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APPLICATIONS OF CONVERGENCE SPACES TO GROUP ACTION

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Abstract: One of the popular settings for group action is the category of groups. However, if we blend in the notion of continuity with the study of action, it is even more appreciated with applications in topological dynamics. A continuous action of a convergence group G on a convergence space X is well known [13]. In this case, X is called a G-space. In this paper, we study the group actions preserved by various convergence structures such as the quotient structure on the coset space. Also, attempts have been made to show that a few modifications of a G-space preserve the continuous group action.

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1. Introduction

Study of group actions first started in the setting of groups in algebra, mainly with the discovery of the natural action of a permutation group on a set. Later it was viewed as a tool for classifying objects up to isomorphism. This initiated the idea of blending in the continuity concept of topology and group action (see [6], [7] and [11]). The result was the study of topological dynamical system [5]. However, it was much restricted to locally compact topological spaces since continuous group actions of a topological group on a locally compact topological space X could be identified with objects in the homeomorphism group $H(X)$. The reason for this was the unavailability of a minimal admissible topological

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group structure on $H(X)$ in the general case.

In the last century, development of the appropriate techniques of convergence theory in set-theoretic topology opened the whole era of possibilities for convergence groups (see [5], [8] and [12]). One of the advantages was that $H(X)$ could have a minimal admissible group convergence structure which could lead to identifying the continuous group actions on X to objects in $H(X)$ [13]. This justified the generalization of continuous group action on a much larger setting, i.e., on the objects of the category CONV of convergence space.

The object of this paper is to suggest the prospect of using the category of CONV of convergence spaces and continuous maps for the study of group actions. In Section 4, we show that sum, product and projective limit of an indexed family of $G-$spaces are $G-$spaces. In addition, any $G-$space is shown to be the convergence sum of the family of orbits of group G in X. In Section 5, the coset space G/H of all right cosets of a subgroup H in G, with the quotient convergence structure is shown to be a $G-$space. In Section 6, we show that continuous action on a convergence space X is preserved by its pretopological and topological modifications under certain special conditions.

2. Preliminaries

For basic definitions and terminologies related to filters the reader is referred to [10] and [13], though a few of the notations, which are frequently used will be mentioned here. For a set X, let $\mathbf{F}(X)$ denote the collection of all filters on X and$\mathbf{P}(X)$ denote all the subsets of X. For each $x \in X$, $\dot{x}$ denotes the fixed ultra filter containing $\{x\}$. If $(X, \cdot)$ is a group, then we can define a binary operation '$\cdot$' on $\mathbf{F}(X)$ as $\mathcal{F} \cdot \mathcal{G} = [\{FG : F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}]$, and $\mathcal{F}^{-1} = [\{F^{-1} : F \in \mathcal{F}\}]$. We will write $\mathcal{F}\mathcal{G}$ for $\mathcal{F} \cdot \mathcal{G}$ throughout this paper. It may be noted that $(\mathbf{F}(X),\cdot)$ is a monoid, not a group in general since $\dot{e} \geq \mathcal{F} \cdot \mathcal{F}^{-1}$, where e is the identity element in the group $(X, \cdot)$.

For a nonempty set X, consider a function $q : \mathbf{F}(X) \to \mathbf{P}(X)$. Conventionally, we write $\mathcal{F} \longrightarrow^q x$, and say that $\mathcal{F}$ 'q-converges to x' whenever $x \in q(\mathcal{F})$. For $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$ and $x \in X$, consider the following conditions:

(c_1) $\dot{x} \to^q x$, $\forall x \in X$.

(c_2) $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F} \to^q x \Longrightarrow \mathcal{G} \to^q x$.

(c_3) $\mathcal{F} \to^q x \Longrightarrow \mathcal{F} \cap \dot{x} \to^q x$.

(c_4) $\mathcal{F} \to^q x, \mathcal{G} \to^q x \Longrightarrow \mathcal{F} \cap \mathcal{G} \to^q x$.

(c_5) If for each ultrafilter $\mathcal{G} \geq \mathcal{F}, \mathcal{G} \rightarrow^q x$, then $\mathcal{F} \rightarrow^q x$.

(c_6) For each $x \in X, \nu_q(x) \rightarrow^q x$, where $\nu_q(x) = \cap\{\mathcal{F} : \mathcal{F} \rightarrow^q x\}$.

(c_7) For each $x \in X$, and for each $V \in \nu_q(x)$, $\exists\, W \in \nu_q(x)$ such that $W \subseteq V$ and $y \in W$ implies $V \in \nu_q(y)$.

The function q is called a *convergence structure* (respectively, *limit*, *pseudotopology*, *pretopolog*, *topology*) if it satisfies $(c_1) - (c_3)$ (respectively, $(c_1) - (c_4)$, $(c_1) - (c_5)$, $(c_1) - (c_6)$, $(c_1) - (c_7)$). It should be noted that all these axioms are not independent. For example, (c_1)and (c_4) imply (c_3) and also (c_2) and (c_6) imply (c_5).The pair (X, q) is called a *convergence space* and it isdenoted by$X$ if there is no ambiguity of the convergence structure q. *Limit space, pseudotopological space, pretopological space and topological space* are defined likewise. Note that limit spaces used to be called convergence spaces by Binz, Park (see [2] and [12]) and several others.

A *closure operator* on a convergence space (X, q) is a mapping $cl_q : \mathbf{P}(X) \rightarrow \mathbf{P}(X)$ defined as

$$cl_q(A) = \{x \in X \mid \exists \mathcal{F} \in \mathbf{F}(X) \text{ such that } \mathcal{F} \rightarrow^q x \text{ and } A \in \mathcal{F}\},$$

for all $A \subseteq X$. A subset $A \subseteq X$ is *closed* if and only if $cl_q(A) = A$. Note that $A \subseteq cl_q A$ and unlike topological spaces, $cl_q A$ is not necessarily a closed subset of the convergence space X. The mapping I_q :$\mathbf{P}(X) {\rightarrow} \mathbf{P}(X)$ defined as

$$I_q(A) = \{x \in A \mid \mathcal{F} \rightarrow^q x \Longrightarrow A \in \mathcal{F}\}$$

is called the *interior operator*. A subset $A \subseteq X$ is *open* if and only if $I_q(A) = A$. Note that $X \backslash cl_q(A) = I_q(X \backslash A)$, so that $A \subseteq X$ is open if and only if $X \backslash A$ is closed. The open subsets on the other hand form a topology on X.

Let q_1 and q_2 be two convergence structures on X. We say q_2 is *finer than* q_1, written $q_2 \geq q_1$ if and only if a filter $\mathcal{F} \rightarrow^{q_2} x$ implies $\mathcal{F} \rightarrow^{q_1} x$. The *pseudotopological modification* γq of a convergence space (X, q) is the finest pseudotopology on X coarser than q. Likewise, we define the *pretopological modification* πq and the *topological modification* τq of the convergence structure q on X. In fact, the set of all open subsets of a convergence space form the finest topolgy on X coarser than q, that is, the topological modification of q. The space $(X, \gamma q)$ (respectively, $(X, \pi q)$ and $(X, \tau q)$) is denoted by γX (respectively, πX and τX). The following lemma explicitly states the characteristics of γq, πq and τq convergence of a filter.

Lemma 2.1. *Let (X, q)be a convergence space and $F \in F(X)$. Then,*

(a) $F \rightarrow^{\gamma q} x$ if and only if $G \rightarrow^q x$ for every ultrafilter G finer than F.

(b) $F \rightarrow^{\pi q} x$ if and only if $F \geq \nu_q(x)$.

(c) $F \to^{\tau q} x$ *if and only if* $F \geq \nu_{\tau q}(x)$, *where* $\nu_{\tau q}(x) = \{V \subseteq X \mid \exists U \subseteq X$ *such that* $I_q(U) = U\}$.

A convergence space (X, q) is said to be:

— T_2 or $Hausdorff$ iff $x = y$, whenever any filter $\mathcal{F} \to^q x, y$.

— *Regular* iff $cl_q(\mathcal{F}) \to^q x$, whenever $\mathcal{F} \to^q x$.

— *Compact* iff each ultrafilter on X converges to at least one point in X.

A subset $A \subseteq X$ is compact iff each ultrafilter containing A converges to a point in A.

A mapping $f : (X, q) \to (Y, p)$ is *continuous* if and only if whenever $\mathcal{F} \to^q x$, $f(\mathcal{F}) \to^p f(x)$. Furthermore, a continuous mapping f is a homeomorphism, if it is bijective and f^{-1} is also continuous. The set $H(X)$ (respectively, $C(X)$) denotes the set of all self homeomorphisms (respectively, continuous functions) on X. It is well-known that a continuous bijective map from a pseudotopological space onto a T_2 space is a homeomorphism.

We denote by $CONV$ (respectively, $PSTOP$, $PRETOP$, TOP) the category of convergence (respectively, pseudotopological, pretopological and topological) spaces and continuous maps.

Lemma 2.2. *Each of PSTOP, PRETOP and TOP is a bireflective subcategory of CONV.*

It is possible to construct new convergence spaces such as product, sum, subspace and quotient space of given convergence spaces. These are special cases of initial and terminal structures. Let $\{(X_i, q_i)\}_{i \in I}$ be any indexed family of convergence spaces and for each $i \in I$, $f_i : X \to X_i$ be a mapping on a set X. The *initial convergence structure* q on X with respect to $\{(X_i, q_i), f_i\}_{i \in I}$ is defined as $\mathcal{F} \to^q x$ in X if and only if $f_i(\mathcal{F}) \to^{q_i} f_i(x)$ in X_i for each $i \in I$. Note that q is the coarsest convergence structure with respect to which each f_i is continuous on X. The subspace, product and projective limit structures are special cases of initial convergence structures. The dual notion of an initial convergence structure is the *terminal structure* on a set with respect to an indexed family of convergence spaces. The sum and quotient structures are special cases of terminal structure. For further details the reader is referred to [3] and [9]. Later in Section 4, we will discuss the sum, product, quotient and subspaces of $G-spaces$.

Lemma 2.3. (see [3]) *Let* (X, q), (Y, s) *be convergence spaces and let* p *denote the product convergence on* $X \times Y$. *Then the following are true for all* $(x, y) \in X \times Y$:

(i) $\nu_p(x,y) \geq \nu_q(x) \times \nu_s(y)$.

(ii) $P_1(\nu_p(x,y)) = \nu_q(x)$ *and* $P_2(\nu_p(x,y)) = \nu_s(y)$, *where* P_1 *and* P_2 *denote the canonical projection of* $X \times Y$ *onto* X *and* Y *respectively.*

Next, we give a brief introduction to convergence groups. Convergence groups are generalisations of topological groups which have many applications in manifolds, Lie groups and global analysis. These form a nice blend of algebraic and geometric structures in a very generalised form, which have been studied in detail in the last half century. For clarity of notations and terminologies for convergence groups the reader is referred to [5], [8]. Precisely, a triplet $(X, q, \cdot)$ is a *convergence group*, if:

(cg_1) (X, q) is a convergence space,

(cg_2) $(X, \cdot)$ is a group,

(cg_3) $\mathcal{F} \to^q x$ and $G \to^q y$ implies that $\mathcal{F}G^{-1} \to^q xy^{-1}$.

Note that the binary operation '$\cdot$' and the inverse operation in the group $(X, .)$ are continuous with respect to the convergence structure q. A convergence group is a limit group (respectively, *pseudotopological group*, *pretopological group*, *topological group*) if and only if (X, q) is a limit space (respectively, pseudotopological space, pretopological space, topological space). For any $x \in X$, let $q(x) = \{\mathcal{F} \in \mathbf{F}(X) \,|\mathcal{F} \to^q x\}$. The following is true for any convergence group.

Lemma 2.4. (see [13]) *Let* $x \neq y$ *be two distinct elements in the convergence group* $(X, q, \cdot)$. *Then either* $q(x) = q(y)$ *or* $q(x) \cap q(y) = \phi$.

The subgroups, products and quotients of convergence groups are convergence groups. Every convergence group is homogeneous. So some of the local properties can be proved at a single point, in particular at the identity element e in X. A convergence group is T_2 if and only if the singleton set $\{e\}$ containing the identity element is closed. Also, every compact, regular, T_2 pseudotopological group is a topological group. Every topological group is a convergence group, however the converse statement is false.

Lemma 2.5. *If* $(X, q, \cdot)$ *is a convergence group and* q *is a pretopology, then* $(X, q, \cdot)$ *is a topological group.*

We denote by $CONVGR$ (respectively, $PSTOPGR$, $PRETOP$- GR, $TOPGR$) the category of convergence groups (respectively, pseudotopological group, pretopological group and topological group) and continuous homomorphisms. So the two subcategories $TOPGR$ and $PRETOPGR$ of the category $CONVGR$ are identical. However, the pretopological modification $(X, \pi q)$ and the topological modification $(X, \tau q)$ of a convergence group $(X, q, \cdot)$ do not preserve the continuity of the group operation on X. So, in general, $(X, \pi q)$ and $(X, \tau q)$ are not convergence groups with respect to the group operation. In view of this, neither $PRETOPGR$ nor $TOPGR$ is a reflective subcategory of $CONVGR$. However, pseudotopological modifications preserve the group operations making the subcategory $PSTOPGR$ a reflective subcategory of $CONGR$.

Lemma 2.6. *The pseudotopological modification* $(X, \gamma q)$ *of a convergence group* $(X, q, \cdot)$ *is a pseudotopological group.*

It is well known that subgroups and products of convergence groups are convergence groups, but terminal convergence structures with respect to continuous group homomorphisms are not compatible in general. However, there are some exceptions in case of quotient convergence structures on the coset space which we will discuss in Section 5.

The set of all homeomorphisms $H(X)$ forms a group with respect to composition of maps '$\circ$'. If (X, q) is a pseudotopological (respectively, limit, regular convergence) space, then (H(X), $\sigma, \circ$) is a pseudotopological (respectively, limit, regular) space where the convergence structure σ on $H(X)$ is defined by $\Phi \to^{\sigma} f$, iff $\Phi(\mathcal{F}) \to^{q} f(x)$ in X and $\Phi^{-1}(\mathcal{F}) \to^{q} f^{-1}(x)$ whenever $\mathcal{F} \to^{q} x$ for $x \in X$. This convergence structure σ is referred to as the *double convergence on* $H(X)$, see [12].

3. Continuous Group Action on a Convergence Space

Action of algebraic groups on arbitrary sets, in particular on finite sets, is a very important aspect of algebra from the application point of view. If we blend in the topological notions with the study of action, it even gets more interesting with applications in topological dynamics [5]. In this section, a brief introduction on action of convergence groups on convergence spaces is given. This notion was introduced by the author in an earlier paper [13] in which an one-to-one correspondence between continuous actions and homeomorphic representations of convergence groups was given.

Consider a function $\mu : X \times G \longrightarrow X$, denoted by $\mu(x, g) = x^g$ for all $x \in X$ and $g \in G$. The following are a few notations which we will use throughout the paper. For any $F \subseteq X$, $g \in G$, $F^g = \{f^g : f \in F\}$ and for any $A \subseteq G$, $F^A = \cup\{F^g : g \in A\}$. If $\mathcal{F} \in \mathbf{F}(X)$ and $g \in G$, then $\mathcal{F}^g = [\{F^g : F \in \mathcal{F}\}]$, and for any subset $A \subseteq G$, $\mathcal{F}^A = [\{F^A : F \in \mathcal{F}\}]$. If $\kappa \in \mathbf{F}(G)$, then $\mathcal{F}^\kappa = [\{F^A : F \in \mathcal{F}, A \in \kappa\}]$.

Definition 3.1. Let $(G, \Lambda, .)$ be a convergence group and (X, q) be a convergence space. G is said to *act continuously* on X if the function $\mu : X \times G \to X$ denoted by $\mu(x, g) = x^g$, $\forall x \in X$ and $\forall g \in G$, satisfies the following conditions:

(a_1) for all $F \to^q x$ and $\kappa \to^\Lambda g$, $F^\kappa \to^q x^g$.

(a_2) $x^e = x, \ \forall x \in X$.

(a_3) $(x^g)^h = x^{gh}, \forall x \in X$ and $\forall\ g, h \in G$.

The pair $((X, q), \mu)$, usually denoted by χ is called a $G-$*space*. If there is no ambiguity of the convergence structure q, we write $\chi = (X, \mu)$.

Note that condition (a_1) implies the continuity of the group action μ with respect to the product convergence structure on $X \times G$. G is said to act *algebraically* on X if μ satisfies only the conditions (a_2) and (a_3). The action is said to be *transitive* if for any $x, y \in X$, there exists $g \in G$ such that $y = x^g$. A $G-$space is *transitive* if the continuous action of G is transitive. For any $x \in X$, the subset $G_x = \{g \in G \mid x^g = x\}$ is called the *stabiliser* of x in G. It can be easily verified that G_x is a subgroup of G. The conditions (a_2) and (a_3) are equivalent to (a_2') $\mathcal{F}^e = \mathcal{F}$ and (a_3') $(\mathcal{F}^g)^h = \mathcal{F}^{gh}$, $\forall \mathcal{F} \in \mathbf{F}(X)$ and $\forall\ g, h \in G$. This can be verified by substituting $\dot{x}$ for $\mathcal{F}$ in (a_2') and (a_3'). So, if group G acts algebraically on a set X, then G acts algebraically on the set $\mathbf{F}(X)$ of all filters on X, the action being defined as $(\mathcal{F}, g) \longmapsto \mathcal{F}^g$ on $\mathbf{F}(X)$.

Any convergence group $(G, \Lambda, .)$ acts on itself continuously, the action being defined by the *right multiplication*, $(a, x) \mapsto a^x = ax$, $\forall a, x \in G$. The left multiplication $(a, x) = xa$, $\forall a, x \in G$ is not an action, but $(a, x) \mapsto a^x = x^{-1}a$, $\forall a, x \in G$ is an action of G on itself. Both these actions are transitive. G also acts on itself by conjugation, i.e., $a^x = x^{-1}ax$. But this is not transitive in general.

The homeomorphism group $(H(X), \sigma)$ acts continuously on (X, q) with respect to the double convergence σ on $H(X)$. In this case, the action is defined by the map $(x, f) \mapsto x^f = f^{-1}(x)$, for all $x \in X$ and $f \in H(X)$. Let (X, q) and (Y, p) be two limit spaces. If G acts on (X, q) or (Y, p) then G acts on $C(X, Y) = \{f \mid f : X \to Y,\ f \text{ is continuous}\}$, with respect to the continuous convergence structure Λ (see [2]) on $C(X, Y)$. In particular, G acts on $C(X)$, the space of all self continuous maps on X.

Proposition 3.2. (see [13]) *If* $(G, \Lambda, \cdot)$ *acts on the limit space* (X, q), *then* G *acts on* $(H(X), \sigma)$, *the group of all self homeomorphisms.*

If a convergence group G acts continuously on a convergence space X, then for each $g \in G$, the mapping $\bar{g} : X \to X$, defined by $\bar{g}(x) = x^g$ is a homeomorphism. Let X be a limit space and $(G, \Lambda, .)$ be a convergence group which acts continuously on X. Then, the map $\rho : G \to H(X)$ defined by $\rho(g) = \bar{g}$ for all $g \in G$ is a continuous group homomorphism. The mapping ρ may be called the *homeomorphic representation* of group G on the limit space X. So every continuous group action on X is associated with a homeomorphic representation. If the continuous homomorphism ρ is one to one, i.e. $\text{Kernel}_\rho = \{e\}$, then it can be easily checked that $G \approxeq \rho(G)$. On the other hand, any continuous group homomorphism $\theta : G \to \mathbf{H}(X)$ on G induces a continuous group action of G on X.

Proposition 3.3. (see [13]) *For any limit space X, there exists an one-to-one correspondence between the continuous action of the convergence group G on X and the homeomorphic representation of G on X.*

However, this is not true for topological groups in general, see [7]. If X is a locally compact topological space, then we can establish such an one-to-one correspondence.

4. Some Initial and Terminal Structures on *G*-Spaces

Let the class of all $G-$spaces be denoted by $\tilde{G}$. For two $G-$spaces (X, μ) and (X', μ'), a continuous map $\theta : X \to X'$ is called a $G-morphism$ if $\theta(\mu(x, g)) = \mu'(\theta(x), g)$, for all $x \in X$ and $g \in G$. More precisely, the $G-$morphism θ satisfies the condition $\theta(x^g) = (\theta(x))^g$.

Proposition 4.1. *$\tilde{G}$ with the $G-$ morphisms forms a category.*

Proof. Let $X, Y, Z \in \tilde{G}$, and let $f : X \to Y$ and $h : Y \to Z$ be $G-$morphisms. Then $h \circ f : X \to Z$ is continuous. Also, for any $g \in G$ and $x \in X$, $(h \circ f)(x^g) = h(f(x^g)) = h((f(x))^g) = (h(f(x)))^g = (h \circ f(x))^g$. So, $h \circ f$ is a G-morphism. For any $X \in \tilde{G}$, the identity map $I_X(x^g) = x^g = (I_X(x))^g$, $\forall x \in X$ and $g \in G$. So I_X is a $G-$morphism for any $X \in \breve{G}$. This shows that $\tilde{G}$ is a category. $\square$

Note that $\tilde{G}$ can be considered as a subcategory of the category $CONV$ of all convergence spaces and continuous maps. But $\tilde{G}$ is not a full subcategory of $CONV$, since the morphisms in $\tilde{G}$ are $G-$morphisms which are special types of continuous maps.

In this section, we intend to study some of the initial and terminal structures on the $G-$category $\tilde{G}$. This category is not a topological category since it may not preserve an initial structure in general. However, the following partial result holds for the initial convergence structure. It turns out that $\tilde{G}$ is closed under product and sum of its objects.

Proposition 4.2. *Let G act algebraically on a set X and $\{((X_i, q_i), \mu_i), f_i\}_{i\in I}$ be an indexed family of $G-$ spaces and maps $f_i : X \to X_i$ such that for each i, $f_i(x^g) = (f_i(x))^g$, for all $x \in X$ and $g \in G$. If p is the initial convergence structure on X with respect to the family $\{(X_i, q_i), f_i\}_{i\in I}$, then (X, p) is a $G-$ space.*

Proof. To show that (a_1) holds, let $\mathcal{F} \to^p x$ in (X, p) and $\Psi \to^\Lambda g$ in (G, Λ). Since p is initial convergence structure, $f_i(\mathcal{F}) \to^{q_i} f_i(x)$ for all $i \in I$. Since (X_i, q_i) are $G-$spaces, $(f_i(\mathcal{F}))^\Psi \to^{q_i} (f_i(x))^g$, which implies $f_i(\mathcal{F}^\Psi) \to^{q_i} f_i(x^g)$ for all i. Hence $f(\mathcal{F}^\Psi) \to^p f(x^g)$, which proves the proposition. □

The category $\tilde{G}$ is closed under product as shown in the following proposition.

Proposition 4.3. *Let $\{((X_i, q_i), \mu_i)\}_{i\in I}$ be an indexed family of $G-$ spaces, with action $\mu_i : X_i \times G \to X_i$, for each i. Let $Y = \Pi_i X_i$ be the Cartesian product of the sets X_i.Then (Y, p), where p is the product convergence structure on Y is a $G-$ space.*

Proof. The product convergence structure p on Y is defined by $\mathcal{F} \to^p x$ iff $p_i(\mathcal{F}) \to^{q_i} p_i(x)$ for all i, where p_i denotes the i-th projection map on Y. Define $\mu : Y \times G \to Y$ by $\mu(x, g) = x^g = ((p_i(x))^g)_{i\in I}$, that is $p_i \circ \mu(x, g) = \mu_i(p_i(x), g)$ for all $x \in Y$ and $g \in G$. It is easy to verify that (a2) and (a3) hold. To show that (a1) holds, let $\mathcal{F} \to^p x$ in (Y, p) and $\Psi \to^\Lambda g$ in (G, Λ). $\mathcal{F} \to^p x \implies p_i(\mathcal{F}) \to^{q_i} p_i(x)$, for all i. Since X_i are $G-$spaces, $(\ p_i(\mathcal{F}))^\Psi \to^{q_i} (p_i(x))^g$ for each i. By definition of μ, $p_i(\mathcal{F}^\Psi) \geq (\ p_i(\mathcal{F}))^\Psi$. So $p_i(\mathcal{F}^\Psi) \to^{q_i} p_i(x^g)$ for each i, which implies $\mathcal{F}^\Psi \to^p x^g$. □

Proposition 4.4. *Let (Y, q) be a $G-$ space and (X, p) be a subspace of Y which is closed under the group action, that is, if $x \in X$, then $x^g \in X$. Then (X, p) is a $G-$ space.*

Proof. Let $\mu : Y \times G \to Y$ be the continuous group action of Y on G. Then $\mu|_{X\times G} : X \times G \to Y$ is continuous. Since (X, p) is a subspace of Y, $X \times G$ is a subspace of $Y \times G$. Let $I : X \to Y$ be the inclusion map. For any map $\Phi : X \times G \to X$ is continuous if and only if $I \circ \Phi : X \times G \to Y$ is continuous. So for $\Phi = \mu|_{X\times G}$, it follows that X is a $G-$ space. □

The subspace (X, p) is called a $G-$ *subspace* of (Y, q).

Definition 4.5. Let $(I, \prec)$ be an upward directed set. Let $\{((X_i, q_i), \mu_i)\}_{i\in I}$ be an indexed family of $G-$ spaces, with action $\mu_i : X_i \times G \to X_i$, for each i and let $p_{j,i} : X_i \to X_j$be a $G-$ morphism for each $i \prec j$. The pair $((X_i)_{i\in I}, (p_{j,i}))$ is called a projective system of $G-$spaces if $p_{j,i} \circ p_{j,k} = p_{k,i}$ for all $i \prec j \prec k$.

The set

$$X = \{x \in \Pi_i X_i : p_i(x) = p_{j,i}(p_j(x)), \quad \forall i \prec j\},$$

where p_i denotes the i th projection map is called the projective limit of this projective system.

Proposition 4.6. *The projective limit of a projective system of $G-$ spaces is a $G-$ space.*

Proof. From Proposition 4.2 it follows that for $Y = \Pi_i X_i$, $\mu\mid_X : X \times G \to Y$ defined by $x^g = ((p_i(x))^g)_{i\in I}$ is continuous, where X (as above) has the subspace convergence structure. Next, we show that $\mu' = \mu\mid_X$ is an action on X. Let $x \in X$, then $p_i(x) = p_{j,i}(p_j(x))$ for all $i \prec j$. For any $g \in G$, $\mu'(x, g) = x^g$, which implies $p_i(x^g) = (p_i(x))^g$, for each $i \in I$. So, $p_i(x^g) = [p_{j,i}(p_j(x))]^g = p_{j,i}([(p_j(x)]^g)$, since for each $i \prec j$, $p_{j,i}$ maps are all $G-$morphisms. This implies $p_i(x^g) = p_{j,i}(p_j(x^g)$, so that $x^g \in X$. So, $\mu' : X \times G \to X$ and since $\mu' = \mu\mid_X$, it satisfies (a_2) and (a_3). $\square$

Next, we discuss the properties of the terminal convergence structure on a set Y with respect to a family of $G-$spaces.

Proposition 4.7. *Let G act algebraically on a set Y and $\{((X_i, q_i), \mu_i), f_i\}_{i\in I}$ be an indexed family of $G-$ spaces where the maps $f_i : X_i \to Y$ for each i, satisfy $f_i(x^g) = (f_i(x))^g$, for all $x \in X_i$ and $g \in G$. If q is the terminal convergence structure on Y with respect to the family $\{(X_i, q_i), f_i\}_{i\in I}$, then (Y, q) is a $G-$ space.*

Proof. We need to show that (a_1) holds for the action μ of G on Y. Let $\mathcal{F} \to^q y$ in (Y, q) and $\Psi \to^\Lambda g$ in G. Since q is the final convergence structure, there exists $i \in I$ such that there is $x_i \in f_i^{-1}(y)$ and there is a filter $\mathcal{L}_i \to^{q_i} x_i$ such that $\mathcal{F} \geq f_i(\mathcal{L}_i)$. Since $f_i(x_i) = y$, it follows that $f_i(x_i^g) = y^g$, that is, $x_i^g \in f_i^{-1}(y^g)$. Also $\mathcal{L}_i \to^{q_i} x_i$ implies $\mathcal{L}_i^\Psi \to^{q_i} x_i^g$ and $\mathcal{F} \geq f_i(\mathcal{L}_i)$ implies $\mathcal{F}^\Psi \geq (f_i(\mathcal{L}_i))^\Psi = f_i(\mathcal{L}_i^\Psi)$. Therefore, $\mathcal{F}^\Psi \to^q y^g$ in (Y, q), which shows that (Y, q) is a $G-$space. $\square$

The proof of the following corollary is immediate.

Corollary 4.8. *Let G act algebraically on a set Y , X be a $G-$ space and map $f : X \to Y$ be a surjection such that $f(x^g) = (f(x))^g$, for all $x \in X$ and $g \in G$. If q is the quotient convergence structure on Y, then (Y, q) is a $G-$ space.*

Let $\{((X_i, q_i)\}_{i\in I}$ be an indexed family of convergence spaces and let $X = \sum_i X_i$ denote their disjoint union. If $\sum_i X_i$ is endowed with the terminal convergence structure q with respect to $\{(X_i, q_i), e_i\}_{i\in I}$, where $e_i : X_i \to X$ denotes the embedding of X_i in X for each $i \in I$, then (X, q) is called the *convergence sum* of the family $(X_i)_{i\in I}$. The next proposition shows that the subcategory $\tilde{G}$ is closed under convergence sum.

Proposition 4.9. *Let $\{((X_i, q_i), \mu_i)\}_{i\in I}$ be an indexed family of $G-$ spaces, with action $\mu_i : X_i \times G \to X_i$, for each i. Then the convergence sum (X, q) is a $G-$ space.*

Proof. Define a map $\mu : (X \times G) \to X$ by $\mu(x, g) = e_i(\mu_i(x_i, g))$ where $e_i : X_i \to X$ is the embedding map with $e_i(x_i) = x$. The map μ is well defined since e_i is an embedding. Let $x \in X$ and $g, h \in G$. Then $\mu(x, e) = e_i(\mu_i(x_i, e)) = e_i(x_i) = x$.

Also,

$$\begin{aligned}\mu[\mu(x, g), h] &= \mu[e_i(\mu_i(x_i, g)), h] = \mu[e_i(x_i^g), h]\\ &= \mu[x^g, h] = e_i(\mu_i(x_i^g, h)) = e_i(x_i^{gh}) = e_i(\mu_i(x_i, gh) = \mu(x, gh).\end{aligned}$$

This proves that μ defines an algebraic action on X. Next, let $\mathcal{F} \to^q x$ in (X, q) and $\Psi \to^\Lambda g$ in G. Since $\mathcal{F} \to^q x$ in (X, q), there exists unique $i \in I$ such that there exists $x_i \in X_i$ with $e_i(x_i) = x$ and $\mathcal{L}_i \to^{q_i} x_i$ in X_i with $\mathcal{F} \geq e_i(\mathcal{L}_i)$. Since $((X_i, q_i), \mu_i)$ is a $G-$space, $\mathcal{L}_i^\Psi \to^{q_i} x_i^g$. Also, $\mathcal{F} \geq e_i(\mathcal{L}_i)$ implies $\mathcal{F}^\Psi \geq [e_i(\mathcal{L}_i)]^\Psi = e_i(\mathcal{L}_i{}^\Psi)$. So for a unique $i \in I$ there exists $x_i^g \in X_i$ such that $e_i(x_i^g) = x^g$ and $\mathcal{L}_i^\Psi \to^{q_i} x_i^g$ in X_i with $\mathcal{F}^\Psi \geq e_i(\mathcal{L}_i^\Psi)$. So $\mathcal{F}^\Psi \to^q x^g$ in (X, q). This implies that (X, q) is a $G-$space. □

From the above results it is established that the category $\tilde{G}$ is closed under product and sum.

If (X, q) is a $G-$space, then for any $\alpha, \beta \in X$, define a relation $\alpha \approx \beta$ iff 'there exists $g \in G$ such that $\alpha^g = \beta$'. It can be easily verified that this defines an equivalence relation on X. The corresponding equivalence classes are called the *orbits of* G in X, see [11]. Let $\{X_i\}_{i\in I}$ denote the class of all orbits of G in X. Each orbit X_i is a transitive closed $G -$ *subspace* of X. Then X can be uniquely expressed as the disjoint union of the orbits X_i. Let q_i be the subspace structure on X_i for each i. Since each orbit X_i is a transitive closed $G-$subspace of X, the following result can be immediately established.

Theorem 4.10. *Any $G-$ space X can be uniquely expressed as a convergence sum of the family of orbits of G in X.*

5. A Coset Space is a G-Space

A *right coset space* (respectively, *left coset space*) of a group G is the set of all right cosets (respectively, left cosets) of a subgroup of G. Since the group G is not necessarily Abelian, the right coset space and left coset space are distinct. In this section, we consider only the right coset space and call it *coset space.* The theory can be duplicated for the left coset space. Equipped with the quotient structure, the coset space turns out to be a special type of quotient convergence space which also allows a transitive group action of G. In this section, the objective is to establish that pseudotopological transitive T_2 $G-$spaces are similar in structure to certain quotient convergence groups.

Let $(G, \Lambda, \cdot)$ be a convergence group and let H be any subgroup of G. The set of all right cosets Ha of H for any $a \in G$ is denoted by G/H. There is a natural map

$$Q : G \to G/H$$

defined by $Q(a) = Ha$ for any $a \in G$. It can be easily verified that Q is a well defined surjective map. With respect to this map Q, the convergence structure Λ on G induces a convergence structure q on G/H, which is the quotient convergence structure. It is the largest convergence structure with respect to which Q is continuous. Explicitly, a filter $\Phi \to^q t$ in G/H, if there exists a filter $\mathcal{F} \in \mathbf{F}(G)$ such that $\Phi \geq Q(\mathcal{F})$ and $\mathcal{F} \to^\Lambda r$, where $r \in Q^{-1}(t)$ in G. Note that when H is a normal subgroup of G, the group operations on G/H are continuous with respect to the the quotient convergence structure and so G/H becomes a convergence group. In particular, if $(G, \cdot)$ is an Abelian group, then $(G/H, \circ)$ is also a group with respect to the group operation $Ha \circ Hb = Hab$. Kent has shown that the convergence structure on the quotient space G/H is inherited from the convergence structure on G, see [9].

Proposition 5.1. *If $(G, \Lambda, \cdot)$ is an Abelian convergence group (respectively, pseudoconvergence group, topological group), then $(G/H, q, \circ)$ is a convergence group (respectively, pseudoconvergence group, topological group).*

A convergence space (X, q) is said to be *homogeneous* if for any two elements x, $x' \in X$, there exists a homeomorphism $T : X \to X$ such that $T(x) = x'$. Any convergence group is homogeneous. Note that the coset space $(G/H, q)$ is not a convergence group in general, but it is homogeneous, since for any two elements Ha and Hb in G/H, there is a map $T : X \to X$ defined by $T(Hg) = Hga^{-1}b$, for all $Hg \in G/H$, which is a homeomorphism. It is known that if G is any topological group then the quotient topology on G/H is regular, see [11]. However, in general, this is not true when G is a convergence group.

The following proposition summarizes some of the properties of the coset space G/H.

Proposition 3.2. *The coset space* $(G/H, q)$

(a) is T_2 *if and only if* H *is a closed subgroup of* G *.*

(b) is a regular convergence space if and only if G *is a regular convergence group.*

(c) is compact, if G *is a compact group.*

(d) is a convergence group if H *is a normal subgroup of* G.

Proof.(a) Let H be a closed subgroup of G. If possible, let $\Psi \to^q t$ and $\Psi \to^q s$ in G/H with $s \neq t$. So, there exist filters $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(G)$ such that $\Psi \geq Q(\mathcal{F}_1)$, $\Psi \geq Q(\mathcal{F}_2)$ and $\mathcal{F}_1 \to^\Lambda x$, $\mathcal{F}_1 \to^\Lambda y$ where $x \in Q^{-1}(t)$ and $y \in Q^{-1}(s)$ in G. This implies that $\mathcal{F}_1\mathcal{F}_2^{-1} \to^\Lambda xy^{-1}$, with $xy^{-1} \notin H$ since $s \neq t$. Further, since H is closed $H \notin \mathcal{F}_1\mathcal{F}_2^{-1}$. However, $\Psi \geq Q(\mathcal{F}_1)$, $\Psi \geq Q(\mathcal{F}_2)$ imply that $Q(\mathcal{F}_1) \vee Q(\mathcal{F}_2)$ exists, which leads to a contradiction. The converse part follows from the same argument as in the topological case.

(b) Let $\mathcal{F} \to^\Lambda x$ in G. Since the quotient map $Q : G \to G/H$ is continuous, $Q(cl_\Lambda \mathcal{F}) \geq cl_q Q(\mathcal{F})$ and $Q(\mathcal{F}) \to^q Q(x) = t$ (say). So by regularity of G/H, $cl_q Q(\mathcal{F}) \to^q t$, which implies $Q(cl_\Lambda \mathcal{F}) \to^q t$. So there exists $\mathcal{L} \to^\Lambda g$ in G such that $Q(cl_\Lambda \mathcal{F}) \geq Q(\mathcal{L})$ and $g \in Q^{-1}(t)$. Since $cl_\Lambda \mathcal{F} \geq \mathcal{L} \to^\Lambda g$, $\mathcal{F} \to^\Lambda g$. If $x \neq g$ in the convergence group G, then since $\mathcal{F} \in \Lambda(x) \cap \Lambda(g)$, by Lemma 2. 4, $\Lambda(x) = \Lambda(g)$. So, $cl_\Lambda \mathcal{F} \to^\Lambda g \Rightarrow cl_\Lambda \mathcal{F} \to^\Lambda x$, which proves that $(G, \Lambda, \cdot)$ is regular.

Conversely, let $(G, \Lambda, \cdot)$ be a regular convergence group. Let $\Psi \to^q s$ in GH. Then, there exists $\Im \to^\Lambda t$ in G such that $\Psi \geq Q(\Im)$ and $t \in Q^{-1}(s)$, where Q is the quotient map. Since G is regular, $cl_\Lambda(\Im) \to^\Lambda t$ and Q is continuous implies $Q(cl_\Lambda(\Im)) \to^q Q(t) = s$. Now, $\Psi \geq Q(\Im) \Longrightarrow cl_q \Psi \geq cl_q(Q(\Im)) \geq Q(cl_\Lambda(\Im))$, since Q is continuous. So $cl_q \Psi \to^q s$, which implies G/H is regular.

(c) If $(G, \Lambda, \cdot)$ is compact, then since the quotient map Q is continuous and onto, $(G/H, q)$ is compact.

(d) If H is a normal subgroup of G, then G/H is a subgroup and it is easy to show that the group operations are continuous with respect to the quotient convergence structure. □

There is a natural action of G on G/H defined by $(Ha, g) \longmapsto (Ha)^g = Hag$ which is also transitive. The next proposition establishes that the coset space $(G/H, q)$ is a $G-$ space.

Proposition 5.3. *Let* $(G, \Lambda, \cdot)$*be a convergence group. Then* G *acts* *continuously on the coset space* $(G/H, q)$.

Proof. Let $\Phi \to^q t$ in G/H and $K \to^\Lambda g$ in G. We need to show that $\Phi^K \to^q t^g$ in G/H. Since $\Phi \to^q t$ in G/H, $\exists \mathcal{F} \in \mathbf{F}(G)$ such that $\Phi \geq Q(\mathcal{F})$ and $\mathcal{F} \to^\Lambda r$, where $r \in Q^{-1}(t\ ) \in G$. Since $r \in Q^{-1}(t\ )$, $Q(r) = Hr = t$. So, it follows from the definition of Q and the action of G on G/H that $Q(rg) = Hrg$ $= (Hr)^g = t^g$. This implies that $r^g = rg \in Q^{-1}(t^g\ )$. Since G acts continuously on itself by right multiplication, $\mathcal{F}^K \to^\Lambda r^g\ (= rg)$. Also, $\Phi \geq Q(\mathcal{F}) \Rightarrow \Phi^K \geq [Q(\mathcal{F})]^K \geq Q(\mathcal{F}^K)$. So there exists a a filter $\Phi^K \geq Q(\mathcal{L})$ and $\mathcal{L} = \mathcal{F}^K \to^\Lambda r^g$, where $r^g \in Q^{-1}(t^g\ )$. This implies that $\Phi^K \to^q t^g$ in G/H. $\square$

Recall that for two $G-$spaces X and X' a map $\theta : X \to X'$ is called a $G-$morphism if θ is continuous and $\theta(x^g) = (\theta(x))^g$ for all $x \in X$ and $g \in G$. Furthermore, if θ is a homeomorphism, then it is called a $G-$*homeomorphism*. Let G_α be the stabiliser of α in G.

Proposition 5.4. *Let* (X, μ) *be a transitive* $G-$ *space and* $\alpha \in X$. *Then there is a bijective* $G-$*morphism between* G/G_α *and* X.

Proof. Define $\theta : G/G_\alpha \to X$ by $\theta(G_\alpha g) = \alpha^g$ for all $g \in G$. Since X is a transitive $G-$space, it follows that θ is a well-defined bijective map. Next, we show that θ is continuous. Let $\Psi \to^q s$ in G/G_α, where q is the quotient convergence structure and let Q be the corresponding quotient map. Then, there exists $\Im \to^\Lambda t$ in G such that $\Psi \geq Q(\Im)$ and $Q(t) = s$. Consider the map $\Phi : G \to X$, defined by $\Phi(g) = \alpha^g$, for all $g \in G$. So by definition of θ, $\theta \circ Q = \Phi$. Since G acts continuously on X, $\Phi = \mu \mid_{\{\alpha\}\times G}$is continuous, which implies that $\Phi(\Im) \to^p \Phi(t\ )$, where p is the convergence structure on X. So, $\theta(\Psi) \geq \theta(Q(\Im)) = \Phi(\Im) \to^p \Phi(t)$. Since $\Phi(t) = \theta \circ Q(t) = \theta(s)$, it follows that $\theta(\Psi) \to^p \theta(s)$, which shows that θ is continuous. $\square$

It is well-known that if the algebraic group action of a group G on a set X is transitive, then there is an one-to-one correspondence between X and the coset space G/G_α for any $\alpha \in X$. In the following propositions we attempt to establish that transitive pseudotopological T_2 $G-$spaces are similar in structure to the coset space G/G_α equipped with the quotient convergence structure.

Theorem 5.5. *If* G *is a compact pseudotopological group and* X *is a* T_2 $G-$ *space, then for any* $\alpha \in X$, G/G_α *is* $G-$ *homeomorphic to* X.

Proof. Since the quotient map Q is continuous and surjective, G/G_α is compact. Also, the quotient structure preserves the pseudotopological structure on G. So G/G_α is a pseudotopological compact space. By Proposition 5.4, $\theta : G/G_\alpha \to X$ is a bijective continuous map. Since X is T_2, it follows that θ is a homeomorphism. $\square$

Corollary 5.6. *If G is a compact topological group, then any T_2 topological $G-$ space X is homeomorphic to a quotient space of G.*

6. Modifications of *G*-Spaces

The pseudotopological, pretopological and topological modifications of objects in the category $CONV$ of convergence spaces are well known. In this section, we study some special conditions on the convergence space X with respect to which these modifications preserve the continuous group action. This leads to some modifications of the objects in the $G-$category $\tilde{G}$. Let $\widetilde{PsG}$ (respectively, $\widetilde{PrG}$ and $\widetilde{TopG}$) denote the subcategory of pseudotopological $G-$spaces (respectively, pretopological and topological $G-$spaces). Considering $\tilde{G}$ as a subcategory of $CONV$ we study similar modifications of objects in $\tilde{G}$. Throughout this section $X = (X, q)$ is a convergence space and $G = (G, \Lambda, \cdot)$ is a convergence group, unless otherwise stated.

Proposition 6.1. *If $\chi = (X, \mu)$ is a $G-$space, then $\gamma\chi = (\gamma X, \mu)$ is a $G-$space.*

Proof. Since X is a $G-$space, $\mu : X \times G \longrightarrow X$ is continuous, where $X \times G$ has the product convergence structure. Since the pseudotopological modification preserves continuous maps and products of convergence spaces, the mapping μ on $\gamma(X \times G) = \gamma(X) \times \gamma(G)$ into γX is continuous. Also, for the same reason γG is a convergence group. This implies γG acts continuously on γX. We need to show that G acts continuously on γX. Let $\mathcal{F} \longrightarrow^{\gamma q} x$ in γX and $\Psi \longrightarrow^{\Lambda} g$ in G. Since $\gamma\Lambda \leq \Lambda$, this implies $\Psi \longrightarrow^{\gamma\Lambda} g$ in γG. So $\mathcal{F}^{\Psi} \longrightarrow^{\gamma q} x^g$ in γX , since γG acts continuously on γX. Hence, G acts continuously on γX making $\gamma\chi$ a $G-$space. $\square$

From Proposition 6.1.it follows thatthe subcategory $\widetilde{PsG}$ of all pseudotopological $G-$spaces is a reflective subcategory of $\tilde{G}$.

However, this is not true for the subcategories $\widetilde{PrG}$ and $\widetilde{TopG}$ of $\tilde{G}$, because, the pretopological and topological modifications of $G-$spaces may not be $G-$spaces in general. This is due to the fact that pretopological and topological modifications of a convergence group are not necessarily convergence groups. It turns out that only a special type of convergence spaces, namely, the *pretopologically coherent* $G-$spaces allow the same group action on their pretopological modification $\pi X = (X, \pi q)$ and topological modification $\tau X = (X, \tau q)$. First we investigate the pretopological modification πX of a $G-$space (X, μ).

Definition 6.2. *Two convergence spaces* (X, q) *and* (Y, s) *are said to be pretopologically coherent iff for each* $(x, y) \in (X \times Y), \nu_q(x) \times \nu_s(y) = \nu_p(x, y)$, *where* p *is the product convergence structure (see [3]) on* $X \times Y$.

Proposition 6.3. *If* (X, μ) *is a* $G-$ *space pretopologically coherent with* (G, Λ), *then* $(\pi X, \mu)$ *is a* $G-$ *space.*

Proof. Let $\mathcal{F} \rightarrow^{\pi q} x$ and $\Psi \longrightarrow^{\Lambda} g$. Then by definition of πq, $\mathcal{F} \geq \nu_q(x)$ and since $\nu_\Lambda(g) = \cap_{\mathcal{L}}\{\mathcal{L} \rightarrow^{\Lambda} g\}$, $\Psi \geq \nu_\Lambda(g)$.This implies that $\mathcal{F} \times \Psi \geq \nu_q(x) \times \nu_\Lambda(g)$. Since (X, q) and (G, Λ) are pretopologically coherent, $\nu_q(x) \times \nu_\Lambda(g) = \nu_p(x, g)$, where p is the product convergence structure on $X \times G$. If μ denotes the action of G on X, then $\mu(\nu_q(x) \times \nu_\Lambda(g)) = \mu(\nu_p(x, g)) \geq \nu_q(\mu(x, g))$, since $\mu : X \times G \longrightarrow X$ is continuous. This shows $\mu(\mathcal{F} \times \Psi) \geq \nu_q(\mu(x, g))$, that is, $\mathcal{F}^\Psi \geq \nu_q(x^g)$, which implies $\mathcal{F}^\Psi \rightarrow^{\pi q} x^g$. This proves that πX is a $G-$ *space.* $\square$

Next, we discuss the topological modification of a $G-$space (X, μ). We obtain the topological modification of a $G-$space by using a decomposition series of the corresponding convergence space. The decomposition series of a convergence space is an ordinal sequence of pretopological spaces which lead to the topological modification of the original space. Kent was the first to construct such a series which was used to establish the topological equivalence of convergence quotient maps and pseudo-open maps on convergence spaces. A brief discussion on a decomposition series is provided here for clarification. For more details the reader is referred to [9].

Let $X = (X, q)$ be a convergence space. For any subset $U \subseteq X$ and for each ordinal α, define an interior operator I_q^α as follows :

$I_q^0(U) = U$,

$I_q^1(U) = \{x \in U \mid U \in \nu_q(x)\}$,.......,

$I_q^\alpha(U) = I_q^1(I_q^{\alpha-1}(U))$ if α is a non-limit ordinal and

$I_q^\alpha(U) = \cap\{I_q^\sigma(U)) \mid \sigma < \alpha\}$ if α is a limit ordinal.

Note that the topological modification $\tau q = \{U \subset X \mid I_q^1(U) = U\}$. It can be easily verified that I_q^α has the same properties as the interior operator described in Section 1 and Section 2. Also, if $\alpha < \beta$, then $I_q^\alpha(U) \supseteq I_q^\beta(U)$. Since $\{I_q^\alpha(U)\}$ is a transfinite family of shrinking subsets, it must eventually terminate. The least ordinal number α for which $I_q^\alpha(U) = I_q^{\alpha+1}(U)$ is called the *length* of the decomposition series and it is denoted by l. For each $x \in X$ and each ordinal α, $\nu_q^\alpha(x) = \{U \subset X \mid x \in I_q^\alpha(U)\}$ forms a filter. For each α, taking $\nu_q^\alpha(x)$ as the neighborhood filter of $x \in X$ an ordinal family $\{\pi^\alpha q\}$ of pretopologies is defined recursively as follows:

$\pi^0 q = q$,

$\mathcal{F} \to^{\pi^1 q} x$ iff $\mathcal{F} \geq \nu_q(x)$, ..., ,

$\mathcal{F} \to^{\pi^\alpha q} x$ iff $\mathcal{F} \geq \nu_q^{\alpha-1}(x)$ when α is a non-limit ordinal and

$\mathcal{F} \to^{\pi^\alpha q}$ iff $\mathcal{F} \geq \cap_{\sigma<\alpha} \nu_q^\sigma(x)$ when α is a limit ordinal.

Note that $\pi^1 q = \pi q$, the pretopological modification of q. Also, $\pi^l q = \tau q$ which is the topological modification of q. If $1 \leq \alpha < \beta \leq l$, then $\tau q \leq \pi^\beta q < \pi^\alpha q \leq \pi q$. This implies that $\nu_q^\alpha(x) > \nu_q^\beta(x)$ if $\alpha < \beta$. Continuous maps preserve the decomposition series which can be established by using the following.

Lemma 6.4. *Let $f(X,q) \to (Y,s)$ be a continuous map. Then for each $x \in X$ and each ordinal α,*

$$f(\nu_q^\alpha(x)) \geq \nu_s^\alpha(f(x)).$$

Lemma 6.5. *Let $f(X,q) \to (Y,s)$ be an onto map. Then for each $x \in X$ and each ordinal α, the following are equivalent:*

(a) $f(\nu_q^\alpha(x)) \leq \nu_s^\alpha(f(x))$.

(b) For each $V \subset X$, $f(I_q^\alpha(B)) \subset I_s^\alpha(f(B))$.

For each ordinal α, let p_α denote the product convergence structure on $(X, \pi^\alpha q) \times (G, \Lambda)$. Then from the above discussion it follows that for any non-limit ordinal $\alpha \geq 2$, $p_\alpha \leq p_{\alpha-1}$. If α is a limit ordinal, then $p_\alpha \leq p_\sigma$ for all $\sigma < \alpha$.

Proposition 6.6. *If (X,q) is pretopologically coherent with (G,Λ), then for each ordinal α, $(X, \pi^\alpha q)$ is pretopologically coherent with (G,Λ).*

Proof. For each ordinal α, we need to show that $\nu_q^\alpha(x) \times \nu_\Lambda(g) = \nu_{p_\alpha}^\alpha(x,g)$, where p_α denotes the product convergence structure on $(X, \pi^\alpha q) \times (G,\Lambda)$. For $\alpha = 1$, since $\nu_q^1(x) = \nu_q(x)$, and since (X,q) and (G,Λ) are pretopologically coherent, $(X, \pi^1 q)$ is pretopologically coherent with (G,Λ).

Let α be a non-limit ordinal. Assume that $(X, \pi^{\alpha-1} q)$ is pretopologically coherent with (G,Λ). Next let $U \in \nu_{p_\alpha}^\alpha(x,g)$. This implies $(x,g) \in I_{p_\alpha}^\alpha(U) = I_{p_\alpha}(I_{p_\alpha}^{\alpha-1}(U))$, which in turn implies that $I_{p_\alpha}^{\alpha-1}(U) \in \nu_{p_\alpha}(x,g)$. So $P_1(I_{p_\alpha}^{\alpha-1}(U)) \in P_1(\nu_{p_\alpha}(x,g))$, where $P_1 : (X \times G, p_\alpha) \to (X, \pi^\alpha q)$ is the projection map. However, by Lemma 2.3, $P_1(\nu_{p_\alpha}(x,g)) = \nu_q^\alpha(x)$, which implies $P_1(\nu_{p_\alpha}(x,g)) \leq \nu_q^1(x) = \nu_q(x)$. Hence, $P_1(I_{p_\alpha}^{\alpha-1}(U)) \in \nu_q(x)$. Similarly for the projection map $P_2 : (X \times G, p_\alpha) \to (G,\Lambda)$, $P_2(I_{p_\alpha}^{\alpha-1}(U)) \in \nu_\Lambda(g)$. Now, by induction hypothesis for all $(x,g) \in X \times G$, $\nu_q^{\alpha-1}(x) \times \nu_\Lambda(g) = \nu_{p_{\alpha-1}}^{\alpha-1}(x,g)$. This implies $P_1(\nu_{p_{\alpha-1}}^{\alpha-1}(x,g)) = P_1(\nu_q^{\alpha-1}(x) \times \nu_\Lambda(g)) = \nu_q^{\alpha-1}(x) = \nu_q^{\alpha-1}(P_1(x,g))$. Since the projection map $P_1 : (X \times G, p_{\alpha-1}) \to (X, \pi^{\alpha-1} q)$ is onto, by Proposition 6.6,

$P_1(I^{\alpha-1}_{p_{\alpha-1}}(U)) \subseteq I^{\alpha-1}_q P_1(U)$. Since for any non-limit ordinal α, $p_\alpha \leq p_{\alpha-1}$, it follows that $I^{\alpha-1}_{p_\alpha}(U) \subseteq I^{\alpha-1}_{p_{\alpha-1}}(U)$. So, $P_1(I^{\alpha-1}_{p_\alpha}(U)) \subseteq P_1(I^{\alpha-1}_{p_{\alpha-1}}(U)) \subseteq I^{\alpha-1}_q P_1(U)$, which shows that $I^{\alpha-1}_q(P_1(U)) \in \nu_q(x)$. So, $x \in I_q(I^{\alpha-1}_q(P_1(U)) = I^\alpha_q(P_1(U))$ and therefore $P_1(U) \in \nu^\alpha_q(x)$. Also, since $P_2(I^{\alpha-1}_{p_\alpha}(U)) \subseteq P_2(U)$, $P_2(U) \in \nu_\Lambda(g)$. So $U \in \nu^\alpha_q(x) \times \nu_\Lambda(g)$ from which it follows that $\nu^\alpha_q(x) \times \nu_\Lambda(g) \geq \nu^\alpha_{p_\alpha}(x,g)$.

Next, we consider the case when α is a limit ordinal and $\nu^\sigma_q(x) \times \nu_\Lambda(g) = \nu^\sigma_{p_\sigma}(x,g)$ for all $\sigma < \alpha$. Let $U \in \nu^\alpha_{p_\alpha}(x,g)$. This implies $(x,g) \in I^\alpha_{p_\alpha}(U) = \cap\{I^\sigma_{p_\sigma}(U) \mid \sigma < \alpha\}$, so that $(x,g) \in I^\sigma_{p_\sigma}(U)$ for all $\sigma < \alpha$, that is, $U \in \nu^\sigma_{p_\sigma}(x,g)$ for all $\sigma < \alpha$. So by induction hypothesis $U \in \nu^\sigma_q(x) \times \nu_\Lambda(g)$, which implies $P_1(U) \in \nu^\sigma_q(x)$ for all $\sigma < \alpha$. Hence, $P_1(U) \in \cap\{\nu^\sigma_q(x) \mid \sigma < \alpha\} = \nu^\alpha_q(x)$. Also, as shown above $P_2(U) \in \nu_\Lambda(g)$ which implies $U \in \nu^\alpha_q(x) \times \nu_\Lambda(g)$. This shows that $\nu^\alpha_q(x) \times \nu_\Lambda(g) \geq \nu^\alpha_{p_\alpha}(x,g)$ for ordinals α. By Lemma 2.3, it follows that $(X, \pi^\alpha q)$ is pretopologically coherent with (G,Λ). This proves Proposition 6.6. □

Theorem 6.7. *Let (X,q) be a $G-$ space pretopologically coherent with (G,Λ). If $\nu^l_q(x) \times \nu_\Lambda(g) \geq \nu^l_p(x,g)$, then $\tau X = (X, \tau q)$ is a $G-$ space.*

Proof. By Proposition 6.6, for each ordinal α, $(X, \pi^\alpha q)$ is pretopologically coherent with (G,Λ). So $\nu^\alpha_q(x) \times \nu_\Lambda(g) = \nu^\alpha_{p_\alpha}(x,g)$. Let $\mathcal{F} \to^{\tau q} x$ and $\Psi \longrightarrow^\Lambda g$. Then $\mathcal{F} \geq \nu^l_q(x)$ and $\Psi \geq \nu_\Lambda(g)$. So, $\mathcal{F} \times \Psi \geq \nu^l_q(x) \times \nu_\Lambda(g) \geq \nu^l_p(x,g)$ which implies $\mu(\mathcal{F} \times \Psi) \geq \mu(\nu^l_p(x,g))$, where μ is the action of G on X. Since μ is continuous by Lemma 6.4, $\mu(\nu^l_p(x,g)) \geq \nu^l_q(\mu(x,g)) = \nu^l_q(x^g)$. Therefore, $\mathcal{F}^\Psi \longrightarrow^{\pi^l q} x^g$,that is, $\mathcal{F}^\Psi \longrightarrow^{\tau q} x^g$, which proves that τX is a $G-$space. □

In view of the above results, it seems that the category $\tilde{G}$ is an interesting category. Several of its characteristics are still to be investigated. For example, the completeness, co-completeness and reflectivity of $\tilde{G}$ as a subcategory of the categories $CONV$ are still to be determined. These properties of $\tilde{G}$ will be investigated in a follow up paper.

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