International Journal of Pure and Applied Mathematics
Volume 48 No. 2 2008, 37-41

NOTE ON THE ALMOST PERIODIC SOLUTIONS OF THE ABSTRACT CAUCHY PROBLEM

Saud M.A. Alsulami
Department of Mathematics
King Abdulaziz University
P.O. Box 138381, Jeddah, 21323, KINGDOM OF SAUDI ARABIA
e-mail: alsulami@kau.edu.sa

Abstract: In this note we present a sufficient condition for a bounded solution of the abstract Cauchy problem to be almost periodic.

AMS Subject Classification: 34C27
Key Words: almost periodic solutions, abstract Cauchy problem

1. Introduction

Consider the abstract Cauchy problem:

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{1}$$

where A is a bounded linear operator on a Banach space E and f is a function from $\mathbb{R}$ to E.

The question of asymptotic behavior of (1) is intensively studied by many authors during the last decades, see e.g. Alsulami [1], Arendt and Batty [2], Daleckii and Krein [4] and Vu and Schuler [10].

Of concern to us is the question of almost periodicity of solutions u, provided f is an almost periodic function. Let us recall the definition of *almost-periodic functions*. A set $\Lambda \subset \mathbb{R}$ is said to be *relatively dense* in $\mathbb{R}$ if there exists a number $\ell > 0$ such that the intersection $(t, t + \ell) \cap \Lambda \neq \emptyset$.

Received: June 17, 2008

Definition 1. A continuous function $f(t) : \mathbb{R} \to E$ is called *almost periodic* if for every $\varepsilon > 0$ there exists a relative dense set $\Lambda \subset \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| < \varepsilon, \forall \tau \in \Lambda.$$

We will denote the space of E-valued almost periodic functions on $\mathbb{R}$ by $AP(\mathbb{R}, E)$.

Note that H. Bohr was the first one, in 1925, who used the relatively dense sets to define the complex valued almost periodic functions defined on the whole line.

In 1926, S. Bochner established his first equivalent definition for the complex valued almost periodic functions defined on the whole line.

Definition 2. The function $f(t), t \in \mathbb{R}$, is almost periodic if and only if any sequence $\{\lambda_n^{'}\} \subset \mathbb{R}$ contains a subsequence $\{\lambda_n\}$ for which

$$\lim_{n\to\infty} f(\lambda_n + t) = g(t)$$

exists uniformly in $\mathbb{R}$.

For convenience, let us denote a sequence $\{\nu_n\}$ by $\boldsymbol{\nu}$. Correspondingly, if $\lim_{n\to\infty} f(\nu_n + t) = g(t)$ then we write $T_{\boldsymbol{\nu}} f = g$.

In 1962, S. Bochner [3] gave his second equivalent definition of complex valued almost periodic functions defined on the whole line.

Definition 3. $f(t), t \in \mathbb{R}$, is almost periodic if and only if for any two sequences $\boldsymbol{\beta}^{'} \subset \mathbb{R}$ and $\boldsymbol{\nu}^{'} \subset \mathbb{R}$, there exist subsequences $\boldsymbol{\beta}$ of $\boldsymbol{\beta}^{'}$ and $\boldsymbol{\nu}$ of $\boldsymbol{\nu}^{'}$ such that

$$T_{\boldsymbol{\beta}+\boldsymbol{\nu}} f = T_{\boldsymbol{\beta}} T_{\boldsymbol{\nu}} f\,. \tag{2}$$

Note that equation (2) is equivalent to

$$\lim_{n\to\infty} f(t + \beta_n + \nu_n) = \lim_{n\to\infty} \lim_{m\to\infty} f(t + \beta_n + \nu_m),\ t \in \mathbb{R}\,.$$

There are some other equivalent definitions of the almost periodicity by Markoff [6] and Seifert [7]. We refer the reader to an interesting paper of Fink [5], in which equivalent definitions of almost periodicity and their analysis are presented.

Next, we give a definition of almost automorphic functions.

Definition 4. A function $f(t)$, $t \in \mathbb{R}$, is called almost automorphic if for any sequence $\boldsymbol{\beta}^{'} \subset \mathbb{R}$, there exists a subsequence $\boldsymbol{\beta}$ such that

$$T_{-\boldsymbol{\beta}} T_{\boldsymbol{\beta}} f = f\,,$$

where if $\boldsymbol{\beta} = \{\beta_n\}$, then $-\boldsymbol{\beta} = \{-\beta_n\}$.

Note that Bochner's original consideration of scalar functions was subsequently extended to vector-valued functions, with applications to differential equations in mind, by G. Sell [8], Sibuya [9] and Zaidman [11] and others. We will recall some known properties of almost periodic functions:

Proposition 5. *(a) If $f(t) \in AP(\mathbb{R}, E)$, then $f(t) \in BUC(\mathbb{R}, E)$ (the space of all bounded uniformly continuous functions from $\mathbb{R}$ to E).*

(b) Let $f(t) \in AP(\mathbb{R}, E)$ and assume that g is continuous function from the range of f in E into a Banach space Y. Then, the composed function $g \circ f := g(f(t))$, $t \in \mathbb{R}$, is almost periodic from $\mathbb{R}$ into Y.

(c) Let $\{f_n(t)\}_{n=1}^{\infty}$ be a sequence of almost periodic functions from $\mathbb{R}$ into E such that $\lim_{n\to\infty} f_n(t) = f(t)$ uniformly on $\mathbb{R}$. Then, $f(t)$ is an almost periodic function.

For the proof and related facts, we refer the reader to Zaidman [11, Chapter 9].

Letting $A = 0$ in (1) and denoting $\frac{d}{dt}$ by L, G. Sell [8] noted that if u is differentiable and bounded, by choosing a subsequence if necessary, $Lu = f$ implies $T_{\boldsymbol{\nu}}(Lu) = LT_{\boldsymbol{\nu}}u = T_{\boldsymbol{\nu}}f$. Thus, by the continuity of the operator A and the above observation, we conclude that if $u \in BUC(\mathbb{R}, E)$, then $LT_{\boldsymbol{\nu}}u = T_{\boldsymbol{\nu}}(Lu)$ where $L = \frac{d}{dt} - A$.

2. The Main Theorem

Theorem 6. *Let $f(t) : \mathbb{R} \to E$ be an almost periodic (resp. almost automorphic) and A be a bounded operator. Assume that the differential equation*

$$u'(t) = Au(t) + f(t)$$

has a unique bounded solution u. Then, u is almost periodic (resp. almost automorphic).

Proof. We give the proof for the case of almost periodicity. The proof for almost automorphicity is analogous and therefore is omitted.

Let $\boldsymbol{\beta}'$ and $\boldsymbol{\nu}'$ be arbitrary sequences in $\mathbb{R}$. Since f is almost periodic, there exist subsequences $\boldsymbol{\beta} \subset \boldsymbol{\beta}'$ and $\boldsymbol{\nu} \subset \boldsymbol{\nu}'$ such that the limits $T_{\boldsymbol{\beta}+\boldsymbol{\nu}}f$ and $T_{\boldsymbol{\beta}}T_{\boldsymbol{\nu}}f$ are exist.

Furthermore, $T_{\boldsymbol{\beta}+\boldsymbol{\nu}}u$ is the unique bounded solution of

$$v'(t) = Av(t) + T_{\boldsymbol{\beta}+\boldsymbol{\nu}}f(t) \tag{3}$$

and $T_{\boldsymbol{\beta}}T_{\boldsymbol{\nu}}u$ is the unique bounded solution of

$$v'(t) = Av(t) + T_{\boldsymbol{\beta}}T_{\boldsymbol{\nu}}f(t)\,. \tag{4}$$

Since f is almost periodic, then (3) = (4). Thus, $T_{\boldsymbol{\beta}+\boldsymbol{\nu}}u = T_{\boldsymbol{\beta}}T_{\boldsymbol{\nu}}u$. Therefore, u is almost periodic. □

Acknowledgements

I would like to thank Professor Qouc-Phong Vu for years of guidance and friendship.

References

[1] S.M. Alsulami, *On Evolution Equations in Banach Spaces and Commuting Semigroups*, Ph.D. Dissertation, Ohio University, June (2005).

[2] W. Arendt, C.J.K. Batty, Almost periodic solutions of first- and second-order Cauchy problems, *J. Differential Equations*, **137**, No. 2 (1997), 363-383.

[3] S. Bochner, A new approach to almost periodicity, *Proc. of the National Academy of Sciences of USA*, **48**, No. 12 (1962), 2039-2043.

[4] J. Daleckii, M.G. Krein, *Stability of Solutions of Differential Equations on Banach Spaces*, Amer. Math. Soc., Providence, RI (1974).

[5] A.M. Fink, Almost periodic functions invented for specific purposes, *SIAM Review*, **14**, No. 4 (October 1972), 572-581.

[6] A. Markoff, Stabilitat in Liapounoffschen sinne und Fastperiodizitat, *Math. Z.*, **36** (1933), 708-738.

[7] G. Seifert, A condition for almost periodicity with some applications to functional-differential equations, *J. Differential Equations*, **1** (1965), 393-408.

[8] G.R. Sell, Almost periodic solutions of linear partial differential equations, *J. Math. Anal. Appl.*, **42** (1973), 302-312.

[9] Y. Sibuya, Almost periodic solutions of Poisson's equation, *Proc. Amer. Math. Soc.*, **28** (1971), 195-198.

[10] Quoc-Phong Vu, E. Schuler, The operator equation $AX-XB=C$, admissibility, and asymptotic behavior of differential equations, *J. of Differential Equations*, **145** (1998), 394-419.

[11] S. Zaidman, *Almost Periodic Functions in Abstract Spaces* (1985).