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NEW EXACT TRAVELLING WAVE SOLUTIONS FOR THE MODIFIED BEJAMIN-BONA-MAHONEY EQUATION

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Abstract: In this paper, a new general algebraic method is presented to construct exact solutions for nonlinear evolution equations. The efficiency of the method can be demonstrated on the modified Bejamin-Bona-Mahoney equation. The method can be applied to other nonlinear evolution equations in mathematical physics.

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1. Introduction

It is important to search for solutions to nonlinear evolution equations (NLEEs). In the line with the development of computerized symbolic computation, much work has been focused on the various extensions and application of the known algebraic methods to construct the solutions of NLEEs [2]-[7], [9]-[13]. We know that the auxiliary equation plays an important role in searching exact solutions of NLEEs. In order to find more general style exact solutions of NLEEs we need to seek more general auxiliary equations. However in traditional auxiliary equation methods [2], [4], [5], [7], [11]-[13] the variables used in an ansatz always satisfy the same auxiliary equation or auxiliary equations.

The present work is motivated by the desire to extend the above work to

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set up a new general algebraic method, to construct new style solutions of NLEEs. We use three or more variables which satisfy a new auxiliary equation system to replace the variables in the above work. These new variables satisfy the conditions that the derivatives of them should be polynomials in themselves. For illustration, we apply the new general algebraic method to the modified Bejamin-Bona-Mahoney equation and successfully construct new and more general rational formal solutions.

2. Summary of the New General Algebraic Method

In the following we would like to outline the main steps of our method.

Step 1. For a given NLEE system with some physical fields $u_i(x, y, t)$ in three variables x, y, t,

$$\Phi(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixx}, u_{ixy}, u_{iyy}, ...) = 0 \quad (i = 1, 2, ...), \tag{2.1}$$

by using the wave transformation

$$u_i(x, y, t) = U_i(\xi), \quad \xi = x + ly + \lambda t, \tag{2.2}$$

where l and λ are constants to be determined later. Then the NLEE system (2.1) is reduced to an ordinary differential system

$$\Theta(U_i, U_i', U_i'', ...) = 0\,. \tag{2.3}$$

Step 2. We introduce a new ansatz in the following forms

$$\begin{aligned} U_i(\xi) &= P_i(F(\xi), G(\xi), H(\xi)) \\ &= a_{i0} + \sum_{j=1}^{m_i}[\sum_{r_1+r_2+r_3=j} a^{ij}_{r_1r_2r_3} F^{r_1}(\xi)G^{r_2}(\xi)H^{r_3}(\xi)], \end{aligned} \tag{2.4}$$

where a_{i0}, $a^{ij}_{r_1r_2r_3}$ $(i = 1, 2...;\ j = 1, ...m_i)$ are constants to be determined later and the new variables $F = F(\xi)$, $G = G(\xi)$, $H = H(\xi)$ satisfy

$$\frac{dF}{d\xi} = K_1(F, G, H), \quad \frac{dG}{d\xi} = K_2(F, G, H), \quad \frac{dH}{d\xi} = K_3(F, G, H), \tag{2.5}$$

where K_1, K_2 and K_3 are polynomials of F, G and H.

Step 3. By balancing the highest order derivative term and the nonlinear terms in system (2.1) or system (2.3), we can gain values of m_i. If m_i is a nonnegative integer, then we first make the transformation $U_i = V_i^{m_i}$.

Step 4. Substitute (2.4) into (2.3) along with (2.5). Then set all coefficients of $F^{r_1}(\xi)G^{r_2}(\xi)H^{r_3}(\xi)$ $(r_1 = 0, 1, 2, ...; r_2 = 0, 1, 2...; r_3 = 0, 1, 2, ...)$ to be zero. We get an over-determined nonlinear algebraic system with respect to $\lambda, l,$ a_{i0},

$a^{j}_{r_1 r_2 r_3}(i = 1, 2...; j = 1, ...m_i)$.

Step 5. Solving the over-determined nonlinear algebraic system with the help of *Maple*, we can gain the explicit expressions for $\lambda, l,\ a_{i0},\ a^{ij}_{r_1 r_2 r_3}(i = 1, 2...; j = 1, ...m_i)$.

Step 6. According to system (2.2), (2.4), the conclusions in Step 5 and the solutions of system (2.5) which can be seen in Appendix A, we can obtain rational formal exact solutions of system (2.1).

3. Exact Solutions of the mBBM Equation

The BBM equation was advocated by Benjiamin, Bona and Mahoney. It modeled the same physical phenomena equally well as the KdV equation, see [8]. Now let us consider the modified BBM (mBBM) equation, see [1].

$$u_t + cu_x + \alpha u^2 u_x + \beta u_{xxt} = 0. \tag{3.1}$$

By considering the wave transformations:

$$u(x, t) = u(\xi), \quad \xi = x + \lambda t, \tag{3.2}$$

where λ is a constant to be determined later, we change equation (3.1) to the form

$$\lambda u' + cu' + \alpha u^2 u' + \lambda \beta u''' = 0. \tag{3.3}$$

By balancing the highest order derivative term and nonlinear terms of (3.1) or (3.3), we obtain $n = 1$. According to the proposed method, we suppose (3.3) has the following formal travelling wave solution:

$$u = a_0 + a_1 F + a_2 G + a_3 H, \tag{3.4}$$

and the variables $F = F(\xi),\ G = G(\xi),\ H = H(\xi)$ satisfy

$$F' = G(2BG + 2CH + 2\epsilon AH - 1) - BF^2, \tag{3.5a}$$

$$G' = F(BG + 2CH - 1), \tag{3.5b}$$

$$H' = -F(2AG + BH), \tag{3.5c}$$

$$AF^2 = H(1 - BG - \epsilon AH - CH), \tag{3.5d}$$

$$AG^2 = H(1 - BG - CH), \tag{3.5e}$$

where $A \neq 0, B$ and C are arbitrary constants, $\epsilon = \pm 1$ and "$'$" $= \frac{d}{d\xi}$. a_0, a_1, a_2, a_3 and λ are constants to be determined later.

With the aid of *Maple*, substituting (3.4) along with (3.5) into (3.3), and setting the coefficients of $F^i G^j H^l$ $(i = 0, 1, j = 0, 1, l = 0, 1, 2, 3, 4...)$ to be zero yields a set of over-determined algebraic equations with respect to a_0, a_1, a_2, a_3 and λ. For simplity we omit them.

By use of *Maple*, solving the over-determined algebraic equations, we get the following results.

When $\epsilon = 1$, we obtain:

Case 1. $B = a_2 = 0$, $A = -\frac{4}{3}$, $a_0 = \sqrt{-\frac{\beta c}{3\alpha\beta+2\alpha}}$, $a_3 = -2\sqrt{-\frac{\beta c}{3\alpha\beta+2\alpha}}$, $a_1 = \pm 4\sqrt{\frac{-\beta c}{3\alpha\beta+2\alpha}}$, $C = 1$, $\lambda = -\frac{2c}{3\beta+2}$.

Case 2. $B = a_0 = a_3 = 0$, $C = 1$, $\lambda = -\frac{c}{\beta+1}$, $a_1 = -\pm\sqrt{-6\frac{\beta cA}{\alpha\beta+\alpha}}$, $a_2 = \sqrt{-\frac{6\beta cA+6\beta cA^2}{\alpha\beta+\alpha}}$, where A is an arbitrary constant.

Case 3. $C = 1$, $a_2 = -\frac{3\beta c}{2\alpha(3\beta+2)}\sqrt{\frac{-A\alpha(3A+4)(3\beta+2)}{\beta c}}$, $a_3 = -\frac{3\beta Ac}{2\alpha(3\beta+2)}\sqrt{-\frac{\alpha(3\beta+2)}{\beta c}}$, $a_0 = \sqrt{-\frac{\beta c}{3\alpha\beta+2\alpha}}$, $B = \frac{\sqrt{3A^2+4A}}{2}$, $a_1 = \pm 3\sqrt{-\frac{-\beta cA^2+4\beta cA}{12\alpha\beta+8\alpha}}$, $\lambda = -\frac{2c}{3\beta+2}$, where A is an arbitrary constant.

When $\epsilon = -1$, we obtain:

Case 4. $B = a_2 = 0$, $A = \frac{4}{3}$, $a_0 = \sqrt{\frac{-c\beta}{3\alpha\beta+2\alpha}}$, $a_3 = -2\sqrt{\frac{-c\beta}{3\alpha\beta+2\alpha}}$, $a_1 = \pm 4\sqrt{\frac{-c\beta}{3\alpha\beta+2\alpha}}$, $C = 1$, $\lambda = -\frac{2c}{3\beta+2}$.

Case 5. $a_3 = B = a_0 = 0$, $\lambda = -\frac{c}{\beta+1}$, $a_1 = \pm\sqrt{-\frac{6\beta Ac}{\alpha\beta+\alpha}}$, $C = 1$, $a_2 = \sqrt{\frac{6c\beta A^2-6\beta Ac}{\alpha\beta+\alpha}}$, where A is an arbitrary constant.

Case 6. $C = 1$, $\lambda = -\frac{2c}{3\beta+2}$, $B = \frac{1}{2}\sqrt{4A-3A^2}$, $a_2 = \frac{3\beta c}{2\alpha(3\beta+2)} \times\sqrt{\frac{A\alpha(3A-4)(3\beta+2)}{\beta c}}$, $a_1 = \pm 3\sqrt{-\frac{4\beta Ac+c\beta A^2}{12\alpha\beta+8\alpha}}$, $a_0 = -\sqrt{-\frac{c\beta}{3\alpha\beta+2\alpha}}$, $a_3 = -\frac{3\beta Ac}{\alpha(3\beta+2)} \times\sqrt{\frac{\alpha(3\beta+2)}{-\beta c}}$, where A is an arbitrary constant.

According to (3.2), (3.4) and the general solutions of (3.5) listed in Appendix A, we will obtain the following wave solutions for mBBM equation.

When $\epsilon = 1$, we obtain the following solutions of the modified BBM equation:

$$u_1 = \sqrt{-\frac{\beta c}{3\alpha\beta+2\alpha}} \pm 16\sqrt{-\frac{\beta c}{3\alpha\beta+2\alpha}}\frac{\sinh(\xi)}{3-4\cosh^2(\xi)} - \frac{6\sqrt{-\frac{\beta c}{\alpha\beta+2\alpha}}}{3-4\cosh^2(\xi)},$$

where $\xi = x - \frac{2c}{3\beta+2}t$.

$$u_2 = \pm\sqrt{-\frac{6\beta cA}{\alpha\beta+\alpha}}\frac{\sinh(\xi)}{1+A\cosh^2(\xi)} + \sqrt{-\frac{6\beta cA+6\beta cA^2}{\alpha\beta+\alpha}}\frac{\cosh(\xi)}{1+A\cosh^2(\xi)},$$

where $\xi = x - \frac{c}{\beta+1}t$.

$$u_3 = \sqrt{-\frac{\beta c}{3\alpha\beta+2\alpha}} \pm 6\sqrt{-\frac{-\beta cA^2+4\beta cA}{12\alpha\beta+8\alpha}}\frac{\sinh(\xi)}{2+\sqrt{3A^2+4A}\cosh(\xi)+2A\cosh^2(\xi)}$$
$$-\frac{3\beta c}{\alpha(3\beta+2)}\sqrt{\frac{-A\alpha(3A+4)(3\beta+2)}{\beta c}}\frac{\cosh(\xi)}{2+\sqrt{3A^2+4A}\cosh(\xi)+2A\cosh^2(\xi)}$$
$$-\frac{3\beta Ac}{\alpha(3\beta+2)}\sqrt{-\frac{\alpha(3\beta+2)}{\beta c}}\frac{1}{2+\sqrt{3A^2+4A}\cosh(\xi)+2A\cosh^2(\xi)},$$

where $\xi = x - \frac{2c}{3\beta+2}t$.

When $\epsilon = -1$, we obtain the following solutions of the modified BBM equation:

$$u_4 = \sqrt{\frac{-c\beta}{3\alpha\beta+2\alpha}} \pm 16\sqrt{\frac{-c\beta}{3\alpha\beta+2\alpha}}\frac{\cosh(\xi)}{3+4\sinh^2(\xi)} - 6\sqrt{\frac{-c\beta}{3\alpha\beta+2\alpha}}\frac{1}{3+4\sinh^2(\xi)},$$

where $\xi = x - \frac{2c}{3\beta+2}t$.

$$u_5 = \pm\sqrt{-\frac{6\beta Ac}{\alpha\beta+\alpha}}\frac{\cosh(\xi)}{1+A\sinh^2(\xi)} + \sqrt{\frac{6c\beta A^2-6\beta Ac}{\alpha\beta+\alpha}}\frac{\sinh(\xi)}{1+A\sinh^2(\xi)},$$

where $\xi = x - \frac{c}{\beta+1}t$.

$$u_6 = -\sqrt{-\frac{c\beta}{3\alpha\beta+2\alpha}} \pm 6\sqrt{-\frac{4\beta Ac+c\beta A^2}{12\alpha\beta+8\alpha}}\frac{\cosh(\xi)}{2+\sqrt{4A-3A^2}\sinh(\xi)+2A\sinh^2(\xi)}$$
$$+\frac{3\beta c}{\alpha(3\beta+2)}\sqrt{\frac{A\alpha(3A-4)(3\beta+2)}{\beta c}}\frac{\sinh(\xi)}{2+\sqrt{4A-3A^2}\sinh(\xi)+2A\sinh^2(\xi)}$$
$$-\frac{3\beta Ac}{\alpha(3\beta+2)}\sqrt{-\frac{\alpha(3\beta+2)}{\beta c}}\frac{1}{2+\sqrt{4A-3A^2}\sinh(\xi)+2A\sinh^2(\xi)},$$

where $\xi = x - \frac{2c}{3\beta+2}t$.

4. Conclusions

In this paper, a new general algebraic method is presented to find new exact solutions of NLEEs. We have derived many exact solutions of the modified Bejamin-Bona-Mahoney equation based upon the new method. The paper is shown that the method is sufficient to seek more new exact solutions of nonlinear evolution equations in mathematical physics. Further generalization about the new algebraic method needs us to find more general auxiliary equations or more general ansätzes.

References

[1] T.B. Benjamin, J.L. Bona, J.L. Mahoney, *Phil. Trans. Roy. Soc. Lond. A*, **272** (1972).

[2] Y. Chen, B. Li, *Chaos, Solitons and Fractals*, **19** (2004).

[3] R. Conte, M. Musette, *J. Phys. A: Math. Gen.*, **25** (1992).

[4] E. Fan, *Phys. Lett. A*, **277** (2000).

[5] Y.T. Gao, B. Tian, *Comput. Phys. Commun.*, **21** (2001).

[6] S.Y. Lou, G.X. Huang, H.Y. Ruan, *J. Phys. A*, **24** (1991), 587.

[7] Z.S. Lü, H.Q. Zhang, *Chaos, Solitons and Fractals*, **17** (2003).

[8] V.G. Makhakov, *Phys. Rep.*, **35** (1978).

[9] V.B. Matveev, M.A. Salle, *Darbooux Transformation and Soliton*, Spinger, Berlin (1991).

[10] M. Wadati, *J. Phys. Soc. Jpn.*, **38** (1975), 673 and 681.

[11] M.L. Wang, Y.B. Zhou, *Phys. Lett. A*, **216** (1996).

[12] F.D. Xie, J. Chen, Z.S. Lü, *Commun. Theor. Phys.*, **43** (2005).

[13] Z.Y. Yan, *Chaos, Solitons and Fractals*, **16** (2003).

Appendix A

The general solutions of the new auxiliary equations (3.5)

$$F' = G(2BG + 2CH + 2\epsilon AH - 1) - BF^2, \tag{3.5a}$$

$$G' = F(BG + 2CH - 1), \tag{3.5b}$$

$$H' = -F(2AG + BH), \tag{3.5c}$$

$$AF^2 = H(1 - BG - \epsilon AH - CH), \tag{3.5d}$$

$$AG^2 = H(1 - BG - CH) \tag{3.5e}$$

are:

(1) When $\epsilon = 1$:

$F(\xi) = \frac{\sinh(\xi)}{C+B\cosh(\xi)+A\cosh^2(\xi)}$,

$G(\xi) = \frac{\cosh(\xi)}{C+B\cosh(\xi)+A\cosh^2(\xi)}$, $\quad H(\xi) = \frac{1}{C+B\cosh(\xi)+A\cosh^2(\xi)}$.

(2) When $\epsilon = -1$:

$F(\xi) = \frac{\cosh(\xi)}{C+B\sinh(\xi)+A\sinh^2(\xi)}$,

$G(\xi) = \frac{\sinh(\xi)}{C+B\sinh(\xi)+A\sinh^2(\xi)}$, $\quad H(\xi) = \frac{1}{C+B\sinh(\xi)+A\sinh^2(\xi)}$.