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CLASSIFICATION OF HOPF ALGEBRAS OF DIMENSION p^2

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Abstract: In this paper we give a simpler, more direct proof of the result that a non-semisimple Hopf algebra of dimension p^2 is isomorphic to a Taft algebra.

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1. Introduction

In recent years, much effort has been focussed on classifying finite-dimensional Hopf algebras over an algebraically closed field k of characteristic 0. While the classification of finite-dimensional Hopf algebras of a given dimension seems to be a very difficult task, some subclasses, like semisimple or pointed Hopf algebras, have been classified for certain dimensions.

In 1994, Zhu showed that Hopf algebras of prime dimension p are isomorphic to the group algebra $k\mathbb{Z}_p$ [9].

For dimension p^2 it was proved in [3] that all semisimple Hopf algebras are group algebras, namely $k\mathbb{Z}_{p^2}$ or $k\left[\mathbb{Z}_p \times \mathbb{Z}_p\right]$.

The only known example of non-semisimple Hopf algebras of dimension p^2, during the last thirty-five years was Taft algebras. However, there were some partial answers to the question whether Taft algebras are the only non-semisimple Hopf algebras of dimension p^2. Andruskiewitsch and Chin proved that non-semisimple pointed Hopf algebras of dimension p^2 are indeed Taft

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algebras [7]. Another proof of this result is given in [1], where authors also proved that a non-semisimple Hopf algebra of dimension p^2, with the antipode of order $2p$, must be pointed. Finaly, in 2002, Ng showed that the order of the antipode in a non-semisimple Hopf algebra of dimension p^2 must be $2p$.

Hence, the classification of Hopf algebras of dimension p^2 is completed, and we have $p+1$ non-isomorphic Hopf algebras of dimension p^2 over k:

(a) $k\mathbb{Z}_{p^2}$

(b) $k\left[\mathbb{Z}_p \times \mathbb{Z}_p\right]$

(c) $T(\xi)\,,\ \xi \in k$ a primitive pth root of unity.

In this paper first we give basic notation and preliminaries and then we prove more directly that a non-semisimple Hopf algebra of dimension p^2 is isomorphic to a Taft algebra.

2. Notation and Preliminaries

Throughout this paper we work over an algebraically closed field k of characteristic 0. Let H be a finite-dimensional Hopf algebra over k with antipode S. Its comultiplication and counit are, respectively, denoted by Δ and ϵ. A non-zero element $g \in H$ is called group-like if $\Delta(g) = g \otimes g$. The set of all group-like elements $G(H)$ of H is a linearly independent set and it forms a group under the multiplication of H. The divisibility of $\dim H$ by $|G(H)|$ is an immediate consequence of the following generalization of Lagrange Theorem, due to Nichols and Zoeller [6].

Theorem 1. *If B is a Hopf subalgebra of H, then H is a free B-module. In particular, $\dim B$ divides $\dim H$.*

The semisimplicity of a finite-dimensional Hopf algebra can be characterized by the antipode [2].

Theorem 2. *Let H be a finite-dimensional Hopf algebra with antipode S over a field of characteristic 0. Then the following statements are equivalent:*

(i) *H is semisimple.*

(ii) *H^* is semisimple.*

(iii) $Tr\left(S^2\right) \neq 0$.

(iv) $S^2 = id_H$.

A basic result in classification of finite-dimensional Hopf algebras is Zhu

Theorem.

Theorem 3. *A Hopf algebra of dimension p is necessarilly isomorphic to the group algebra $k\mathbb{Z}_p$, of cyclic group of order p.*

Let $g \in H$ and $\alpha \in H^*$ be the modular (or distinguished) group-like elements. Then we have the following formula for S^4 in terms of α and g, due to Radford:

$$S^4(h) = g\left(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1}\right) g^{-1}, \qquad h \in H,$$

where $\rightharpoonup$ and $\leftharpoonup$ denote the natural actions of the Hopf algebra H^* on H described by

$$\beta \rightharpoonup h = \sum h_{(1)} \beta\left(h_{(2)}\right) \text{ and } h \leftharpoonup \beta = \sum \beta\left(h_{(1)}\right) h_{(2)},$$

for $\beta \in H^*$ and $h \in H$. We know that Hopf algebra H (respectively H^*) is unimodular iff α (respectively g) is trivial. If both H and H^* are unimodular, then $S^4 = id_H$.

The following lemma is one step in the proof of Zhu Theorem.

Lemma 4. *Let H be a finite-dimensional non-semisimple Hopf algebra with antipode S of odd dimension over k. Then H and H^* cannot be both unimodular and $S^4 \neq id$.*

Proof. Suppose that both H and H^* are unimodular. Then $S^4 = id_H$. Thus $H = H_+ \oplus H_-$, where $H_\pm$ is the eigenspace of S^2 of eigenvalue ± 1. Hence $\dim H = \dim H_+ + \dim H_-$. By 2, $Tr\left(S^2\right) = 0$, and hence $\dim H_+ = \dim H_-$. Therefore $\dim H$ is even, a contradiction! □

We will need the version of the Taft-Wilson Theorem proved in [4]. If H is a Hopf algebra, then $G(H)$ will denote the group of grouplike elements of H, and $H_0, H_1, \ldots$ will denote the coradical filtration of H. H is called pointed if $H_0 = kG(H)$, where H_0 is the coradical. If $g, h \in G(H)$, then $P_{g,h} = \{x \in H : \Delta(x) = x \otimes g + h \otimes x\}$ is the set of (g,h)-primitive elements. If H is finite dimensional, there are no nonzero $(1,1)$-primitive elements.

Since $g - h \in P_{g,h}$ we can choose a subspace $P'_{g,h}$ of $P_{g,h}$ such that $P_{g,h} = k(g-h) \oplus P'_{g,h}$.

Theorem 5. *Let H be a pointed Hopf algebra. Then:*

(1) $H_1 = H_0 \oplus \left(\oplus_{g,h \in G(H)} P'_{g,h}\right)$.

(2) *For any $n \geq 1$ and $c \in H_n$, there exist $\left(c_{g,h}\right)_{g,h \in G(H)}$ in H and $w \in H_{n-1} \otimes H_{n-1}$ such that $c = \sum_{g,h \in G(H)} c_{g,h}$ and $\Delta\left(c_{g,h}\right) = c_{g,h} \otimes g + h \otimes c_{g,h} + w$.*

Since $P_{1,1} = 0$, the first part of the theorem shows that $G(H)$ is not trivial if H is pointed, nontrivial, finite dimensional Hopf algebra.

In order to compute comultiplication on products and powers of (g,h)-primitives, we will require Quantum Binomial Formula.

Lemma 6. *If x and y are elements of a $k-$algebra that $q-$commute, i.e. $xy = qyx,\ q \in k$, then the following formula holds for every $n \in \mathbb{N}$:*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i}_q y^i x^{n-i}.$$

Here $\binom{n}{i}_q = \frac{(n)!_q}{(i)!_q(n-i)!_q}$, where $(n)!_q = (n)_q \ldots (2)_q (1)_q$, and $(n)_q = 1 + q + \ldots + q^{n-1}$. By definition, $(0)!_q = 1$.

Corollary 7. *If q is a primitive n-th root of unity, then $(x+y)^n = x^n + y^n$.*

We remind definition and basic facts about Taft algebras.

Definition 8. Let $\omega \in k$ be a primitive n-th root of unity. The Taft algebra $T(\xi)$ is generated by elements g and x, as a k-algebra, subject to the relations

$$g^n = 1, \quad x^n = 0, \quad gx = \xi xg.$$

The Hopf algebra structure is given by

$$\Delta(g) = g \otimes g, \qquad S(g) = g^{-1}, \qquad \epsilon(g) = 1,$$
$$\Delta(x) = 1 \otimes x + x \otimes g, \qquad S(x) = -xg^{-1}, \qquad \epsilon(x) = 0.$$

The Hopf algebra $T(\xi)$ is not semisimple and $\dim T(\xi) = n^2$ (the set $\left\{x^i g^j \mid \ 0 \leq i,j \leq n-1\right\}$ forms a basis for $T(\xi)$). It is also known that $T(\xi) \cong T(\xi)^*$ as Hopf algebras, and that $T(\xi) \cong T\left(\xi'\right)$ only if $\xi = \xi'$.

3. Main Theorems

In [1] Andruskiewitsch and Schneider proved the next theorem. Now we give simpler, more direct proof.

Theorem 9. *If H is a pointed non-semisimple Hopf algebra of dimension p^2, then H is a Taft algebra.*

Proof. Let H_0 denotes the coradical of H, a Hopf subalgebra isomorphic to the group algebra. By the Nichols-Zoeller Theorem we have $\dim H_0 \in$

$\{1, p, p^2\}$. But we have seen that the coradical of H cannot be of dimension 1. Also, $\dim H_0 \neq p^2$, because H is not semisimple. So, $\dim H_0 = p$, and coradical of H is isomorphic to $k\mathbb{Z}_p$ by Zhu's Theorem.

Further, the Taft-Wilson Theorem ensures the existence of some $g \in G(H)$ such that $P_{1,g} \neq k(1-g)$. If $\phi : P_{1,g} \longrightarrow P_{1,g}$ is the map defined by $\phi(a) = g^{-1}ag$ for every $a \in H$, then $\phi^p = id$, so $P_{1,g}$ has a basis of eigenvectors for ϕ. Let x be such an eigenvector, which is not in $k(1-g)$, and let λ be the corresponding eigenvalue. If $\lambda = 1$, then the Hopf subalgebra A generated by g and x is commutative, and hence involutory, i.e. $S^2 = id$. Now, by 2, this is equivalent with semisimplicity of H, but Hopf algebra over an algebraically closed field of characteristic zero is semisimple if and only if it is cosemisimple [2]. Thus A is cosemisimple of dimension bigger than p, and this is in contradiction with fact that coradical of H is $k\mathbb{Z}_p$. Therefore, $\lambda \neq 1$.

Now from $\Delta(x) = 1 \otimes x + x \otimes g$, using the quantum binomial formula, we obtain that $\Delta(x^p) = 1 \otimes x^p + x^p \otimes g$. Thus, $x^p \in P_{1,1}$, so $x^p = 0$. Together with $\Delta(g) = g \otimes g$, $g^p = 1$ and $xg = \lambda gx$, this shows that A is a Taft algebra. Since, $\dim A = p^2 = \dim H$, it follows that H is a Taft algebra. □

We need the next two results to prove that every non-semisimple Hopf algebra of dimension p^2 is isomorphic to a Taft algebra. First of them is Proposition 5.1 from [1].

Proposition 10. *Let H be a finite-dimensional Hopf algebra whose antipode S has order $2p$. Assume also that H contains a cosemisimple Hopf subalgebra B such that $\dim H = p \dim B$. Then B is the coradical of H.*

The second is the main result of [5].

Theorem 11. *If H is a non-semisimple Hopf algebra of dimension p^2, with antipode S, then the order of S^2 is p.*

Now we have the main classification result.

Theorem 12. *Let H be a non-semisimple Hopf algebra of dimension p^2, where p is any prime number. Then H is isomorphic to a Taft algebra.*

Proof. *Case* $p = 2$. Let $C \subset H$ be simple subcoalgebra. As k is algebraically closed, any simple coalgebra C over k is isomorphic to the dual of the matrix algebra $M_n(k)$, for some n. Therefore, $\dim C$ is a square. In our case ($\dim H = 4$), that implies that $\dim C = 1$, because H is not cosimple. Hence, every simple subcoalgebra of H is of dimension 1, and, by definition, H is pointed. Now we can apply Theorem 9. So, if H is a non-semisimple Hopf algebra of dimension 4, then H is isomorphic to the Taft algebra $T(-1)$.

Case $p > 2$. By Lemma 4, H or H^* have a nontrivial group-like element; say H. This element g is of order p, because H is not isomorphic to $k\mathbb{Z}_{p^2}$. Also, by Theorem 11, we know that the order of the antipode of H is p. Then Theorem 10 applies, with $B = k[g]$. Therefore, coradical of H is group algebra, and H is pointed. By Theorem 9, H is isomorphic to a Taft algebra. Because Taft algebras are self-dual, the assertion also follows in the case when H^* has a nontrivial group-like element. □

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References

[1] N. Andruskiewitsch, H.-J. Schneider, Hopf algebras of order p^2 and braided Hopf algebras of order p, *J. Algebra*, **199**, No. 2 (1998), 430-454.

[2] R.G. Larson, D.E. Radford, Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple, *J. Algebra*, **117**, No. 2 (1988), 267-289.

[3] A. Masuoka, The p^n theorem for semisimple Hopf algebras, *Proc. Amer. Math. Soc.*, **124**, No. 3 (1996), 735-737.

[4] S. Montgomery, Hopf algebras and their actions on rings, In: *CBMS Regional Conference Series in Mathematics*, **82**, Washington DC (1993).

[5] S.-H. Ng, Non-semisimple Hopf algebras of dimension p^2, *J. Algebra*, **255**, No. 1, (2002), 182-197.

[6] W.D. Nichols, M.B. Zoeller, A Hopf algebra freeness theorem, *Amer. J. Math.*, **111**, No. 2 (1989), 381-385.

[7] D. Stefan, Hopf subalgebras of pointed Hopf algebras and applications, *Proc. Amer. Math. Soc.*, **125**, No. 11 (1997), 3191-3193.

[8] M.E. Sweedler, *Hopf Algebras*, W.A. Benjamin, Inc., New York (1969).

[9] Y. Zhu, Hopf algebras of prime dimension, *Internat. Math. Res. Notices*, **1**, (1994), 53-59.