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STRONG AND SUPERSTRONG VERTICES IN A FUZZY GRAPH

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Abstract: The concept of fuzzy sets was introduced by L.A Zadeh in 1965. The concept of fuzzy graph was introduced by A. Rosenfeld [6] in 1975. Many of the crisp graph concepts have been extended to fuzzy graph theory. Here we define the strong and superstrong vertices of a fuzzy graph and observe some of their properties.

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1. Preliminaries

A *fuzzy graph* $G = (\sigma, \mu)$ is a pair of functions $\sigma : S \to [0, 1]$ and $\mu : SXS \to [0, 1]$ such that for all x, y in S we have $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ where S is the underlying (vertex) set (in this paper we always consider S as a finite set). For the fuzzy graph G, $G^* = (\sigma^*, \mu^*)$ is called the corresponding (crisp)*support graph* where σ^* =Support of σ and μ^* =Support of μ. The fuzzy graph $H = (\tau, \nu)$ is called a *spanning fuzzy subgraph* of G if $\tau(x) = \sigma(x)$ for all x in S. A fuzzy graph $G = (\sigma, \mu)$ is called a *complete fuzzy graph* if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all x and y; it is called *quasi complete* if G^* is complete (clearly complete implies quasi complete). A complete fuzzy graph with n vertices is denoted by σ_n (clearly it is not unique).

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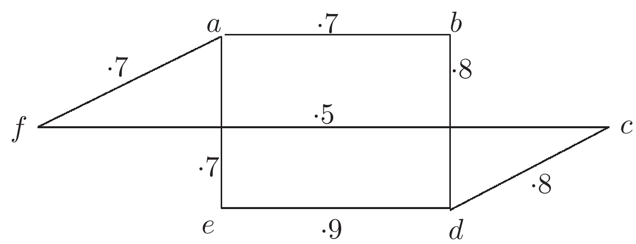
2. Strong and Superstrong Vertices of a Fuzzy Graph

In this paper all fuzzy graphs under consideration are finite and connected.

Definition 2.1. Let $G = (\sigma, \mu)$ be a fuzzy graph and let $v \in \sigma^*$. v is called a *superstrong vertex* if $\mu^\infty(v, x) = \alpha$ for every $x(\neq v) \in \sigma^*$ and for some $\alpha \in (0, 1]$.

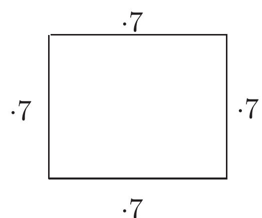
Example 2.2.



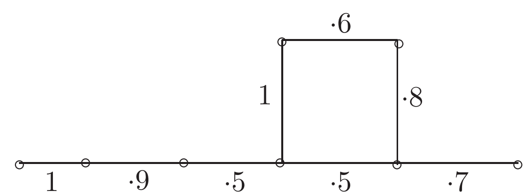
G

Here a and f are the only superstrong vertices.

Example 2.3.



G_1



G_2

In G_1 every vertex is superstrong while in G_2 no vertex is superstrong.

Definition 2.4. Let $G = (\sigma, \mu)$ be a fuzzy graph and let $v \in \sigma^*$. v is called a *strong vertex* if (v, x) is strong for every $(v, x) \in \mu^*$. If every edge incident at v is a strong edge then v is called a *strong vertex*.

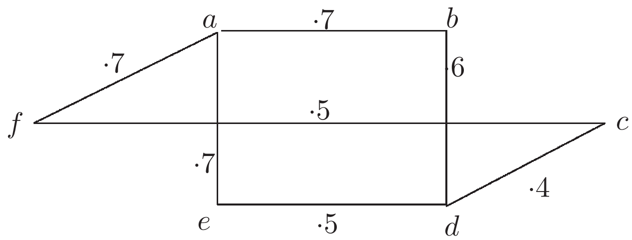
Proposition 2.5. *Let $G = (\sigma, \mu)$ be a fuzzy graph and fix v in σ^*. For some $\alpha \in (0, 1]$, if $\mu(v, x) = \alpha$, for all $(v, x) \in \mu^*$ then v is a strong vertex, i.e. if every edge incident at v has same weight then v is a strong vertex.*

Proof. Suppose every edge incident at v has same weight α, i.e. $\mu(v, x) = \alpha, \quad \forall (v, x) \in \mu^*$. Let $(v, x) \in \mu^*$. If (v, x) is the only path from v to x then clearly (v, x) is strong and hence v is strong. If there is another path P from v to x then clearly strength of $P \leq \alpha$. If Strength of $P < \alpha$ then

$\mu^\infty(v,x) = \alpha = \mu(v,x) \Rightarrow (v,x)$ is strong. If Strength of $P = \alpha$ then also $\mu^\infty(v,x) = \alpha = \mu(v,x) \Rightarrow (v,x)$ is strong. Hence v is a strong vertex. □

Remark 2.6. The converse of the above proposition need not be true. In the above example G, b is strong but $\mu(b,a) \neq \mu(b,d)$.

Remark 2.7. If the hypothesis of the above statement is true then v need not be a superstrong vertex. For example,



G_3

Here $\mu(a,b) = \mu(a,e) = \mu(a,f) = \cdot 7$ but $\mu^\infty(a,b) = \cdot 7$ and $\mu^\infty(a,c) = \cdot 5$ so that a is not a superstrong vertex.

Proposition 2.8. *If the hypothesis of the above statement is true in which α is the minimum of all edges then v is a superstrong vertex.*

But the converse is not true.

Proposition 2.9. *Let $G = (\sigma, \mu)$ be a fuzzy graph and fix $v \in \sigma^*$. For some $\alpha \in (0,1]$, if $\mu(v,x) = \alpha, \forall x \in \sigma^*(x \neq v)$ then v is a superstrong vertex.*

Proof: Suppose v is not a superstrong vertex. Then there exist $x, y \in \sigma^*$ such that $\mu^\infty(v,x) \neq \mu^\infty(v,y)$. Without loss of generality, take $\mu^\infty(v,x) > \mu^\infty(v,y)$. Since (v,y) is strong by , $\mu^\infty(v,y) = \mu(v,y) = \mu(v,x)$. $\Rightarrow \mu^\infty(v,x) > \mu(v,x)$. Which is contradiction to the fact that (v,x) is strong. Hence v is a superstrong vertex. □

Remark 2.10. But the converse of the above proposition need not be true. For example, consider the fuzzy graph G given in Example 2.2. a is a superstrong vertex there but it does not satisfy the hypothesis of the above proposition.

Remark 2.11. There is no relationship between strong vertices and superstrong vertices, i.e. a vertex may be strong without being superstrong and vice versa.

Proposition 2.12. *Let $G = (\sigma, \mu)$ be a connected fuzzy graph. If μ is constant (on μ^*) then every vertex of G is both strong and superstrong. But* *the converse need not be true.*

Proposition 2.13. *Every strong edge of a connected fuzzy graph has same weight iff every vertex is superstrong.*

Proof. If every strong edge of a connected fuzzy graph has same weight α then every weak edge has weight less than α. Let $v \in \sigma^*$. We know that there is a strong path between every pair of vertices of a connected fuzzy graph. Therefore $\mu^\infty(v,x) = \alpha, \ \forall x(\neq v) \in \sigma^*$. Hence v is superstrong. Conversely, supose every vertex of G is superstrong. To prove every strong edge has same weight. Suppose not. Then there exist strong edges (x,y) and (u,v) of G such that $\mu(x,y) \neq \mu(u,v)$. $\Rightarrow \mu^\infty(x,y) = \mu(x,y) \neq \mu(u,v) = \mu^\infty(u,v) \quad \Rightarrow \mu^\infty(x,y) \neq \mu^\infty(u,v) \quad \Rightarrow$ Not every vertex of G is superstrong. Which is a contradiction. □

Proposition 2.14. *If v is a pendant vertex in G^* where G^* is connected and if the edge incident at v has the minimum weight then v is a superstrong vertex.*

Proof. Let $G = (\sigma, \mu)$ be a connected fuzzy graph and let v be a pendant vertex in G^*. Suppose (v,z) is the minimum weight edge of G. To prove that v is superstrong. Suppose not. Then there exist $x, y \ \in \sigma^*$ such that $\mu^\infty(v,x) \neq \mu^\infty(v,y)$. Without loss of generality, assume that $\mu^\infty(v,x) > \mu^\infty(v,y)$. If $\mu^\infty(v,x) = \mu(v,z)$ then $\mu(v,z) > \mu^\infty(v,y)$, i.e. there is an edge in the $v - y$ path whose weight is less than $\mu(v,z)$. Which is a $\Rightarrow\Leftarrow$. If $\mu^\infty(v,x) \neq \mu(v,z)$ then there is an edge in the $v - x$ path whose weight is less than $\mu(v,z)$. Which is also a $\Rightarrow\Leftarrow$. Hence v is superstrong. □

Definition 2.15. In [3], the powers of a given fuzzy graph has been defined as follows: Let $G = (\sigma, \mu)$ be a connected fuzzy graph. Then for some positive integer n,

$$G^n = (\sigma, \bigvee_{k=1}^{n} \mu^k) = (\sigma, \mu \vee \mu^2 \vee \cdots \vee \mu^n),$$

i.e.

$$G = (\sigma, \mu), \quad G^2 = (\sigma, \mu \vee \mu^2), \quad G^3 = (\sigma, \mu \vee \mu^2 \vee \mu^3), \cdots .$$

Proposition 2.16. *If v is superstrong in G then it is superstrong in $G^n, \ n = 1, 2, \cdots \text{diam}G^*$.*

Definition 2.17. In [5], the fusion of two vertices in a fuzzy graph has been defined as follows: Let $G = (\sigma, \mu)$ be a fuzzy graph and let $u, v \ \in \ \sigma^*$. By the *fusion* of two vertices u and v we mean,

(i) Fuse the vertices u and v as uv in the corresponding crisp graph $G^* =$

(σ^*, μ^*) and then consider its underlying simple graph.

(ii) The resulting fuzzy graph is $G_{uv} = (\sigma_{uv}, \mu_{uv})$, where

$$\sigma_{uv}(x) = \begin{cases} \max[\sigma(u), \sigma(v)], & \text{if } x = uv\,, \\ \sigma(x), & \text{if } x \neq uv\,, \end{cases}$$

and

$$\mu_{uv}(x,y) = \begin{cases} \max[\mu(x,u), \mu(x,v)], & \text{if } y = uv\,, \\ \max[\mu(u,y), \mu(v,y)], & \text{if } x = uv\,, \\ \mu(x,y), & \text{if } x \neq uv \text{ and } y \neq uv\,. \end{cases}$$

Proposition 2.18. *The fusion of two superstrong vertices is again superstrong.*

Definition 2.19. In [4], the spanning fuzzy supergraph has been defined as: Let $G = (\sigma, \mu)$ be a fuzzy graph. The *spanning fuzzy supergraph* of G is defined as a fuzzy graph $G' = (\sigma, \mu')$, where

$$\mu'(x,y) = \begin{cases} \mu(x,y), & \text{if } (x,y) \in \mu^*\,, \\ \sigma(x) \wedge \sigma(y), & \text{if } (x,y) \notin \mu^*\,. \end{cases}$$

Remark 2.20. If v is superstrong in G then it need not to be so in G' or G^c.

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References

[1] K.R. Bhutani, A. Rosenfeld, Strong arcs in fuzzy graphs, *Information Sciences*, **152** (2003), 319-322.

[2] J.N. Mordeson, P.S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica-Verlag, Heidelberg (2000).

[3] P.V. Ramakrishnan, T. Lakshmi, Fuzzy bipartite* fuzzy graphs and powers, *International Journal of Mathematics, Game Theory and Algebra,* **17**, No-s. 1, 2 (2007), 19-26.

[4] P.V. Ramakrishnan, T. Lakshmi, Spanning fuzzy supergraphs, In: *Proceedings of the International Conference on Mathematics and Computer*

Science, Held at Loyola College, Chennai (1-3 March, 2007), 67-69; *Journal of Mathematics and System Sciences*.

[5] P.V. Ramakrishnan, T. Lakshmi, A note on the fusion of two vertices in a fuzzy graph, *Tsukuba Journal of Mathematics,* **32**, No. 1 (2008), 1-10.

[6] A. Rosenfeld, Fuzzy graphs, In: *Fuzzy Sets and their Applications to Cognitive and Decision Processes* (Ed-s: L.A. Zadeh, K.S. Fu, K. Thanaka, M. Shimura), Academic Press, New York (1975), 77-95.

[7] M.S. Sunitha, A. Vijaya Kumar, Complement of a fuzzy graph, *Indian J. Pure and Appl. Math.*, **33**, No. 9 (2002), 1451-1464.