BLOW-UP PHENOMENA FOR
SOME NONLINEAR PARABOLIC SYSTEMS

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Abstract: A lower bound on the blow-up time for the solution of a system of parabolic equations coupled through their nonlinearities is determined by a first order differential inequality technique. A criterion which ensures that blow-up does not occur and a criterion which ensures that blow-up does occur are also given.

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1. Introduction

In a series of papers the authors have studied the phenomena of blow-up of solutions in initial-boundary value problems for nonlinear parabolic equations (see [3], [4], [5], [6]). The results which were obtained there included the determination of lower bounds for the blow-up time when blow-up of the solution does occur as well as the determination of sufficient conditions which imply that blow-up does occur or does not occur. In this note we extend such results to nonlinear systems of parabolic equations under homogeneous Dirichlet boundary conditions.

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We note that there is a vast literature dealing with blow-up of solutions to parabolic and hyperbolic partial differential equations. One can find an extensive list of references in [1], [2], [8], and [9]. The authors considered the existence or nonexistence of global solutions, blow-up of the solutions in finite time, bounds on the blow-up rate or on the blow-up time, the structure of the blow-up set, and the asymptotic behavior of solutions at blow-up. A variety of physical, chemical, and biological applications in which blow-up occurs are also discussed in [1] and [9]. In a recent paper [7], the authors determined a lower bound for blow-up time for the temperature dependent Navier-Stokes equations.

In Section 2 we consider a system of parabolic equations which are coupled in their nonlinear terms and determine a lower bound on the blow-up time assuming the vector solution becomes unbounded at some finite time. Many upper bounds for blow-up time have appeared in the literature; however, lower bounds for blow-up time are more important due to the explosive nature of the solution in applications. A criterion which ensures that blow-up does not occur is presented in Section 3. In Section 4 we impose alternative conditions on the nonlinearities in the system and derive a constraint on the initial data which implies that blow-up does occur as well as indicate the procedure for determining a lower bound on the blow-up time.

2. Lower Bound on Blow-Up Time

Let $\Omega$ be a bounded domain in Euclidean 3-space with smooth boundary $\partial \Omega$. We consider the following initial-boundary value problem for the nonlinear system of $n$ parabolic equations which are coupled through the nonlinear terms:

$$
\frac{\partial u_\alpha}{\partial t} = \Delta u_\alpha + f_\alpha(u) \quad \text{in} \quad \Omega \times (0, t^*),
$$

$$
u_\alpha(x, t) = 0 \quad \text{on} \quad \partial \Omega \times (0, t^*),
$$

$$
u_\alpha(x, 0) = g_\alpha(x) \quad \text{in} \quad \Omega,
$$

where $x = (x_1, x_2, x_3)$ and $u = (u_1, \cdots, u_n)$. We shall use comma $i$ notation to denote partial differentiation with respect to $x_i$, $i = 1, 2, 3$, and the summation convention on repeated indices, where the Greek indices sum over 1, 2, ..., $n$. We impose the following constraint on the nonlinear terms in (2.1)

$$
u_\alpha f_\alpha(u) \leq K(u_\beta u_\beta)^{p+1},
$$

(2.2)
for positive constants $K$ and $p > 1$, and assume the existence of solutions $u_\alpha$, $\alpha = 1, 2, \ldots, n$, of the system (2.1), one or more of which becomes unbounded in finite time $t^*$. Our aim is to determine a lower bound on the blow-up time $t^*$.

We begin by defining the auxiliary function

$$\varphi(t) = \int_\Omega (u_\alpha u_\alpha)^{2p} \, dx$$

and computing

$$\varphi'(t) = 2p \int_\Omega (u_\alpha u_\alpha)^{2p-1} u_\beta u_\beta, i \, dx$$

$$= 4p \int_\Omega (u_\alpha u_\alpha)^{2p-1} u_\beta [\Delta u_\beta + f_\beta(u)] \, dx$$

$$= -8p(2p - 1) \int_\Omega (u_\alpha u_\alpha)^{2p-2} u_\gamma u_\gamma, i u_\beta u_\beta, i \, dx$$

$$-4p \int_\Omega (u_\alpha u_\alpha)^{2p-1} u_\beta, i u_\beta, i \, dx + 4p \int_\Omega (u_\alpha u_\alpha)^{2p-1} u_\beta f_\beta(u) \, dx$$

on integration by parts. By (2.2) and the inequality $u_\alpha u_\alpha u_\beta, i u_\beta, i \geq u_\alpha u_\alpha, i u_\beta u_\beta, i$, it follows that

$$\varphi'(t) \leq -4p^2 \left(4 - \frac{1}{p}\right) \int_\Omega (u_\alpha u_\alpha)^{2p-2} u_\gamma u_\gamma, i u_\beta u_\beta, i \, dx + 4pK \int_\Omega (u_\alpha u_\alpha)^{3p} \, dx.$$ 

Since

$$[(u_\alpha u_\alpha)^p], i [(u_\beta u_\beta)^p], i = 4p^2 (u_\alpha u_\alpha)^{2p-2} u_\gamma u_\gamma, i u_\beta u_\beta, i,$$

we have

$$\varphi'(t) \leq - \left(4 - \frac{1}{p}\right) \int_\Omega [(u_\alpha u_\alpha)^p], i [(u_\beta u_\beta)^p], i \, dx + 4pK \int_\Omega (u_\alpha u_\alpha)^{3p} \, dx.$$ \hfill (2.4)

For simplicity, we now let

$$v = (u_\alpha u_\alpha)^p$$

and use Schwarz’s inequality and the Sobolev inequality [10] to compute

$$\int_\Omega v^3 \, dx \leq \left(\int_\Omega v^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega v^4 \, dx\right)^{\frac{1}{4}}$$

$$\leq \left(\int_\Omega v^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega v^6 \, dx\right)^{\frac{1}{4}}$$ \hfill (2.6)
\[ \leq C \left( \int_{\Omega} v^2 dx \right)^{\frac{3}{4}} \left( \int_{\Omega} v_i v_j dx \right)^{\frac{3}{4}}, \]

where \( C = 2\pi^{-1} 3^{-\frac{3}{4}} \) since we have used \( q = 6, \ p = 2, \ m = 3 \) in [10]. We note that the restriction \( \Omega \subset \mathbb{R}^3 \) arises from the use of the Sobolev inequality here. We can now replace (2.4) by

\[ \varphi'(t) \leq -\left( 4 - \frac{1}{p} \right) \int_{\Omega} v_i v_j dx + 4pCK \left( \int_{\Omega} v^2 dx \right)^{\frac{3}{4}} \left( \int_{\Omega} v_i v_j dx \right)^{\frac{3}{4}}. \quad (2.7) \]

In order to determine a lower a bound on the blow-up time, we rewrite (2.7) as

\[ \varphi'(t) \leq -\left( 4 - \frac{1}{p} - \frac{3}{4\tau} \right) \int_{\Omega} v_i v_j dx + \frac{1}{4\tau^3} \left( 4pCK \right)^4 \left( \int_{\Omega} v^2 dx \right)^3, \]

where \( \tau \) is a positive constant to be determined, and then use the basic inequality

\[ a^r b^s \leq ra + sb, \quad r + s = 1, \quad a, b > 0, \quad (2.8) \]

on the latter term. It follows that

\[ \varphi'(t) \leq -\left( 4 - \frac{1}{p} - \frac{3}{4\tau} \right) \int_{\Omega} v_i v_j dx + \frac{1}{4\tau^3} \left( 4pCK \right)^4 \left( \int_{\Omega} v^2 dx \right)^3, \]

and on choosing

\[ \tau = \frac{4}{3} \left( 4 - \frac{1}{p} \right), \]

we obtain

\[ \varphi'(t) \leq C_1 \varphi(t)^3, \quad C_1 = \frac{1}{4\tau^3} \left( 4pCK \right)^4. \quad (2.9) \]

An integration of (2.9) from 0 to \( t \) results in

\[ \frac{1}{\varphi(0)^2} - \frac{1}{\varphi(t)^2} \leq 2C_1 t \]

and implies that if the solution blows up at time \( t^* \), then

\[ t^* \geq \frac{1}{2C_1 [\varphi(0)]^2}, \quad \varphi(0) = \int_{\Omega} (g_\alpha g_\alpha)^{2p} dx. \quad (2.10) \]

We summarize this result in the following theorem.

**Theorem 1.** If the vector solution \( u \) of (2.1), (2.2) becomes unbounded in the norm \( \varphi \) at some finite \( t^* \), then \( t^* \) is bounded below by (2.10) where \( C_1 \) is given in (2.9).

We note that this result can be extended to the cases of parabolic inequali-
ities and/or more general second order parabolic operators with appropriate changes to the argument above.

We remark that it is possible to obtain a result analogous to (2.10) when a homogeneous Neumann condition is prescribed in (2.1) rather than the Dirichlet boundary condition. In this case, Talenti’s Sobolev inequality [10] which was used in (2.6) is no longer applicable and (2.6) is replaced by an integral inequality derived in [4] - see (2.16) in [4] - namely,

\[
\int_{\Omega} v^3 \, dx \leq \frac{1}{3^2} \left\{ \frac{3}{2p_0} \int_{\Omega} v^2 \, dx + \left( \frac{d}{p_0} + 1 \right) \left( \int_{\Omega} v^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} v_i v_i \, dx \right)^{\frac{1}{2}} \right\}^\frac{3}{2}, \tag{2.11}
\]

where \( \Omega \) is now assumed to be convex, and for some origin inside \( \Omega \)

\[
p_0 = \min_{\partial \Omega} x_i \nu_i > 0, \quad d^2 = \max_{\Omega} x_i x_i,
\]

for \( \nu_i \) the \( i \)-th component of the unit outer normal vector on the boundary. We then use the inequality

\[(a + b)^\frac{3}{2} \leq 2^{\frac{1}{2}} (a^\frac{3}{2} + b^\frac{3}{2})\]

on the brace in (2.11) and substitute into (2.4) to obtain

\[
\varphi'(t) \leq - \left( 4 - \frac{1}{p} \right) \int_{\Omega} v_i v_i \, dx + \frac{4pK_2^\frac{2}{3}}{3^4} \frac{1}{d} \left( \int_{\Omega} v^2 \, dx \right)^{\frac{2}{3}} \left( \int_{\Omega} v_i v_i \, dx \right)^{\frac{1}{3}} \left( \int_{\Omega} v_i v_i \, dx \right)^{\frac{1}{3}} \} \right. \tag{2.12}
\]

Using (2.8) on the second term of the brace in (2.12) as done previously with an undetermined weight factor to be chosen appropriately, we have

\[
\varphi'(t) \leq K_1 [\varphi(t)]^\frac{2}{3} + K_2 [\varphi(t)]^3 \tag{2.13}
\]

for computable constants \( K_1 \) and \( K_2 \). An integration of (2.13) them leads to a lower bound on blow-up time \( t^* \) of the from

\[
t^* \geq \int_0^\infty \frac{d\eta}{\varphi(0) K_1 \eta^2 + K_2 \eta^3}.
\]
3. Criterion for Global Existence

We again consider problem (2.1) under the condition (2.2) on the nonlinear functions \( f_\alpha \) in the system. In this section we seek sufficient conditions which imply that blow-up cannot occur.

For the function \( \varphi \) defined by (2.3) and \( v \) as in (2.5), we can rewrite (2.7) as

\[
\varphi'(t) \leq \left( \int_\Omega v_i v_i dx \right)^{\frac{3}{4}} \left\{ - \left( 4 - \frac{1}{p} \right) \left( \int_\Omega v_i v_i dx \right)^{\frac{1}{4}} + 4pCK \left( \int_\Omega v^2 dx \right)^{\frac{3}{4}} \right\}. \tag{3.1}
\]

By the membrane inequality

\[
\lambda \int_\Omega v^2 dx \leq \int_\Omega v_i v_i dx, \tag{3.2}
\]

where \( \lambda \) is the first positive eigenvalue of the fixed membrane problem

\[
\Delta w + \lambda w = 0, \quad w > 0, \quad \text{in } \Omega, \quad w = 0 \text{ on } \partial \Omega,
\]

we have

\[
\varphi'(t) \leq \left( \int_\Omega v_i v_i dx \right)^{\frac{3}{4}} \left( \int_\Omega v^2 dx \right)^{\frac{1}{4}} \left\{ - \left( 4 - \frac{1}{p} \right) \lambda^{\frac{3}{4}} + 4pCK \left( \int_\Omega v^2 dx \right)^{\frac{1}{2}} \right\}. \tag{3.3}
\]

Consequently, if

\[
\int_\Omega v^2 dx \leq \left[ \frac{\left( 4 - \frac{1}{p} \right) \lambda^{\frac{1}{4}}}{4pCK} \right]^2, \tag{3.4}
\]

for all \( t > 0 \), then \( \varphi'(t) \leq 0 \) for all \( t > 0 \), and \( \varphi(t) \) remains bounded for all time. Hence, if initially

\[
\int_\Omega (g_\alpha g_\alpha)^{2p} dx \leq \left[ \frac{\left( 4 - \frac{1}{p} \right) \lambda^{\frac{1}{4}}}{4pCK} \right]^2, \tag{3.5}
\]

it follows that the vector solution \( u \) of (2.1) is bounded in \( \varphi \) norm for all time.

We formulate this result in the following theorem.

**Theorem 2.** If the initial data in the problem (2.1), (2.2) satisfies condition (3.5), then the vector solution \( u \), assumed to exist, cannot blow up in finite time.
One can, in fact, show that if (3.4) holds, then \( \phi(t) \) decays exponentially in time. To see this, we use (3.2) in (3.3) and write

\[
\phi'(t) \leq \lambda^\frac{3}{4} \left( \int_\Omega v^2 dx \right) \left\{ 4 - \left( \frac{1}{p} \right) \lambda^\frac{1}{4} + 4pCK \left( \int_\Omega v^2 dx \right)^\frac{1}{2} \right\},
\]

since the expression in the brace is negative. Thus,

\[
\phi'(t) \leq \phi(t) \left\{ - \left( 4 - \left( \frac{1}{p} \right) \lambda + 4pCK\lambda^\frac{3}{4}[\phi(t)]^\frac{1}{2} \right) \leq -\phi(t) \left\{ K_1 - K_2[\phi(t)]^\frac{1}{2} \right\},
\]

where the positive constants \( K_1 \) and \( K_2 \) have the obvious meaning. Upon the substitution \( \eta = \phi^\frac{1}{2} \) and integration of the separable differential inequality by partial fractions, one then obtains an inequality of the form

\[
\phi(t) \leq K_3e^{-K_4t}
\]

for computable constant \( K_3 \) and \( K_4 \).

### 4. Criterion for Blow-Up

In this section we consider the system (2.1) where the nonlinearities are of the special form

\[
f_\alpha(u) = u_\alpha F(u_\beta u_\beta)
\]

for some function \( F \) which satisfies

\[
u_\alpha u_\alpha F(u_\beta u_\beta) \geq \delta \int_0^{u_\alpha u_\alpha} F(\eta) d\eta
\]

for a constant \( \delta > 1 \). We seek conditions which imply that blow-up of the solution does occur and obtain an upper bound on the blow-up time.

We begin by defining

\[
\Phi(t) = \int_\Omega u_\alpha u_\alpha dx
\]

and computing

\[
\Phi'(t) = 2 \int_\Omega u_\alpha [\Delta u_\alpha + u_\alpha F(u_\beta u_\beta)] dx
\]

\[
= -2 \int_\Omega u_{\alpha,i} u_{\alpha,i} dx + 2 \int_\Omega u_\alpha u_\alpha F(u_\beta u_\beta) dx
\]

\[
\geq -2\delta \int_\Omega u_{\alpha,i} u_{\alpha,i} dx + 2\delta \int_\Omega \left[ \int_0^{u_\beta u_\beta} F(\eta) d\eta \right] dx.
\]
We now define
\[
\Psi(t) = -2\delta \int_\Omega u_{\alpha,i} u_{\alpha,i} dx + 2\delta \int_\Omega \left[ \int_0^{u_\beta} F(\eta) d\eta \right] dx
\] (4.4)
and compute
\[
\Psi'(t) = -4\delta \int_\Omega u_{\alpha,i} u_{\alpha,i} dx + 4\delta \int_\Omega F(u_\beta u_\beta) u_{\alpha} u_{\alpha,t} dx
\]
\[
= 4\delta \int_\Omega u_{\alpha,t} \Delta u_{\alpha} dx + 4\delta \int_\Omega u_{\alpha,t} u_{\alpha} u_{\alpha,t} F(u_\beta u_\beta) dx
\]
\[
= 4\delta \int_\Omega u_{\alpha,t} u_{\alpha,t} dx \geq 0
\]
for all \( t \geq 0 \). Thus, if \( \Psi(0) \geq 0 \), i.e., if
\[
\int_\Omega g_{\alpha,i} g_{\alpha,i} dx \leq \int_\Omega \left[ \int_0^{g_{\alpha,g_{\alpha}}} F(\eta) d\eta \right] dx,
\] (4.5)
then \( \Psi(t) \geq 0 \) for all \( t \geq 0 \).

We now relate the derivatives of \( \Phi \) and \( \Psi \) by means of
\[
[\Phi'(t)]^2 = \left( 2 \int_\Omega u_{\alpha} u_{\alpha,t} dx \right)^2 \leq 4 \int_\Omega u_{\alpha} u_{\alpha} dx \int_\Omega u_{\alpha,t} u_{\alpha,t} dx
\]
\[
= \frac{1}{\delta} \Phi(t) \Psi'(t).
\]
Since \( \Phi'(t) \geq \Psi(t) \), it follows that
\[
\delta \Phi'(t) \Psi(t) \leq \Phi(t) \Psi'(t)
\]
or that
\[
\frac{\delta \Phi'(t)}{\Phi(t)} \leq \frac{\Psi'(t)}{\Psi(t)}.
\] (4.6)

An integration of (4.6) results in
\[
\frac{\Psi(t)}{[\Phi(t)]^\delta} \geq \frac{\Psi(0)}{[\Phi(0)]^\delta} = M,
\] (4.7)
but since \( \Phi'(t) \geq \Psi(t) \), we can write
\[
\Phi'(t) \geq M[\Phi(t)]^\delta.
\] (4.8)

We now integrate (4.8) and obtain
\[
\frac{1}{[\Phi(t)]^{\delta-1}} \leq \frac{1}{[\Phi(0)]^{\delta-1}} - M(\delta - 1)t,
\]
an inequality which cannot hold for all time \( t \). We conclude that the solution
u blows up at some finite time $t^*$ and that

$$t^* \leq \frac{1}{M(\delta - 1)[\Phi(0)]^{\delta - 1}}$$

(4.9)

when the initial data satisfies (4.5).

We formalize our conclusion in the following theorem.

**Theorem 3.** If $u$ is a vector solution of the system (2.1), where the $f_\alpha$ are defined by (4.1) and the initial data satisfies (4.5), then the solution blows up in $\Phi$ norm at some finite time $t^*$ and an upper bound for $t^*$ is given by (4.9).

We note that an integration of (4.8) from $t$ to $t^*$ results in the decay bound

$$\Phi(t) \leq \left[ \frac{1}{M(\delta - 1)(t^* - t)} \right] \frac{1}{\delta - 1}$$

as $t \to t^*$. Further, we remark that if $2u_\alpha u_\alpha F(u_\beta u_\beta)$ satisfies the condition in [3] then one can determine a lower bound for blow-up time when blow-up does occur directly from Theorem 2.1 in [3]. This follows since the differential equation in (2.1) with nonlinearities of the form (4.1) can be written as

$$2u_\alpha \frac{\partial u_\alpha}{\partial t} = 2u_\alpha \Delta u_\alpha + 2u_\alpha u_\alpha F(u_\beta u_\beta)$$

$$= \Delta(u_\alpha u_\alpha) - 2u_{\alpha,i} u_{\alpha,i} + 2u_\alpha u_\alpha F(u_\beta u_\beta)$$

and leads to

$$\frac{\partial v}{\partial t} \leq \Delta v + 2vF(v)$$

for $v = u_\alpha u_\alpha$.

**References**


