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THE EXISTENCE AND UNIQUENESS OF GLOBAL WEAK SOLUTION FOR A CLASS OF NONLINEAR THERMOELASTIC PLATE EQUATIONS

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Abstract: In this paper, we study the initial-boundary value problems for a class of nonlinear thermoelastic plate equations. Under some certain initial data and boundary conditions, we obtain the existence and uniqueness theorem of global weak solution of the nonlinear thermoelastic plate equations, by means of the Faedo-Galerkin method.

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1. Introduction

The main purpose of this work is to study the existence and uniqueness of the global weak solutions of the following nonlinear time-varying thermoelastic plate equations

$$u_{tt} + \Delta^2 u + \alpha\Delta\theta + N_1(u,\theta) = f\,, \quad t \geq 0, x \in \Omega\,, \tag{1.1}$$

$$\theta_t - \beta\Delta\theta - \alpha\Delta u_t + N_2(u,\theta) = g\,, \quad t \geq 0, x \in \Omega\,, \tag{1.2}$$

with the initial conditions

$$u(x,y,0) = u_0(x,y)\,, \quad u(x,y,0) = u_1(x,y)\,, \quad \theta(x,y,0) = \theta_0(x,y)\,, \tag{1.3}$$

and the boundary conditions

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$$u(0,y,t)=u(l,y,t)=0\,,\quad u(x,0,t)=u(x,l,t)=0\,, \tag{1.4}$$

$$u_{xx}(0,y,t)=u_{xx}(l,y,t)=0\,,\quad u_{yy}(x,0,t)=u_{yy}(x,l,t)=0\,, \tag{1.5}$$

$$\theta(0,y,t)=\theta(l,y,t)=0\,,\quad \theta(x,0,t)=\theta(x,l,t)=0\,, \tag{1.6}$$

where u is the transversal displacement and θ is temperature of the plate, and $\Omega=(0\times l)\times(0\times l)$ be a rectangular domain in R^2, α,β are positive constants, f,g are given functions depending on t,x,y. Here, Δ denotes the two-dimensional Laplacian operator, and Δ^2 is the two-dimensional biharmonic operator.

In the following arguments, the nonlinear terms $N_1(u,\theta)$ and $N_2(u,\theta)$ satisfy

$$\begin{cases} N_1(u,\theta)=N_{11}(u)+N_{12}(\theta)\,, \\ N_{11}(u)\in C^1(R), 0\le N_{11}'(u)\le M_1, N_{11}(0)=0\,, \\ N_{12}(\theta)\in C^1(R), 0\le N_{12}'(\theta)\le M_2, N_{12}(0)=0\,, \end{cases} \tag{1.7}$$

$$\begin{cases} N_2(u,\theta)=N_{21}(u)+N_{22}(\theta)\,, \\ N_{21}(u)\in C^1(R), 0\le N_{21}'(u)\le M_3, N_{21}(0)=0\,, \\ N_{22}(\theta)\in C^1(R), 0\le N_{22}'(\theta)\le M_4, N_{22}(0)=0\,, \end{cases} \tag{1.8}$$

where $M_i(i=1,\cdots,4)$ are positive constants.

In [1], Kim studied the following linear thermoelastic system

$$\begin{cases} w_{tt}+\Delta^2 w+\alpha\Delta\theta=0\,,\quad t\ge 0, x\in\Omega\,, \\ \theta_t-\beta\Delta\theta-\alpha\Delta w_t=0\,,\quad t\ge 0, x\in\Omega\,, \end{cases}$$

with the homogeneous Dirichlet boundary conditions

$$w=\frac{\partial w}{\partial \eta}=0\,,$$

where Ω is a regular domain in R^n and he proved the exponential decay of the energy as time tends to infinity.

In [2], J.E. Muñoz Rivera considered the thermoelastic plate system

$$u_{tt}-\gamma\Delta u_{tt}+\Delta^2 u+\alpha\Delta\theta=0\,,\quad t\ge 0, x\in\Omega\,,$$
$$\theta_t-\beta\Delta\theta-\alpha\Delta u_t=0\,,\quad t\ge 0, x\in\Omega\,,$$

and they made a comparison between the models in which $\gamma=0$ and $\gamma>0$. They conclude that in the first case the plate system is of a parabolic type, while when $\gamma>0$ the corresponding system has a hyperbolic behavior.

In [3], Z.Y. Liu proved that the semigroup associated with the elliptic part of the thermoelastic plate system is analytic, provided the boundary conditions is the Dirichlet type.

Recently, in [4], M.L. Santos studied the existence, uniqueness and asymptotic behavior of the solutions to a von Kármán system for Kirchhoff plate equations in the presence of thermal effects with free boundary and boundary conditions of memory type.

In this paper, we study the initial-boudary value problems of the nonlinear thermoelastic plate equations in a rectangular domain of R^2. Under the initial conditions (1.3), we consider a class of rectangular plate in presence of thermal effects with four edges simply supported boundary conditions (1.4)-(1.6). And we get the the existence and uniqueness theorem of weak solution for the system (1.1)-(1.6) by means of the Galerkin method.

Let $S_2 = H_0^1(\Omega)\bigcap H^2(\Omega)$, then S_2 is Hilbert space and $S_2 \subset L^2(\Omega)$. We denote the norm of $L^2(\Omega)$ by $\|\cdot\|$, and the inner product of $L^2(\Omega)$ by $(\cdot,\cdot)$, that is $\|\cdot\|^2 = (\int_0^l\int_0^l |\cdot|^2 dxdy)^{\frac{1}{2}}$, and $(u,v) = \int_0^l\int_0^l uvdxdy$.

2. Existence of Weak Solutions for (1.1)-(1.6)

In this section, we show the existence of weak solutions to the nonlinear thermoelastic system (1.1)-(1.6).

Definition 2.1. The couple (u,θ) is a weak solution of system (1.1)-(1.6) when

$$u \in L^\infty(0,T;S_2), \quad u_t \in L^\infty(0,T;L^2(\Omega)), \quad \theta \in L^\infty(0,T;L^2(\Omega))$$

and satisfy the following identity:

$$-\int_0^T (u_t,\varphi_t)dt + \int_0^T (\Delta u,\Delta\varphi)dt + \alpha\int_0^T (\theta,\Delta\varphi)dt + \int_0^T (N_1(u,\theta),\varphi)dt = \int_0^T (f,\varphi)dt - (u_0,\varphi(\cdot,0)), \quad (2.1)$$

$$-\int_0^T (\theta,\varphi_t)dt - \beta\int_0^T (\theta,\Delta\varphi)dt - \alpha\int_0^T (u_t,\Delta\varphi)dt + \int_0^T (N_2(u,\theta),\varphi)dt = \int_0^T (g,\varphi)dt - (\theta_0,\varphi(\cdot,0)), \quad (2.2)$$

for any $\varphi \in C^1([0,T];S_2(\Omega))$ such that $\varphi(\cdot,T) = 0$.

Theorem 2.1. *Let us take $f,g \in L^2(0,\infty;L^2(\Omega))$. Suppose that $N_1(u,\theta)$ and $N_2(u,\theta)$ satisfy (1.7) and (1.8) and $(u_0,u_1,\theta_0) \in S_2 \times L^2(\Omega) \times L^2(\Omega)$, then for any $T > 0$, there exists a weak solution (u,θ) for the system (1.1)-(1.6).*

Proof. The main idea is to use Galerkin method. Let us denote by $w_j(x,y)$ a basis to $S_2(\Omega)$. Let us take $u^m(\cdot,t)\in W_m$ and $\theta^m(\cdot,t)\in W_m$, where W_m is generated by $w_1,w_2,\cdots,w_m$. Standard results on ordinary differential equations guarantee that there exists only one local solution

$$u^m(t)=\sum_{i=1}^{m} g_{jm}(t)w_j(x,y)\,,\quad \theta^m(t)=\sum_{i=1}^{m} h_{jm}(t)w_j(x,y)$$

of the approximate system

$$(u^m_{tt},w_j)+(\Delta u^m,\Delta w_j)-\alpha(\nabla\theta,\nabla w_j)+(N_1(u^m,\theta^m),w_j)=(f,w_j)\,,\quad (2.3)$$

$$(\theta^m_t,w_j)+\beta(\nabla\theta^m,\nabla w_j)+\alpha(\nabla u^m_t,\nabla w_j)+(N_2(u^m,\theta^m),w_j)=(g,w_j)\,,\quad (2.4)$$

for $j=1,2,\cdots,m$, with the initial conditions

$$u^m(0)=u_{0,m}=\sum_{i=1}^{m}(u_0,w_i),u_{0,m}\longrightarrow u_0(x,y)\,,\quad \text{in}\quad S_2\,,$$

$$u^m_t(0)=u_{1,m}=\sum_{i=1}^{m}(u_1,w_i),u_{1,m}\longrightarrow u_1(x,y)\,,\quad \text{in}\quad L^2(\Omega)\,,$$

$$\theta^m(0)=\theta_{0,m}=\sum_{i=1}^{m}(\theta_0,w_i),\theta_{0,m}\longrightarrow \theta_0(x,y)\,,\quad \text{in}\quad L^2(\Omega)\,.$$

The extension of the solutions to the whole interval $[0,T]$, $0<T<\infty$, is a consequence of the priori estimate which we are going to prove below.

A priori estimate. Multiplying the equations (2.3) and (2.4) by g'_{jm} and h_{jm}, respectively, and summing up the product result in $j=1,2,\cdots,m$, we obtain

$$(u^m_{tt},u^m_t)+(\Delta u^m,\Delta u^m_t)-\alpha(\nabla\theta,\nabla u^m_t)+(N_1(u^m,\theta^m),u^m_t)=(f,u^m_t)\,,\quad (2.5)$$

$$(\theta^m_t,\theta^m)+\beta(\nabla\theta^m,\nabla\theta^m)+\alpha(\nabla u^m_t,\nabla\theta^m)+(N_2(u^m,\theta^m),\theta^m)=(g,\theta^m)\,.\quad (2.6)$$

Plus (u_m,u_{mt}) and $-(\Delta u_m,u_{mt})$ in both sides of the equation (2.3) simultaneously, then plus it with (2.6), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u^m_t\|^2+\|\Delta u^m\|^2+\|u^m\|^2+\|\frac{\partial u^m}{\partial x}\|^2+\|\frac{\partial u^m}{\partial y}\|^2+\|\theta^m\|^2\right)\\&+\beta\left(\|\frac{\partial \theta^m}{\partial x}\|^2+\|\frac{\partial \theta^m}{\partial y}\|^2\right)=(f,u^m_t)+(g,\theta^m)+(u_m,u^m_t)\\&-(\Delta u^m,u^m_t)-(N_{11}(u^m),u^m_t)-(N_{12}(\theta^m),u^m_t)\\&-(N_{21}(u^m),\theta^m)-(N_{22}(\theta^m),\theta^m)\,.\end{aligned}$$

Let

$$E_m(t) = \|u_t^m\|^2 + \|\Delta u^m\|^2 + \|u^m\|^2 + \|\frac{\partial u^m}{\partial x}\|^2 + \|\frac{\partial u^m}{\partial y}\|^2 + \|\theta^m\|^2 .$$

Then we get

$$\frac{1}{2}\frac{d}{dt}E_m(t) + \beta\left(\|\frac{\partial \theta^m}{\partial x}\|^2 + \|\frac{\partial \theta^m}{\partial y}\|^2\right) = (f, u_t^m) + (g, \theta^m) + (u^m, u_t^m) - (\Delta u^m, u_t^m) - (N_{11}(u^m), u_t^m) - (N_{12}(\theta^m), u_t^m) - (N_{21}(u^m), \theta^m) - (N_{22}(\theta^m), \theta^m) .$$

So from (1.7) and (1.8), and applying the Cauchy-Schwarz inequality, we have

$$\frac{d}{dt}E_m(t) \le \|f\|^2 + \|g\|^2 + CE_m(t) ,$$

where C is a positive constant, while C denotes various positive constants in different places of this paper. Considering $f, g \in L^2(0,\infty; L^2(\Omega))$ and using the Growall inequality, we have

$$E_m(t) \le (C + E_m(0))e^{CT} \qquad (0 \le t \le T) .$$

Taking into account the definition of the initial data of u^m and θ^m we conclude that $E_m(t)$ is uniformly bounded independent of m and $t \in [0,T]$. That is

$$\|u_{mt}\|^2 + \|\Delta u_m\|^2 + \|u_m\|^2 + \|\frac{\partial u_m}{\partial x}\|^2 + \|\frac{\partial u_m}{\partial y}\|^2 + \|\theta_m\|^2 \le C .$$

Hence

$$\{u_m\} \text{ is bounded in } L^\infty(0,T;S_2) ,$$
$$\{u_{mt}\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) ,$$
$$\{\theta_m\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) .$$

By the Aubin-Lions Theorem, we can extract a subsequence $\{u_\mu\}$ of $\{u_m\}$ and a subsequence $\{\theta_\mu\}$ of $\{\theta_m\}$, such that:

$$\{u_\mu\} \to u \quad \text{weak-}\star \quad \text{in} \quad L^\infty(0,T;S_2) ,$$
$$\{u_{\mu t}\} \to u_t \quad \text{weak-}\star \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) ,$$
$$\{\theta_\mu\} \to \theta \quad \text{weak-}\star \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) .$$

Since $\{u_\mu\} \to u$ weak-$\star$ in $L^\infty(0,T;S_2)$, so when $\mu \to \infty$, we have

$$\left|\int_0^T \int_0^l \int_0^l (u_\mu - u)\,\xi_1 dxdydt\right| \to 0 , \quad \forall \xi_1 \in L^1(0,T;S_2^1) . \tag{2.7}$$

About $N_{11}(u_\mu)$, by (1.7) and $u_\mu \in L^\infty(0,T;S_2)$, we know

$$N_{11}(u_\mu) \to \chi_1 \quad \text{weak-}\star \text{ in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{as} \quad \mu \to \infty .$$

Since $0 \le N_{11}' \le M_1$, and by (2.7) we arrive at

$$\left|\int_0^T\int_0^l\int_0^l \left(N_{11}(u_\mu)-N_{11}(u)\right)\xi_1 dxdydt\right| = \left|\int_0^T\int_0^l\int_0^l N_{11}'\left(u_\mu-u\right)\xi_1 dxdydt\right|$$
$$\leq M_1\left|\int_0^T\int_0^l\int_0^l \left(u_\mu-u\right)\xi_1 dxdydt\right| \to 0 \qquad (\mu\to\infty)\,,$$

where ξ_1 is between u_μ and u. By the uniqueness, we conclude that

$$N_{11}(u_\mu)\to N_{11}(u) \quad \text{weak-}\star \text{ in} \quad L^\infty(0,T;L^2(\Omega)).$$

Since $\theta_\mu \to \theta$ weak-$\star$ in $L^\infty(0,T;L^2(\Omega))$, so when $\mu\to\infty$, we have

$$\left|\int_0^T\int_0^l\int_0^l \left(\theta_\mu-\theta\right)\xi_2 dxdydt\right| \to 0 \qquad \forall \xi_2\in L^1(0,T;L^2(\Omega)) \quad . \tag{2.8}$$

About $N_{12}(\theta_\mu)$, by (1.7) and $\theta_\mu\in L^\infty(0,T;L^2(\Omega))$, we know

$$N_{11}(u_\mu)\to \chi_2 \quad \text{weak-}\star \text{ in} \quad L^\infty(0,T;L^2(\Omega)).$$

With the similar arguments as above, we easily get

$$N_{12}(\theta_\mu)\to N_{12}(\theta) \quad \text{weak-}\star \text{ in} \quad L^\infty(0,T;L^2(\Omega)).$$

Using the similar arguments as above again, we can easily verify

$$N_{21}(u_\mu)\to N_{21}(u) \quad \text{weak-}\star \text{ in} \quad L^\infty(0,T;L^2(\Omega))\,,$$
$$N_{22}(\theta_\mu)\to N_{22}(\theta) \quad \text{weak-}\star \text{ in} \quad L^\infty(0,T;L^2(\Omega))\,.$$

Multiplying equation (2.3) by $\rho\in C^1([0,T])$, such that $\rho(T)=0$ and integrating over $[0,T]$ we have

$$-\int_0^T\int_0^l\int_0^l u_t^m w_j\rho_t dxdydt + \int_0^T\int_0^l\int_0^l \Delta u^m \Delta w_j \rho dxdtdy$$
$$+\alpha\int_0^T\int_0^l\int_0^l \theta^m\rho\Delta w_j dxdydt + \int_0^T\int_0^l\int_0^l N_{11}(u^m) w_j\rho dxdydt$$
$$+\int_0^T\int_0^l\int_0^l N_{12}(\theta^m) w_j\rho dxdydt = \int_0^T\int_0^l\int_0^l f w_j\rho dxdydt$$
$$+\int_0^l\int_0^l u_{1,m} w_j \rho(0) dxdydt\,.$$

Since the convergence we have prove above and using the density of the set $\{w_j\rho : \rho\in C^1([0,T]), j=1,2,\cdots,n,\cdots\}$ in $C^1([0,T];S_2)$, we obtain (2.1) when $m\to\infty$.

Similarly, multiplying (2.4) by ρ we have

$$-\int_0^T\int_0^l\int_0^l \theta^m z_j\rho_t dxdydt - \beta\int_0^T\int_0^l\int_0^l \theta^m \Delta z_j\rho dxdydt$$
$$-\alpha\int_0^T\int_0^l\int_0^l u_t^m\Delta z_j\rho dxdydt + \int_0^T\int_0^l\int_0^l N_{21}(u^m) z_j\rho dxdydt$$

$$+\int_0^T\int_0^l\int_0^l N_{22}(\theta^m)z_j\rho dxdydt = \int_0^T\int_0^l\int_0^l gz_j\rho dxdydt + \int_0^l\int_0^l \theta_{0,m}z_j\rho(0)dxdydt\,,$$

and letting $m \to \infty$ we have (2.2). This completes the proof of Theorem 2.1. □

3. Uniqueness of the Weak Solutions for (1.1)-(1.6)

In this section, we show the uniqueness of the weak solutions to the system (1.1)-(1.6).

Theorem 3.1. *The solution (u, θ) satisfying Theorem 2.1 is unique.*

Proof. Suppose (u, θ) and (v, η) are two solutions for (1.1)-(1.6) with the same initial data and take $(w, \sigma) = (u - v, \theta - \eta)$. So the couple (w, σ) has null initial data. With the similar arguments as the proof of Theorem 2.1, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w_t\|^2 + \|\Delta w\|^2 + \|w\|^2 + \|\frac{\partial w}{\partial x}\|^2 + \|\frac{\partial w}{\partial y}\|^2 + \|\sigma\|^2\right)$$
$$+\beta\left(\|\frac{\partial \sigma}{\partial x}\|^2 + \|\frac{\partial \sigma}{\partial y}\|^2\right) = (w, w_t) - (\Delta w, w_t)$$
$$-(N_{11}(u) - N_{11}(v)), w) - (N_{12}(\theta) - N_{12}(\eta)), w)$$
$$-(N_{21}(u) - N_{21}(v)), \sigma) - (N_{22}(\theta) - N_{22}(\eta)), \sigma)\,.$$

Set

$$Y(t) = \|w_t\|^2 + \|\Delta w\|^2 + \|w\|^2 + \|\frac{\partial w}{\partial x}\|^2 + \|\frac{\partial w}{\partial y}\|^2 + \|\sigma\|^2\,.$$

Then we have

$$\frac{1}{2}\frac{d}{dt}Y(t) + \beta\left(\|\frac{\partial \sigma}{\partial x}\|^2 + \|\frac{\partial \sigma}{\partial y}\|^2\right) = (w, w_t) - (\Delta w, w_t)$$
$$-(N_{11}(u) - N_{11}(v)), w) - (N_{12}(\theta) - N_{12}(\eta)), w)$$
$$-(N_{21}(u) - N_{21}(v)), \sigma) - (N_{22}(\theta) - N_{22}(\eta)), \sigma)\,.$$

From (1.7), (1.8) and applying the Cauchy-Schwarz inequality, it follows that

$$\frac{d}{dt}Y(t) \le CY(t)\,.$$

By the Growall inequality, we have

$$Y(t) \le Y(0)e^{C}T \qquad (0 \le t \le T)\,.$$

Since the couple (w,σ) has null initial data, that is $Y(t)=0$. Hence

$$\|w_t\|^2+\|\Delta w\|^2+\|w\|^2+\|\frac{\partial w}{\partial x}\|^2+\|\frac{\partial w}{\partial y}\|^2+\|\sigma\|^2=0$$

that is

$$\|w_t\|^2=\|\Delta w\|^2=\|w\|^2=\|\frac{\partial w}{\partial x}\|^2=\|\frac{\partial w}{\partial y}\|^2=\|\sigma\|^2=0\,.$$

Now Theorem 3.1 is proved. □

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