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# TORSION FREE SHEAVES ON REDUCIBLE CURVES WITH ONLY PLANAR SINGULARITIES

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**Abstract:** Let $X$ be a reduced projective curve with only planar singularities with some ordinary node or ordinary cusp. Here we give a local stratification of the moduli space of stable torsion free sheaves on $X$ with pure rank, depending only on the singular points which are ordinary nodes or ordinary cusps.

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**Key Words:** torsion free sheaf, reducible curve, planar singularities Gorenstein curve

## 1. Introduction

Let $X$ be a reduced and connected projective curve defined over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. We will say that a torsion free sheaf $E$ on $X$ has pure rank $r$ if for every irreducible component $T$ of $X$ the vector bundle $E|T_{reg}$ has rank $r$. Set $\mathrm{Sing}(E) := \{P \in X : E \text{ is not locally free at } P\}$. Obviously, $\mathrm{Sing}(E)$ is finite and $\mathrm{Sing}(E) \subseteq \mathrm{Sing}(X)$. For any $P \in X$ let $E_P$ denote the germ of $E$ at $P$: Thus $E_P$ is a finite $\mathcal{O}_{X,P}$-module. Let $\widehat{\mathcal{O}}_{X,P}$ denote the completion of the local ring $\mathcal{O}_{X,P}$ with respect to its maximal ideal $m_{X,P}$. Set $\widehat{E}_P := E_P \otimes_{\mathcal{O}_{X,P}} \widehat{\mathcal{O}}_{X,P}$. We will first prove the following result.

**Proposition 1.** *Assume that $X$ has only planar singularities. Let $E$ be a torsion free sheaf on $X$ with pure rank $r$. Assume that $E$ is not locally free and fix $S \subsetneqq \mathrm{Sing}(E)$. Then there exist a flat family of torsion free sheaves $\{E_t\}_{t\in T}$ on $X$ and $o \in T$ such that $T$ is an integral curve, $o \in T$, $E_o \cong E$, $\mathrm{Sing}(F_t) = S$*

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*for every $t \in T\backslash\{o\}$, and for every $P \in S$ and every $t \in T\backslash\{o\}$ the $\widehat{\mathcal{O}}_{X,P}$-module $\widehat{E}_{t\ P}$ is formally equivalent to the $\widehat{\mathcal{O}}_{X,P}$-module $\widehat{E}_P$.*

Taking $S = \emptyset$ in the statement of Proposition 1 we get the following results.

**Corollary 1.** *Assume that $X$ has only planar singularities. Let $E$ be a torsion free sheaf on $X$ with pure rank $r$. Then $E$ is a flat limit of a flat family of vector bundles on $X$.*

Since stability is an open condition even for reducible curves (see [5]), Proposition 1 implies the following results.

**Proposition 2.** *Assume that $X$ has only planar singularities. Fix a polarization $H$ on $X$. Let $E$ be a torsion free sheaf on $X$ with pure rank $r$. Assume that $E$ is not locally free and fix $S \subsetneq \mathrm{Sing}(E)$. Then there exist a flat family of torsion free sheaves $\{E_t\}_{t\in T}$ on $X$ and $o \in T$ such that $T$ is an integral curve, $o \in T$, $E_o \cong E$, $\mathrm{Sing}(F_t) = S$ for every $t \in T\backslash\{o\}$,for every $P \in S$ and every $t \in T\backslash\{o\}$ the $\widehat{\mathcal{O}}_{X,P}$-module $\widehat{E}_{t\ P}$ is formally equivalent to the $\widehat{\mathcal{O}}_{X,P}$-module $\widehat{E}_P$, and each $E_t$ is $H$-stable.*

**Corollary 2.** *Assume that $X$ has only planar singularities. Fix a polarization $H$ on $X$. Let $E$ be a torsion free sheaf on $X$ with pure rank $r$. Then $E$ is a flat limit of a flat family of $H$-stable vector bundles on $X$.*

Then we look at curves for which some of the singular points are ordinary nodes or ordinary cusps and give a stratification of the moduli space of stable reflexive sheaves. We could just avoid the stability condition and work with local deformation spaces.

**Theorem 1.** *Assume that $X$ has only planar singularities. Fix a polarization $H$ on $X$ and an $H$-stable torsion free sheaf $E$ on $X$ with pure rank $r$. Fix $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \subseteq \mathrm{Sing}(F)$ and assume that each point of $\Sigma_1$ (resp. $\Sigma_2$) is an ordinary node of of $X$ lying on one (resp. two) irreducible components of $X$, while every point of $\Sigma_3$ is an ordinary cusp of $X$. For any $P \in \Sigma_i$, $i = 1, 2$, let $c_P$ be the integer $r-a$ associated to $F_P$ in Remark 1 below. For any $P \in \Sigma_3$ le $c_P$ be the integer $c$ associated to $E_P$ in Remark 2 below. There is a neighborhood $\mathcal{U}$ of $E$ in the moduli space $\mathcal{M}$ of $H$-stable torsion free sheaves on $X$ with pure rank $r$ such that $\mathcal{U}$ has a stratification described in the following way. There are $\sum_{P\in\Sigma_1\cup\Sigma_3} c_P + \sum P \in \Sigma_2 c_P$ strata. Each strata is labelled by $\sum_{P\in\Sigma_1\cup\Sigma_3} c_P + \sum P \in \Sigma_2 c_P$ non-negative integers and in the decomposition (codimension and so on) the contribution of different points of $\Sigma$ are independent. If $P \in \Sigma_1 \cup \Sigma_3$ the labeling associated to $P$ is given by the choice of an integer $w_P$ such that $0 \le w_P \le c_P$. If $P \in \Sigma_2$ the labelling associated to $P$ is* *given by a pair of non-negative integers $z_{1,P}, z_{2,P}$ such that $z_{1,P} + z_{2,P} \leq c_P$. See Remarks 1 and 2 for the description of when a stratum is in the closure of another stratum.*

For more general sigularities we only know the following result.

**Theorem 2.** *Assume that $X$ has only planar singularities. Fix a polarization $H$ on $X$ and an $H$-stable torsion free sheaf $E$ on $X$ with pure rank $r$. Fix $S \subseteq \mathrm{Sing}(E)$ and assume that for every $P \in S$ there is an integer $c_P$ such that $1 \leq c_P \leq r$ and $\widetilde{E}_P \cong \widehat{\mathcal{O}}_{X,P}^{\oplus(r-c_P)} \oplus \widehat{m}_{X,P}^{\oplus c_P}$. For each $P \in S$ let $h_P$ be the number of the irreducible components of $X$ containing $P$. There is a neighborhood $\mathcal{U}$ of $E$ in the moduli space $\mathcal{M}$ of $H$-stable torsion free sheaves on $X$ with pure rank $r$ such that $\mathcal{U}$ has a stratification described in the following way. There are $\sum_{P\in S} c_P$ strata. Each strata is labelled by $\sum_{P\in S} h_P c_P$ integers and in the decomposition (codimension and so on) the contribution of different points of $S$ are independent. For every $P \in S3$ the labeling associated to $P$ is given by the choice of $h_P$ non-negative integers $z_{i,P}$, $1 \leq i \leq h_P$, such that $\sum_{i=1}^{h_P} z_{i,P} \leq w_P$.*

Of course, one can also use simultaneously Therorems 1 and 2 at different subsets of $\mathrm{Sing}(X)$.

## 2. Proofs and Related Results

*Proof of Proposition 1.* Fix an ample line bundle $H$ on $X$ and an integer $a$ such that $E^* \otimes H^{\otimes a}$ is spanned. There is an injective morphism $u : \mathcal{O}_X^{\otimes u} \to E^*$, which is surjective at each point of $\mathrm{Sing}(X)\backslash\mathrm{Sing}(E)$ (if any), that it has rank at least $r-1$ at each point of $X_{reg}$ and that at each of these points $P \in X_{reg}$ these $r$ sections of $E^*$ generates a subsheaf formally isomorphic to $\mathcal{O}_{X,P} \oplus \mathbf{m}_{X,P}$, where $\mathbf{m}_{X,P}$ is the maximal ideal of $\mathcal{O}_{X,P}$. Since $X$ has only planar singularities, it is Gorenstein. Hence every torsion free sheaf on $X$ is reflexive (see [3], Remark at p. 230). Hence the natural map $E \to E^{**}$ is an isomorphism. Hence the dual of $u$ induces an exact sequence

$$0 \to E \to \mathcal{O}_X^{\oplus r} \xrightarrow{f} Z \to 0 \tag{1}$$

with $Z$ a torsion sheaf, $Z$ supported by the union of $\mathrm{Sing}(E)$ and a finite set $A \subset X_{reg}$ and $Z_P$ has length 1 at each point of $A$. The proof of [4], Theorem A', is local, and hence we may just deform $Z$ (seen as an element of the Quot-scheme using the map $f$ of the exact sequence (1) at the points of $\mathrm{Sing}(F)\backslash A$. We find a flat family of surjections (all of them starting from $\mathcal{O}_X^{\oplus r}$) and the flat family of their kernels is the flat family $\{E_t\}_{t\in T}$ we were looking for. $\square$

**Remark 1.** Assume that $X$ has only planar singularities. Let $E$ be a torsion free sheaf on $X$ with pure rank $r$. Fix $P \in \mathrm{Sing}(X)$ such that $P$ is an ordinary node of $X$ lying of two irreducible components, $U$ and $V$, of $X$. There are non-negative integers $a, b_U, b_V$ such that $E_P \cong \mathcal{O}_{U\cup V,P}^{\oplus a} \oplus \mathcal{O}_{U,P}^{\oplus b_U} \oplus \mathcal{O}_{V,P}^{\oplus b_V}$ and $r = a + b_U = a + b_V$ (see [5], Proposition 3 of Huitieme Partie). Hence $b_U = b_V$ and $0 \le b_U \le r$. Conversely, for every pair of integers $(r, a)$ there is a torsion free sheaf $F$ on $X$ with pure rank $r$ with $F_P \cong \mathcal{O}_{U\cup V,P}^{\oplus a} \oplus \mathcal{O}_{U,P}^{\oplus(r-a)} \oplus \mathcal{O}_{V,P}^{\oplus(r-a)}$. We may also prescribe the numerical invariants of $F_Q$ at all $Q \in \mathrm{Sing}(X)\backslash\{0\}$ (e.g. we may take $F$ with $\mathrm{Sing}(F) \subseteq \{P\}$) and the degree of $F$. Obviously, $P \in \mathrm{Sing}(F)$ if and only if $a < r$. Now assume $a < r$. and that $E^*$ is spanned. Take a general $r$-dimensional linear subspace $W \subseteq H^0(X, E^*)$. Let $e_W : W \otimes \mathcal{O}_X \to E^*$ denote the evaluation map and $e_W^*$ its dual. $T$ be the connected component of the torsion sheaf $\mathrm{Coker}(e_W^*)$ supported by $P$. We have $\mathrm{length}(T) = r - a$ and $T$ is killed by the maximal ideal $m_{X,P}$ of the local ring $\mathcal{O}_{X,P}$. The map $e_W^*$ induces the exact sequence (1) and $T$ is the connected component of $Z$ supported by $P$. Fix non-negative integers $z_0, z_1, z_2$ such that $z_0+z_1+z_2 = r-a$. As in the proof of Proposition 1 we may deform the map $f$ of (1) to a map $f'$ such that nothing is changed at the points of $\mathrm{Sing}(F)\backslash\{P\}$, while $T$ is deformed to a sheaf $T'$ formed by a length $z_0$ supported by $P$, $z_1$ distinct points of $U \cap X_{reg}$ and $z_2$ distinct points of $V \cap X_{reg}$. Set $E' := \mathrm{Coker}(f')$. $E'_P \cong \mathcal{O}_{U\cup V,P}^{\oplus(r-z_0)} \oplus \mathcal{O}_{U,P}^{\oplus z_0} \oplus \mathcal{O}_{V,P}^{\oplus z_0}$. Roughly speaking, we have distributed $z_1$ of the singularities of $E_P$ along $U$ and $z_2$ of the singularities of $E_P$ along $V$. Now we fix $Q \in \mathrm{Sing}(X)$ and assume that $X$ is an ordinary node of $X$ lying in a unique irreducible component of $X$, say $U$. There is a unique integer $c$ such that $0 \le c \le a$ and $F \cong \mathcal{O}_{U,Q}^{\oplus c} \oplus m_{U,Q}^{\oplus(r-c)}$ (see [5], Proposition 3 of Huitieme Partie). Obviously, $c < r$ if and only if $Q \in \mathrm{Sing}(E)$. Now assume $c < r$ and that $E^*$ is spanned and take again a general $r$-dimensional linear subspace $W \subseteq H^0(X, E^*)$. Set $J$ be the connected component of the torsion sheaf $\mathrm{Coker}(e_W^*)$ supported by $Q$. The map $e_W^*$ induces the exact sequence (1) and $J$ is the connected component of $Z$ supported by $Q$. We have $\mathrm{length}(J) = r - c$ and $J$ is killed by $m_{X,P}$. Fix an integer $w$ such that $0 \le w < r - c$. As in the proof of Proposition 1 we may deform the map $f$ of (1) to a map $f'$ such that nothing is changed at the points of $\mathrm{Sing}(F)\backslash\{Q\}$, while $J$ is deformed to a sheaf $J'$ formed by a length $w$ supported by $Q$ and $r - c - w$ distinct points of $U \cap X_{reg}$ . Set $E' := \mathrm{Coker}(f')$. $E'_P \cong \mathcal{O}_{X,P}^{\oplus(r-w)} \oplus \mathcal{O}_{X,P}^{w}$. Now we fix $\Sigma = \Sigma_1 \cup \Sigma_2 \subseteq \mathrm{Sing}(F)$ and assume that each point of $\Sigma_1$ (resp. $\Sigma_2$) is an ordinary node of of $X$ lying on one (resp. two) irreducible components of $X$. If $E^*$ is spanned, then we may apply the previous discussion simultaneously to all points of $\Sigma$, using only deformations of $E$ which do not change the formal isomorphism class of $E$ at the points of $\mathrm{Sing}(F)\backslash\Sigma$. If $E^*$ is not spanned, then we twist it by a suitable ample line bundle and reduce to the previous case.

**Remark 2.** Assume that $X$ has only planar singularities. Let $E$ be a torsion free sheaf on $X$ with pure rank $r$. ix $Q \in \mathrm{Sing}(X)$ and assume that $X$ is an ordinary cusp of $X$. Hence $P$ lies a unique irreducible component of $X$, say $U$. Let $m_{U,P}$ or $m_{X,P}$ denote the maximal ideal of the local ring $\mathcal{O}_{U,Q} = \mathcal{O}_{U,Q}$. There is a unique integer $c$ such that $0 \le c \le a$ and $F \cong \mathcal{O}_{U,Q}^{\oplus c} \oplus m_{U,Q}^{\oplus(r-c)}$ (see [2] or [1], p. 24). Obviously, $c < r$ if and only if $Q \in \mathrm{Sing}(E)$. Now assume $c < r$ and that $E^*$ is spanned and take a general $r$-dimensional linear subspace $W \subseteq H^0(X, E^*)$. Set $J$ be the connected component of the torsion sheaf $\mathrm{Coker}(e_W^*)$ supported by $Q$. The map $e_W^*$ induces the exact sequence (1) and $J$ is the connected component of $Z$ supported by $Q$. We have $\mathrm{length}(J) = r - c$ and $J$ is killed by $m_{X,P}$. Fix an integer $w$ such that $0 \le w < r - c$. As in the proof of Proposition 1 we may deform the map $f$ of (1) to a map $f'$ such that nothing is changed at the points of $\mathrm{Sing}(F)\backslash\{Q\}$, while $J$ is deformed to a sheaf $J'$ formed by a length $w$ supported by $Q$ and $r-c-w$ distinct points of $U \cap X_{reg}$. Set $E' := \mathrm{Coker}(f')$. $E'_P \cong \mathcal{O}_{X,P}^{\oplus(r-w)} \oplus \mathcal{O}_{X,P}^{w}$. Now look again at Remark 1 and fix $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \subseteq \mathrm{Sing}(F)$ and assume that each point of $\Sigma_1$ (resp. $\Sigma_2$) is an ordinary node of of $X$ lying on one (resp. two) irreducible components of $X$, while every point of $\Sigma_3$ is an ordinary cusp of $X$. If $E^*$ is spanned, then we may apply the previous discussion simultaneously to all points of $\Sigma$, using only deformations of $E$ which do not change the formal isomorphism class of $E$ at the points of $\mathrm{Sing}(F)\backslash\Sigma$. If $E^*$ is not spanned, then we twist it by a suitable ample line bundle and reduce to the previous case.

*Proof of Theorem 1.* Use the proof of Proposition 1 and Remarks 1 and 2. $\square$

*Proof of Theorem 2.* Use the proof of Proposition 1 and then the proof of Remark 1, just using $h_P$ irreducible components. $\square$

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## References

[1] P.R. Cook, *Local and Global Aspects of the Moduli Theory of Singular Curves*, Ph.D. Thesis, Liverpool (1993).

[2] G.-M. Greuel, H. Knörrer, Einfache Kurvensingularitäten und torsionfreie Moduln, *Math. Ann.*, **270** (1985), 417-425.

[3] C.J. Rego, Compactification of the space of vector bundles on a singular curve, *Comm. Math. Helvetici*, **57** (1982), 226-236.

[4] C.J. Rego, Deformation of modules on curves and surfaces, In: *Singularities, Representations of Algebras, and Vector Bundles*, Lamprecht (1985), 157-165; Lecture Notes in Math., **1273**, Springer, Berlin (1987).

[5] C. Seshadri, Fibrés vectoriels sur les courbes algébriques, *Astérisque*, **96** (1982).