International Journal of Pure and Applied Mathematics
Volume 48 No. 2 2008, 43-52

BLOW-UP PHENOMENA FOR SOME NONLINEAR PARABOLIC SYSTEMS

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Abstract: A lower bound on the blow-up time for the solution of a system of parabolic equations coupled through their nonlinearities is determined by a first order differential inequality technique. A criterion which ensures that blow-up does not occur and a criterion which ensures that blow-up does occur are also given.

AMS Subject Classification: 35K50, 35K60
Key Words: parabolic systems, nonlinear problems, blow-up solutions

1. Introduction

In a series of papers the authors have studied the phenomena of blow-up of solutions in initial-boundary value problems for nonlinear parabolic equations (see [3], [4], [5], [6]). The results which were obtained there included the determination of lower bounds for the blow-up time when blow-up of the solution does occur as well as the determination of sufficient conditions which imply that blow-up does occur or does not occur. In this note we extend such results to nonlinear systems of parabolic equations under homogeneous Dirichlet boundary conditions.

Received: June 19, 2008

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We note that there is a vast literature dealing with blow-up of solutions to parabolic and hyperbolic partial differential equations. One can find an extensive list of references in [1], [2], [8], and [9]. The authors considered the existence or nonexistence of global solutions, blow-up of the solutions in finite time, bounds on the blow-up rate or on the blow-up time, the structure of the blow-up set, and the asymptotic behavior of solutions at blow-up. A variety of physical, chemical, and biological applications in which blow-up occurs are also discussed in [1] and [9]. In a recent paper [7], the authors determined a lower bound for blow-up time for the temperature dependent Navier-Stokes equations.

In Section 2 we consider a system of parabolic equations which are coupled in their nonlinear terms and determine a lower bound on the blow-up time assuming the vector solution becomes unbounded at some finite time. Many upper bounds for blow-up time have appeared in the literature; however, lower bounds for blow-up time are more important due to the explosive nature of the solution in applications. A criterion which ensures that blow-up does not occur is presented in Section 3. In Section 4 we impose alternative conditions on the nonlinearities in the system and derive a constraint on the initial data which implies that blow-up does occur as well as indicate the procedure for determining a lower bound on the blow-up time.

2. Lower Bound on Blow-Up Time

Let Ω be a bounded domain in Euclidean 3-space with smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem for the nonlinear system of n parabolic equations which are coupled through the nonlinear terms:

$$\begin{aligned} \frac{\partial u_\alpha}{\partial t} &= \Delta u_\alpha + f_\alpha(u) \quad &&\text{in} \quad \Omega \times (0, t^*), \\ u_\alpha(x,t) &= 0 \quad &&\text{on} \quad \partial\Omega \times (0, t^*), \\ u_\alpha(x,0) &= g_\alpha(x) \quad &&\text{in} \quad \Omega, \end{aligned} \tag{2.1}$$

where $x = (x_1, x_2, x_3)$ and $u = (u_1, \cdots, u_n)$. We shall use comma i notation to denote partial differentiation with respect to x_i, $i = 1, 2, 3$, and the summation convention on repeated indices, where the Greek indices sum over $1, 2, ..., n$. We impose the following constraint on the nonlinear terms in (2.1)

$$u_\alpha f_\alpha(u) \leq K(u_\beta u_\beta)^{p+1}, \tag{2.2}$$

for positive constants K and $p > 1$, and assume the existence of solutions u_α, $\alpha = 1, 2, ..., n$, of the system (2.1), one or more of which becomes unbounded in finite time t^*. Our aim is to determine a lower bound on the blow-up time t^*.

We begin by defining the auxiliary function

$$\varphi(t) = \int_\Omega (u_\alpha u_\alpha)^{2p} dx \tag{2.3}$$

and computing

$$\begin{aligned} \varphi'(t) &= 2p \int_\Omega (u_\alpha u_\alpha)^{2p-1} 2u_\beta u_{\beta,t} dx \\ &= 4p \int_\Omega (u_\alpha u_\alpha)^{2p-1} u_\beta [\Delta u_\beta + f_\beta(u)] dx \\ &= -8p(2p-1) \int_\Omega (u_\alpha u_\alpha)^{2p-2} u_\gamma u_{\gamma,i} u_\beta u_{\beta,i} dx \\ &\quad -4p \int_\Omega (u_\alpha u_\alpha)^{2p-1} u_{\beta,i} u_{\beta,i} dx + 4p \int_\Omega (u_\alpha u_\alpha)^{2p-1} u_\beta f_\beta(u) dx \end{aligned}$$

on integration by parts. By (2.2) and the inequality

$$u_\alpha u_\alpha u_{\beta,i} u_{\beta,i} \geq u_\alpha u_{\alpha,i} u_\beta u_{\beta,i},$$

it follows that

$$\varphi'(t) \leq -4p^2 \left(4 - \frac{1}{p}\right) \int_\Omega (u_\alpha u_\alpha)^{2p-2} u_\gamma u_{\gamma,i} u_\beta u_{\beta,i} dx + 4pK \int_\Omega (u_\alpha u_\alpha)^{3p} dx.$$

Since

$$[(u_\alpha u_\alpha)^p]_{,i} [(u_\beta u_\beta)^p]_{,i} = 4p^2 (u_\alpha u_\alpha)^{2p-2} u_\gamma u_{\gamma,i} u_\beta u_{\beta,i},$$

we have

$$\varphi'(t) \leq -\left(4 - \frac{1}{p}\right) \int_\Omega [(u_\alpha u_\alpha)^p]_{,i} [(u_\beta u_\beta)^p]_{,i} dx + 4pK \int_\Omega (u_\alpha u_\alpha)^{3p} dx. \tag{2.4}$$

For simplicity, we now let

$$v = (u_\alpha u_\alpha)^p \tag{2.5}$$

and use Schwarz's inequality and the Sobolev inequality [10] to compute

$$\begin{aligned} \int_\Omega v^3 dx &\leq \left(\int_\Omega v^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega v^4 dx\right)^{\frac{1}{2}} \\ &\leq \left(\int_\Omega v^2 dx\right)^{\frac{3}{4}} \left(\int_\Omega v^6 dx\right)^{\frac{1}{4}} \end{aligned} \tag{2.6}$$

$$\leq \; C\left(\int_\Omega v^2 dx\right)^{\frac{3}{4}}\left(\int_\Omega v_{,i}v_{,i}dx\right)^{\frac{3}{4}},$$

where $C = 2\pi^{-1}3^{-\frac{3}{4}}$ since we have used $q = 6$, $p = 2$, $m = 3$ in [10]. We note that the restriction $\Omega \subset \mathbb{R}^3$ arises from the use of the Sobolev inequality here. We can now replace (2.4) by

$$\varphi'(t) \leq -\left(4 - \frac{1}{p}\right)\int_\Omega v_{,i}v_{,i}dx + 4pCK\left(\int_\Omega v^2 dx\right)^{\frac{3}{4}}\left(\int_\Omega v_{,i}v_{,i}dx\right)^{\frac{3}{4}}. \tag{2.7}$$

In order to determine a lower a bound on the blow-up time, we rewrite (2.7) as

$$\varphi'(t) \leq -\left(4 - \frac{1}{p}\right)\int_\Omega v_{,i}v_{,i}dx + \left[\frac{(4pCK)^4}{\tau^3}\left(\int_\Omega v^2 dx\right)^3\right]^{\frac{1}{4}}\left[\tau\int_\Omega v_{,i}v_{,i}dx\right]^{\frac{3}{4}},$$

where τ is a positive constant to be determined, and them use the basic inequality

$$a^r b^s \leq ra + sb, \qquad r + s = 1, \qquad a, b > 0\,, \tag{2.8}$$

on the latter term. It follows that

$$\varphi'(t) \leq -\left(4 - \frac{1}{p} - \frac{3}{4}\tau\right)\int_\Omega v_{,i}v_{,i}dx + \frac{1}{4\tau^3}(4pCK)^4\left(\int_\Omega v^2 dx\right)^3,$$

and on choosing

$$\tau = \frac{4}{3}\left(4 - \frac{1}{p}\right),$$

we obtain

$$\varphi'(t) \leq C_1\varphi(t)^3, \qquad C_1 = \frac{1}{4\tau^3}(4pCK)^4. \tag{2.9}$$

An integration of (2.9) from 0 to t results in

$$\frac{1}{\varphi(0)^2} - \frac{1}{\varphi(t)^2} \leq 2C_1 t$$

and implies that if the solution blows up at time t^*, then

$$t^* \geq \frac{1}{2C_1[\varphi(0)]^2}, \qquad \varphi(0) = \int_\Omega (g_\alpha g_\alpha)^{2p} dx. \tag{2.10}$$

We summarize this result in the following theorem.

Theorem 1. *If the vector solution u of* (2.1), (2.2) *becomes unbounded in the norm φ at some finite t^*, then t^* is bounded below by* (2.10) *where C_1 is given in* (2.9).

We note that this result can be extended to the cases of parabolic inequalities and/or more general second order parabolic operators with appropriate changes to the argument above.

We remark that it is possible to obtain a result analogous to (2.10) when a homogeneous Neumann condition is prescribed in (2.1) rather than the Dirichlet boundary condition. In this case, Talenti's Sobolev inequality [10] which was used in (2.6) is no longer applicable and (2.6) is replaced by an integral inequality derived in [4] - see (2.16) in [4] - namely,

$$\int_\Omega v^3 dx \leq \frac{1}{3^{\frac{3}{4}}}\left\{\frac{3}{2p_0}\int_\Omega v^2 dx + \left(\frac{d}{p_0}+1\right)\left(\int_\Omega v^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega v_{,i}v_{,i}dx\right)^{\frac{1}{2}}\right\}^{\frac{3}{2}}, \quad (2.11)$$

where Ω is now assumed to be convex, and for some origin inside Ω

$$p_0 = \min_{\partial\Omega} x_i\nu_i > 0, \qquad d^2 = \max_{\overline{\Omega}} x_i x_i,$$

for ν_i the i-th component of the unit outer normal vector on the boundary. We then use the inequality

$$(a+b)^{\frac{3}{2}} \leq 2^{\frac{1}{2}}(a^{\frac{3}{2}} + b^{\frac{3}{2}})$$

on the brace in (2.11) and substitute into (2.4) to obtain

$$\varphi'(t) \leq -\left(4-\frac{1}{p}\right)\int_\Omega v_{,i}v_{,i}dx + \frac{4pK2^{\frac{1}{2}}}{3^{\frac{3}{4}}} \times\left\{\left(\frac{3}{2p_0}\right)^{\frac{3}{2}}\left(\int_\Omega v^2 dx\right)^{\frac{3}{2}} + \left(\frac{d}{p_0}+1\right)^{\frac{3}{2}}\left(\int_\Omega v^2 dx\right)^{\frac{3}{4}}\left(\int_\Omega v_{,i}v_{,i}dx\right)^{\frac{3}{4}}\right\}. \quad (2.12)$$

Using (2.8) on the second term of the brace in (2.12) as done previously with an undetermined weight factor to be chosen appropriately, we have

$$\varphi'(t) \leq K_1[\varphi(t)]^{\frac{3}{2}} + K_2[\varphi(t)]^3 \quad (2.13)$$

for computable constants K_1 and K_2. An integration of (2.13) them leads to a lower bound on blow-up time t^* of the from

$$t^* \geq \int_{\varphi(0)}^{\infty}\frac{d\eta}{K_1\eta^{\frac{3}{2}} + K_2\eta^3}.$$

3. Criterion for Global Existence

We again consider problem (2.1) under the condition (2.2) on the nonlinear functions f_α in the system. In this section we seek sufficient conditions which imply that blow-up cannot occur.

For the function φ defined by (2.3) and v as in (2.5), we can rewrite (2.7) as

$$\varphi'(t) \leq \left(\int_\Omega v_{,i}v_{,i}dx\right)^{\frac{3}{4}} \left\{-\left(4-\frac{1}{p}\right)\left(\int_\Omega v_{,i}v_{,i}dx\right)^{\frac{1}{4}} + 4pCK\left(\int_\Omega v^2dx\right)^{\frac{3}{4}}\right\}. \quad (3.1)$$

By the membrane inequality

$$\lambda \int_\Omega v^2dx \leq \int_\Omega v_{,i}v_{,i}dx, \quad (3.2)$$

where λ is the first positive eigenvalue of the fixed membrane problem

$$\Delta w + \lambda w = 0, \qquad w > 0, \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega,$$

we have

$$\varphi'(t) \leq \left(\int_\Omega v_{,i}v_{,i}dx\right)^{\frac{3}{4}} \left(\int_\Omega v^2dx\right)^{\frac{1}{4}} \left\{-\left(4-\frac{1}{p}\right)\lambda^{\frac{1}{4}} + 4pCK\left(\int_\Omega v^2dx\right)^{\frac{1}{2}}\right\}. \quad (3.3)$$

Consequently, if

$$\int_\Omega v^2dx \leq \left[\frac{\left(4-\frac{1}{p}\right)\lambda^{\frac{1}{4}}}{4pCK}\right]^2, \quad (3.4)$$

for all $t > 0$, then $\varphi'(t) \leq 0$ for all $t > 0$, and $\varphi(t)$ remains bounded for all time. Hence, if initially

$$\int_\Omega (g_\alpha g_\alpha)^{2p}dx \leq \left[\frac{\left(4-\frac{1}{p}\right)\lambda^{\frac{1}{4}}}{4pCK}\right]^2, \quad (3.5)$$

it follows that the vector solution u of (2.1) is bounded in φ norm for all time.

We formulate this result in the following theorem.

Theorem 2. *If the initial data in the problem* (2.1), (2.2) *satisfies condition* (3.5), *then the vector solution* u, *assumed to exist, cannot blow up in finite time.*

One can, in fact, show that if (3.4) holds, then $\varphi(t)$ decays exponentially in time. To see this, we use (3.2) in (3.3) and write

$$\varphi'(t) \leq \lambda^{\frac{3}{4}} \left(\int_\Omega v^2 dx \right) \left\{ - \left(4 - \frac{1}{p} \right) \lambda^{\frac{1}{4}} + 4pCK \left(\int_\Omega v^2 dx \right)^{\frac{1}{2}} \right\},$$

since the expression in the brace is negative. Thus,

$$\varphi'(t) \leq \varphi(t) \left\{ - \left(4 - \frac{1}{p} \right) \lambda + 4pCK\lambda^{\frac{3}{4}} [\varphi(t)]^{\frac{1}{2}} \right\} \leq -\varphi(t) \left\{ K_1 - K_2 [\varphi(t)]^{\frac{1}{2}} \right\},$$

where the positive constants K_1 and K_2 have the obvious meaning. Upon the substitution $\eta = \varphi^{\frac{1}{2}}$ and integration of the separable differential inequality by partial fractions, one then obtains an inequality of the form

$$\varphi(t) \leq K_3 e^{-K_4 t}$$

for computable constant K_3 and K_4.

4. Criterion for Blow-Up

In this section we consider the system (2.1) where the nonlinearities are of the special form

$$f_\alpha(u) = u_\alpha F(u_\beta u_\beta) \tag{4.1}$$

for some function F which satisfies

$$u_\alpha u_\alpha F(u_\beta u_\beta) \geq \delta \int_0^{u_\alpha u_\alpha} F(\eta) d\eta \tag{4.2}$$

for a constant $\delta > 1$. We seek conditions which imply that blow-up of the solution does occur and obtain an upper bound on the blow-up time.

We begin by defining

$$\Phi(t) = \int_\Omega u_\alpha u_\alpha dx \tag{4.3}$$

and computing

$$\begin{aligned} \Phi'(t) &= 2 \int_\Omega u_\alpha \left[\Delta u_\alpha + u_\alpha F(u_\beta u_\beta) \right] dx \\ &= -2 \int_\Omega u_{\alpha,i} u_{\alpha,i} dx + 2 \int_\Omega u_\alpha u_\alpha F(u_\beta u_\beta) dx \\ &\geq -2\delta \int_\Omega u_{\alpha,i} u_{\alpha,i} dx + 2\delta \int_\Omega \left[\int_0^{u_\beta u_\beta} F(\eta) d\eta \right] dx. \end{aligned}$$

We now define

$$\Psi(t) = -2\delta \int_\Omega u_{\alpha,i}u_{\alpha,i}dx + 2\delta \int_\Omega \left[\int_0^{u_\beta u_\beta} F(\eta)d\eta\right] dx \tag{4.4}$$

and compute

$$\begin{aligned}\Psi'(t) &= -4\delta \int_\Omega u_{\alpha,i}u_{\alpha,it}dx + 4\delta \int_\Omega F(u_\beta u_\beta)u_\alpha u_{\alpha,t}dx \\ &= 4\delta \int_\Omega u_{\alpha,t}\Delta u_\alpha dx + 4\delta \int_\Omega u_{\alpha,t}u_\alpha F(u_\beta u_\beta)dx \\ &= 4\delta \int_\Omega u_{\alpha,t}u_{\alpha,t}dx \geq 0\end{aligned}$$

for all $t \geq 0$. Thus, if $\Psi(0) \geq 0$, i.e., if

$$\int_\Omega g_{\alpha,i}g_{\alpha,i}dx \leq \int_\Omega \left[\int_0^{g_\alpha g_\alpha} F(\eta)d\eta\right] dx, \tag{4.5}$$

then $\Psi(t) \geq 0$ for all $t \geq 0$.

We now relate the derivatives of Φ and Ψ by means of

$$\begin{aligned}[\Phi'(t)]^2 &= \left(2\int_\Omega u_\alpha u_{\alpha,t}dx\right)^2 \\ &\leq 4\int_\Omega u_\alpha u_\alpha dx \int_\Omega u_{\alpha,t}u_{\alpha,t}dx \\ &= \frac{1}{\delta}\Phi(t)\Psi'(t).\end{aligned}$$

Since $\Phi'(t) \geq \Psi(t)$, it follows that

$$\delta\Phi'(t)\Psi(t) \leq \Phi(t)\Psi'(t)$$

or that

$$\delta\frac{\Phi'(t)}{\Phi(t)} \leq \frac{\Psi'(t)}{\Psi(t)}. \tag{4.6}$$

An integration of (4.6) results in

$$\frac{\Psi(t)}{[\Phi(t)]^\delta} \geq \frac{\Psi(0)}{[\Phi(0)]^\delta} = M, \tag{4.7}$$

but since $\Phi'(t) \geq \Psi(t)$, we can write

$$\Phi'(t) \geq M[\Phi(t)]^\delta. \tag{4.8}$$

We now integrate (4.8) and obtain

$$\frac{1}{[\Phi(t)]^{\delta-1}} \leq \frac{1}{[\Phi(0)]^{\delta-1}} - M(\delta-1)t,$$

an inequality which cannot hold for all time t. We conclude that the solution

u blows up at some finite time t^* and that

$$t^* \leq \frac{1}{M(\delta-1)[\Phi(0)]^{\delta-1}} \tag{4.9}$$

when the initial data satisfies (4.5).

We formalize our conclusion in the following theorem.

Theorem 3. *If u is a vector solution of the system* (2.1), *where the f_α are defined by* (4.1) *and the initial data satisfies* (4.5), *then the solution blows up in Φ norm at some finite time t^* and an upper bound for t^* is given by* (4.9).

We note that an integration of (4.8) from t to t^* results in the decay bound

$$\Phi(t) \leq \left[\frac{1}{M(\delta-1)(t^*-t)}\right]^{\frac{1}{\delta-1}}$$

as $t \longrightarrow t^*$. Further, we remark that if $2u_\alpha u_\alpha F(u_\beta u_\beta)$ satisfies the condition in [3] then one can determine a lower bound for blow-up time when blow-up does occur directly from Theorem 2.1 in [3]. This follows since the differential equation in (2.1) with nonlinearities of the form (4.1) can be written as

$$\begin{aligned} 2u_\alpha \frac{\partial u_\alpha}{\partial t} &= 2u_\alpha \Delta u_\alpha + 2u_\alpha u_\alpha F(u_\beta u_\beta) \\ &= \Delta(u_\alpha u_\alpha) - 2u_{\alpha,i}u_{\alpha,i} + 2u_\alpha u_\alpha F(u_\beta u_\beta) \end{aligned}$$

and leads to

$$\frac{\partial v}{\partial t} \leq \Delta v + 2vF(v)$$

for $v = u_\alpha u_\alpha$.

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