i(H, \mathcal{I}_{C \cup J}(k)) = 0$, $i = 0, 1$, for a general $J \subset H$ such that $\sharp(J) = x - g - 1$. Since $k \geq \max\{n, g\}$, it is easy to check that $u \geq g + 1$. Hence we may find C with the additional condition $S \subset E$. For a general such E we may assume $Y \cap E = S$. Notice that $D \cup E$ is a binary curve of genus g. If $k \gg 0$ the term $\lfloor 2u/(n-2) \rfloor$ coming from Remark 5 gives $u + 1 + \lfloor 2u/(n-2) \rfloor \geq g + 1 + (\binom{n+k-1}{n} - (k-1) \cdot a_{n,0,k}/2)/k + 1$. This is the only place in which we are not able to do a precise estimate. Hence $u + 1 + \lfloor 2u/(n-2) \rfloor \geq g + 1 + y$. Since $Y \cap H$ is general and $u + 1 + \lfloor 2u/(n-2) \rfloor \geq g + 1 + y$, we may also assume that each line R_i intersects E. Since $b_{n-1,0,k} \leq k - 1$ and $\deg(D) \geq g + k$, we may also assume that each line E_i intersects D, while $E_i \cap S = \emptyset$ for all i. Set $W := Y \cup C$. Notice that $p_a(W) = g$ and $\deg(W) = a_{n,g,k}$. Since $h^1(\mathbb{P}^n, \mathcal{I}_Y(k-1)) = 0$, $h^1(H, \mathcal{I}_C(k)) = 0$ and $\mathrm{Res}_H(W) = Y$, Remark 2 gives $h^1(\mathbb{P}^n, \mathcal{I}_W(k)) = 0$. Set $T_1 := D \cup \bigcup_{i=1}^{v} E_i$ and $T_2 := E \cup_{i=1}^{y} R_i$. Apply several times Lemma 4 to obtain that we may deform T_i to a smooth rational curve X_i keeping fixed the set $T_1 \cap T_2 = S$. Set $X := X_1 \cup X_2$. Notice that X is a genus g binary curve. By semicontinuity we may assume $h^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$. Notice that $h^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ if and only if $h^0(\mathbb{P}^n, \mathcal{I}_X(k)) = b_{n,g,k}$. X has bidegree (d_1, d_2) with $d_1 = \deg(D) + b_{n-1,0,k}$ and $d_2 = y + u = a_{n,g,k} - d_1$. By construction $|d_1 - d_2| \leq k^2$. □

Proof of Theorem 1. Let $H \subset \mathbb{P}^n$ be a hyperplane. Fix a large integer d. Set $k := c_{n,g,d}$. It is sufficient to prove the existence a binary curve $X = C_1 \cup C_2 \subset$

$\mathbb{P}^n$ with maximal rank such that $p_a(X) = g$, $\deg(X) = d$, $|\deg(\mathcal{O}_X|C_1) - \deg(\mathcal{O}_X|C_2)| \le g+1$, $h^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ and $h^0(\mathbb{P}^n, \mathcal{I}_X(k+1)) = 0$. We assume $k \ge \max\{n, g\}$ and that Theorem 1 is true for the triples (n, g, d') such that $c_{n,g,d'} = k-1$. Let t be the maximal integer such that $t(k-1)+1-g \le \binom{n+k-1}{n-1}$. Set $e := \binom{n+k-1}{n} - t(k-1) - 1 + g$. The maximality of the integer t implies $0 \le e \le k-2$. Since $c_{n,g,t} \le k-1$, the assumption gives the existence of a binary curve $Y \subset \mathbb{P}^n$, such that $h^1(\mathbb{P}^n, \mathcal{I}_Y(k-1)) = 0$ (i.e. $h^0(\mathbb{P}^n, \mathcal{I}_Y(k-1)) = e$) and with $\mathcal{O}_Y(1)$ satisfying the Basic Inequality. Let D_1, D_2 be the irreducible components of Y. Set $k_i := \deg(\mathcal{O}_{D_i}(1))$. Without loosing generality we may assume $k_1 \le k_2$. With this assumption the Basic Inequality for $\mathcal{O}_Y(1)$ gives $k_1 \le k_2 \le k_1+g-1$. Obviously, $t = k_1+k_2$. We also have $h^1(Y, \mathcal{O}_Y(1)) = 0$ (use a Mayer-Vietoris exact sequence and that $k_1 \ge n+g$). By semicontinuity we may assume that Y is the general genus g binary curve embedded by a general n-dimensional linear subsystem of the complete linear system associated to a general line bundle of bidegree (k_1, k_2). By Lemma 3 we may assume that $Y \cap H$ is a general subset of H with cardinality t.

First Claim. *Let $S \subset H$ be a general subset such that $\sharp(S) = e$. Then $h^0(\mathbb{P}^n, \mathcal{I}_{Y\cup S}(k-1)) = 0$.*

Proof of the First Claim. Recall that $h^0(\mathbb{P}^n, \mathcal{I}_Y(k-1)) = e$. Assume that First Claim is not true and let a be the maximal integer such that $0 \le a \le e-1$ and $h^0(\mathcal{P}^n, \mathcal{I}_{Y\cup A}(k-1)) = e-a$ for a general $A \subset H$ such that $\sharp(A) = a$. The maximality of the integer a gives that the linear system $|\mathcal{I}_{Y\cup A}(k-1))|$ has H in its base locus. Since $e-a > 0$ and $h^0(\mathbb{P}^n, \mathcal{I}_Y(k-2)) = 0$, we get a contradiction proving the First Claim.

Lemma 3 implies that we may take Y such that $Y \cap H$ is general. We fix Y. Then we take S general. Hence we may also assume that $(Y\cap H)\cup S$ is a general subset of H with cardinality $t+e$. Since $k := c_{n,g,d}$, $f := \binom{n+k}{k} - (kd+1-g) \ge 0$. Let u, v be the only integers such that $ku - e + v = \binom{n+k-1}{n-1}$ and $0 \le v \le k-1$. Since $\binom{n+k}{n} = \binom{n+k-1}{n} + \binom{n+k-1}{n-1}$, we get $k(d-t) + f = ku + v$. Since f is a non-negative integer, $u \ge d-t$. □

Second Claim. $u \ge 2e$.

Proof of the Second Claim. Since $e \le k-2$, it is sufficient to prove $u \ge 2k-4$. Since $u \ge \lfloor \binom{n+k-1}{n-1}/k \rfloor$, it is sufficient to check the inequality $k(2k-4) \le \binom{n+k-1}{n-1}$, which is obviously true if $n \ne 4$ and $k \ge 2$. □

Since $u \ge 2e$ (Second Claim) and $t \ge u$ (Lemma 1), there is a disjoint union $C \subset H$ of e reducible conic and $u-2e$ lines such that $\mathrm{Sing}(C) = S$ and every irreducible component of C intersects Y at a unique point. Set $W := Y \cup C \cup \chi$, where χ is zero-dimensional scheme with e connected components, each of them supported by a different point of S , but none of them comtained in H. The last assumption is equivalent to $\mathrm{Res}_H(W) = Y \cup S$. Remark 4 gives $h^1(H, \mathcal{I}_C(k)) = 0$. Hence Remark 2 gives $h^1(\mathbb{P}^n, \mathcal{I}_W(k)) = 0$. Let A be a unreduced component of C. We may deform to a disjoint union of 2 lines keeping fixed the intersection points with Y of the two lines of A_{red}. Hence we may deform $Y \cup C$ to a curve $Y \cup C'$, where C' is a disjoint union of u lines, each of them intersecting Y at a unique point and in such a way that $Y \cup C'$ has only ordinary nodes as singularities. Semicontinuity gives $h^1(\mathbb{P}^n, \mathcal{I}_{Y \cup C'}(k)) = 0$. Let C'' be the union of $d-u$ of the lines of C'. Use $d-u$ Mayer-Vietoris exact sequences in which at each time a line is added to get the surjectivity of the restriction map $H^0(Y \cup C', \mathcal{O}_{Y \cup C'}(k)) \to H^0(Y \cup C'', \mathcal{O}_{Y \cup C''}(k))$. Thus the vanishing of $h^1(\mathbb{P}^n, \mathcal{I}_{Y \cup C''}(k))$ implies $h^1(\mathbb{P}^n, \mathcal{I}_{Y \cup C''}(k)) = 0$. Let T_1, T_2 be the irreducible components of Y. We may chose the number n_i of points of $Y_i \cap C'$, $i = 1, 2$, with the following rules; $n_1 + n_2 = u$; $n_1 \leq k_1$; $n_1 + k_1 \leq n_2 + k_2$; n_1 is maximal with these properties. If $k_2 \geq k_1 + u$, we get $n_1 = u$ and $n_2 = 0$; if $k_2 \leq k_1 + u$ we get $n_1 + k_1 \leq n_2 + k_2 \leq n_1 + k_2$. Then we take $C'' \subseteq C$ with the same restrictions and get $d_1 = k_1 + (d-t)$ if $k_2 \geq k_1 + (d-t)$ and $d_1 + k_1 \leq d_2 + k_2 \leq d_1 + k_1$ if $k_2 < k_1 + (d-t)$. Hence $|d_1 - d_2| \leq \max\{1, k_2 - k_1\}$. Let C_i'', $i = 1, 2$, be the union of the lines of C'' intersecting T_i. Apply several times Lemma 4 to obtain that we may deform $T_i \cup C_i$ to a smooth rational curve keeping fixed the set $T_1 \cap T_2$. Call E_i a nearby smooth rational curve and write $X = E_1 \cup E_2$. X is a binary curve of bidegree $(\deg(T_1) + \deg(C_1''), \deg(T_2) + \deg(C_2''))$. The semicontinuity theorem for cohomology groups gives $h^1(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ if E_i is sufficiently near to $C_i \cup C_i''$. In the same way we find a binary curve $X_1 \subset \mathbb{P}^n$ with genus g and bidegree (d_1, d_2) and such that $h^1(\mathbb{P}^n, \mathcal{I}_{X_1}(k)) = 0$. Since the set of all binary curves in $\mathbb{P}^n$ with genus g and bidegree (d_1, d_2) is irreducible, we find X as above with maximal rank. We need to start the inductive step. It is sufficient to start with Lemma 5 for the integer $k' := k - 2$ and then use twice the construction to get a semibalanced hyperplane line bundle. □

Proof of Theorem 2. Just use that in each inductive step of the proof of Theorem 1 the number n_i of points of $Y_i \cap C'$, $i = 1, 2$, are arbitrary non-negative integers with the only restrictions that $n_1 + n_2 = u$, $n_1 \leq k_1$ and $n_2 \leq k_2$. □

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