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SECOND-ORDER QUASI-LINEAR BOUNDARY-VALUE PROBLEMS

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Abstract: We discuss the well-posedness of quasi-linear hyperbolic second-order boundary-value problems in the half-space $\Omega = \Re^{d-1} \times (0, \infty)$, with homogeneous boundary conditions. At first, we consider second-order boundary-value problems for linear systems having smooth coefficients, that depend on the space-variable and on the time-variable. We prove the well-posedness in the space $C^1([0,T], H^{s-2}(\Omega)^d)$, with $s \in \Re$. Next, by applying the previous result, we study quasi-linear boundary-value problems and establish the existence of a non-null solution in $C([0,T], H^{s'}(\Omega)^d)$ with $s' \in (0,s)$ and $s > d/2 + 2$. The main result is proved by approximating the solution through an iteration scheme.

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1. Introduction

We study the well-posedness of the following quasi-linear hyperbolic second-order boundary-value problem, with homogeneous boundary conditions

$$\partial_t^2 u + \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} B^{\alpha,\beta}(u)\partial_\alpha \partial_\beta u + \sum_{\alpha=1}^{d} C^\alpha(u)\partial_\alpha u - D(u)u = 0, \quad x \in \Omega,\ t \in \Re,$$

$$\sum_{\beta=1}^{d-1} B^{d,\beta}(u)\partial_\beta u + C^d(u)u = 0, \quad x \in \partial\Omega,\ t \in \Re; \tag{1}$$

where Ω denotes the half-space $\Omega = \Re^{d-1} \times (0, \infty)$; $u = u(x,t)$ represents the

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unknown vector field with values in R^d; for every α, $\beta = 1, \cdots, d$, the functions $B^{\alpha,\beta}$, C^α, D are defined in R^d with values in the space of $d \times d$ real matrices and belong to $C^\infty(R^d)$; moreover, we assume that the function $B^{d,d}$ is the null-matrix.

In the papers [2] and [3], we discussed the well-posedness of linear evolution boundary-value problems in a half-space, in the case where the coefficients of the associated differential operator are either constant or depend on the space variable x. As for the constant-coefficient case, the well-posedness in the space $C^1((0,+\infty), H^1(\Omega)^d)$ has been established in [2] and [3] by means of the Fourier-Laplace analysis and the application of Hille-Yosida Theorem. In [3] we proved that the existence of the solution to the boundary-value problem with variable coefficients, relies on the well-posedness of the BVP with frozen coefficients in a fixed point of Ω. A sufficient condition has been established similarly to the constant-coefficient problem, by proving that the associated linear differential operator satisfies the assumptions of Hille-Yosida Theorem.

As far as the quasi-linear problem (1) is concerned, we shall prove a preliminary result concerning the well-posedness of boundary-value problems for linear second-order operators, having coefficients which depend on the space-variable x and on the time-variable t

$$\partial_t^2 u + \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} B^{\alpha,\beta}(x,t)\partial_\alpha \partial_\beta u + \sum_{\alpha=1}^{d} C^\alpha(x,t)\partial_\alpha u - D(x,t)u = 0, \qquad x \in \Omega, \quad t \in \Re,$$

$$\sum_{\beta=1}^{d-1} B^{d,\beta}(x,t)\partial_\beta u + C^d(x,t)u = 0, \quad x \in \partial\Omega,\ t \in \Re. \tag{2}$$

By following the procedure introduced in [1] for first-order initial-value problems, we shall assume that the second-order operator satisfies a suitable estimate, in order to show the existence of a non-null function $u \in C^1([0,T], H^{s-2}(\Omega)^d)$, with $s \in \Re$, that provides the unique solution to the IBVP (2). In [1] Benzoni-Gavage and Serre studied the well-posedness of linear Cauchy problems for first-order symmetrizable systems, with variable coefficients. The main tool in their proof is the derivation of a priori estimates, that imply the existence and the uniqueness of the solution $u \in C([0,T], H^s(R^d)^d)$, for initial data $u(0) \in H^s$.

The result on the well-posedness of problem (2) will be applied to discuss the case where the dependence of the matrices $B^{\alpha,\beta}$, C^α, D , with respect to (x,t) occurs via a known but not necessarily smooth function.

The main theorem of the paper deals with the well-posedness of the quasi-linear BVP (1). The techniques applied to perform the proof of the result are similar to those introduced in [1], in the case of first-order quasi-linear Cauchy problems. The authors proved in [1] the existence and uniqueness of the solution in $C([0,T], H^s(R^d)^d)$, for Friedrichs-symmetrizable systems, achieving the initial value in H^s. As for the second-order quasi-linear system (1), we shall define an iteration scheme, whose boundary-value problems at each step, turn out to be well-posed, for they are included in the problems of the form (2). By studying the convergence properties of the sequence of the solutions and passing to the limit in the iteration scheme, we shall prove the existence of a non-null solution of the BVP (1), belonging to the space $C([0,T], H^{s'}(\Omega)^d)$, with $s' \in (0,s)$ and $s > d/2+2$.

2. Variable-Coefficient Systems

We consider the boundary-value problem (2) and define the associated linear second-order differential operator

$$L = \partial_t^2 u + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} B^{\alpha,\beta}(x,t)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} C^\alpha(x,t)\partial_\alpha - D(x,t)\,. \tag{3}$$

We suppose that for every α, $\beta = 1,\cdots,d$, $B^{\alpha,\beta} = B^{\beta,\alpha}$. The adjoint operator of L turns out to be defined by

$$L^\star = \partial_t^2 + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d}\Big(\partial_\beta(\partial_\alpha B^{\alpha,\beta^T}) + 2\partial_\alpha B^{\alpha,\beta^T}\partial_\beta + B^{\alpha,\beta^T}\partial_\alpha\partial_\beta\Big)$$
$$- \sum_{\alpha=1}^{d}\Big(C^{\alpha T}\partial_\alpha + \partial_\alpha(C^{\alpha T})\Big) - D^T\,. \tag{4}$$

As far as the existence of the solution of the BVP (2) is concerned, we prove the following result.

Proposition 2.1. *Consider the boundary-value problem (2). Let $s \in \Re$, $s > d/2+2$, and $T > 0$. Assume the following conditions are satisfied:*

(i) For all α, $\beta = 1,\cdots,d$, $B^{\alpha,\beta}$, C^α, $D \in C^\infty(\Omega\times[0,T])$ and are bounded as well as their derivatives; in addition, $B^{\alpha,\beta} = B^{\beta,\alpha}$ and $B^{d,d}$ is the null matrix.

(ii) There exists a positive real constant c such that for every $\phi \in C^\infty([0,T];$ $H^{-s}(\Omega))$ and for every $t \in [0,T]$,

$$\int_0^T \|L^\star\phi\|^2_{H^{-s}(\Omega)}\,d\tau \geq c\,\|\phi(t)\|^2_{H^{-s}(\Omega)}. \tag{5}$$

Then the boundary-value problem (2) admits a non-null solution

$$u \in C^1([0,T],H^{s-2}(\Omega)^d)\bigcap L^2([0,T],H^s(\Omega)^d).$$

Proof. Let $g \in H^s(\Omega)^d$ a non-null function and define for every $\phi \in C^\infty([0,T];H^{-s}(\Omega))$, the linear functional f as follows:

$$f(L^\star\phi) =< g,\phi(0) >_{H^s,H^{-s}}\ .$$

The functional f turns out to be well-defined on the space

$$L^\star(C^\infty([0,T];H^{-s}(\Omega)));$$

since the operator $L^\star$ is one-to-one due to (ii). Furthermore, f is bounded. Because of (5) and the Cauchy-Schwarz inequality, we have indeed $\mid f(L^\star\phi)\mid\leq c^{-1/2}\|g\|_{H^s}\|L^\star\phi\|^2_L([0,T],H^{-s}(\Omega))$. By means of Hahn-Banach Theorem, f extends to a continuous linear form F, defined in the space $L^2([0,T],H^{-s}(\Omega)^d)$. By Riesz Representation Theorem, there exists a function $u \in L^2([0,T], H^{-s}(\Omega)^d)'$ such that

$$F(v) = \int_0^T < u,v >_{H^s,H^{-s}}\ d\tau.$$

If $\phi \in C^\infty([0,T];H^{-s}(\Omega))$ then

$$\int_0^T < u,L^\star\phi >_{H^s,H^{-s}}\ d\tau =< g,\phi(0) >_{H^s,H^{-s}}\ .$$

Let $\psi \in C_0^\infty(\Omega\times(0,T))$, hence, we obtain

$$\int_0^T < u,L^\star\psi >_{H^s,H^{-s}}\ d\tau = 0.$$

Therefore, $\int_0^T < Lu,\psi >_{H^s,H^{-s}}\ d\tau = 0$; as a consequence of this result, the function u, which belongs to $L^2([0,T],H^s(\Omega)^d)$, turns out to be a solution of system (2), in the sense of distributions. Let $\chi \in C_0^\infty(\Re^{d-1}\times[0,\infty)\times(0,T))$. Then $\int_0^T < Lu,\chi >_{H^s,H^{-s}}\ d\tau = 0$. By integrating by parts, we obtain

$$\int_0^T \int_{\Re^{d-1}} < \sum_{\beta=1}^{d-1} B^{d,\beta}\partial_\beta u + C^d u,\chi > (y,0,\tau)dyd\tau = 0. \tag{6}$$

In view of the previous result, the function u satisfies the homogeneous boundary condition. Since $\partial_t^2 u \in L^2([0,T],H^{s-2}(\Omega)^d)$, the solution u belongs to $C^1([0,T],H^{s-2}(\Omega)^d)$. Moreover, if $\psi \in C_0^\infty(\Omega \times [0,T))$, then integrating by parts, we get the following identity

$$< g, \psi(0) >_{H^s,H^{-s}} = < \partial_t u(0), \psi(0) >_{H^s,H^{-s}} - < u(0), \partial_t \psi(0) >_{H^s,H^{-s}} .$$

Let us consider now a function $v \in H^s(\Omega \times [0,T])^d$, with $s > d/2+2$, and denote by L_v the second-order linear differential operator

$$L_v = \partial_t^2 + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} B^{\alpha,\beta}(v(x,t))\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{d} C^\alpha(v(x,t))\partial_\alpha - D(v(x,t))\,. \tag{7}$$

Let $L_v^\star$ be the adjoint operator of L_v. Similarly to Proposition 2.1, we obtain the following result, which will be applied to establish the well-posedness of the BVP (1).

Proposition 2.2. *Let $v \in H^s(\Omega \times [0,T])^d$, with $s > d/2+2$ and let L_v be the second-order linear differential operator defined in (7). We consider the boundary-value problem*

$$\begin{aligned} L_v u = 0, \qquad & x \in \Omega,\ t \in \Re, \\ \sum_{\beta=1}^{d-1} B^{d,\beta}(v(x,t))\partial_\beta u + C^d(v(x,t))u = 0, \quad & x \in \partial\Omega,\ t \in \Re; \end{aligned} \tag{8}$$

and assume the following conditions are fulfilled:

(i) For every α, $\beta = 1,\cdots,d$, the matrix-valued functions $B^{\alpha,\beta}$, C^α, $D \in C^\infty(R^d)$ and are bounded as well as their derivatives; moreover, $B^{\alpha,\beta} = B^{\beta,\alpha}$ and $B^{d,d}$ is a null matrix-valued function.

(ii) There exists a positive constant C such that for every $\phi \in C_0^\infty(\Omega \times [0,T])$,

$$\int_0^T \|L_v^\star \phi\|^2_{H^{-s}(\Omega)}\, d\tau \geq C\, \|\phi(t)\|^2_{H^{-s}(\Omega)},$$

in the case where $t \in [0,T]$.

Then there exists a non-null function $u \in H^s(\Omega \times [0,T])^d$, which provides a solution to the boundary-value problem (8).

Proof. Because $s > d/2+2$, by Sobolev embedding, the coefficients of the operator L_v and of the adjoint $L_v^\star$ turn out to be bounded in $\Omega \times [0,T]$. Similarly to the proof performed for the Proposition 2.1, one proves that the BVP (8) admits a solution $u \in C^1([0,T],H^{s-2}(\Omega)^d) \bigcap L^2([0,T],H^s(\Omega)^d)$. According to a result proved in [1], since $u \in L^2([0,T],H^s(\Omega)^d)$, by finite induction on the Sobolev exponent, we obtain that the function $u \in H^s(\Omega \times [0,T])^d$.

3. Well-Posedness of Quasi-Linear Problems

We shall be concerned in this section with the well-posedness of the boundary-value problem (1) in the space $C([0,T], H^{s'}(\Omega)^d)$ with $s' \in (0,s)$ and $s > d/2+2$. We shall prove the existence of a unique solution of the initial boundary-value problem by defining an iteration scheme and discussing the convergence. Let us introduce the following notation for the quasi-linear second-order differential operator associated with problem (1)

$$\tilde{L}u = \partial_t^2 u + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} B^{\alpha,\beta}(u)\partial_\alpha\partial_\beta u + \sum_{\alpha=1}^{d} C^\alpha(u)\partial_\alpha u - D(u)u. \tag{9}$$

In addition, let us fix a non-null function $g \in H^s(\Omega)^d$ and let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of smooth mollifiers. We define the functions $g^k = g * \rho_k$, for every $k \in \mathbb{N}$; thus, $g^k \in C^\infty(\Omega)$. Let the sequence $(\rho_k)_{k\in\mathbb{N}}$ be such that $\|g^k - g\|^2_{H^1(\Omega)} \le \text{const}\,\epsilon_k\,\|g\|^2_{H^2(\Omega)}$, and $\epsilon_k \to 0$, as $k \to +\infty$.

Theorem 3.1. *Let $s > d/2+2$ and consider the boundary-value problem (1). We suppose the following conditions are satisfied:*

(i) for every α, $\beta = 1,\cdots,d$, the matrix-valued functions $B^{\alpha,\beta}$, C^α, $D \in C^\infty(R^d)$ and are bounded as well as their derivatives; in addition, $B^{\alpha,\beta} = B^{\beta,\alpha}$; for every $l = 1,\cdots,d$, $\partial_{u_l}B^{\alpha,\beta}(0) = 0$, $\partial_{u_l}C^\alpha(0) = 0$, $\partial_{u_l}D(0) = 0$;

(ii) if $v \in H^s(\Omega\times[0,T])^d$, there exists a positive constant λ such that for every $\phi \in C_0^\infty(\Omega\times[0,T])$,

$$\int_0^T \|L_v^\star\phi\|^2_{H^{-s}(\Omega)}\,d\tau \ge \lambda\,\|\phi(t)\|^2_{H^{-s}(\Omega)},$$

where $t \in [0,T]$;

(iii) if r is a positive real number with $2 \le r \le s$, there is a positive constant Γ such that for all $v \in H^s(\Omega\times[0,T])^d$ and for all $\phi \in H^r(\Omega\times[0,T])^d$,

$$\|\phi(t)\|^2_{H^{r-1}(\Omega)} \le T\,\Gamma(\sup_{t\in[0,T]}\|L_v\phi\|^2_{H^{r-2}(\Omega)}) + \|\phi(0)\|^2_{H^{r-1}(\Omega)} + \|\partial_t\phi(0)\|^2_{H^{r-1}(\Omega)} \tag{10}$$

as $t \in [0,T]$.

Then there exists a non-null function u in the space $C([0,T], H^{s'}(\Omega)^d)$ with $s' \in (0,s)$, that turns out to be the unique solution to the initial boundary-value problem (1), with initial conditions $\partial_t u(x,0) = g(x)$, $u(x,0) = 0$ as $x \in \Omega$.

Proof. The proof of the well-posedness of the BVP (1) will be carried out by studying the following iteration scheme

$$\partial_t^2 u^k + \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} B^{\alpha,\beta}(u^{k-1})\partial_\alpha\partial_\beta u^k + \sum_{\alpha=1}^{d} C^\alpha(u^{k-1})\partial_\alpha u^k - D(u^{k-1})u^k = 0, \qquad x \in \Omega, \quad t \in \Re,$$

$$\sum_{\beta=1}^{d-1} B^{d,\beta}(u^{k-1})\partial_\beta u^k + C^d(u^{k-1})u^k = 0, \quad x \in \partial\Omega, \quad t \in \Re. \tag{11}$$

Let us denote by L_{k-1} the linear differential operator $L_{u^{k-1}}$ associated with system (11), and by S_{k-1} the corresponding boundary operator. Let us set $u^0 = g$. The function u^k is defined by induction process as the unique solution of the initial boundary-value problem, with initial data $\partial_t u^k(x,0) = g^k(x)$ and $u^k(x,0) = 0$, as $x \in \Omega$, where the functions g^k are as defined above. Since for every $k \in \mathbb{N}$ the assumptions of Proposition 2.2 are satisfied by the operators L_k, the BVPs (11) turn out to be well-posed.

Let $\delta > 0$. Assume that $\|g\|_{H^s(\Omega)} \leq \delta$. Due to our choice, $\|u^0\|_{C([0,T];H^s(\Omega))} \leq \delta$. We shall prove below by induction that for every $k \in \mathbb{N}$, $\sup_{t\in[0,T]} \|u^k\|_{H^s(\Omega)} \leq \delta$. Let us assume that the estimate holds true for all $k \leq m$ and prove that it is verified in the case where $k = m+1$, provided that T is small enough. Let us define $\omega_j^{k+1} = \partial_{x_j} u^{k+1}$, for every $j = 1, \cdots, d$.

The functions ω_j^{k+1} solve the problems

$$L_k\omega_j^{k+1} = -\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \partial_{x_j}(B^{\alpha,\beta}(u^k))\partial_\alpha\partial_\beta u^{k+1} - \sum_{\alpha=1}^{d} \partial_{x_j}(C^\alpha(u^k))\partial_\alpha u^{k+1} + \partial_{x_j}(D(u^k))u^{k+1}, \qquad x \in \Omega, \quad t \in \Re,$$

$$S_k\omega_j^{k+1} = -\sum_{\beta=1}^{d-1} \partial_{x_j}(B^{d,\beta}(u^k))\partial_\beta u^{k+1} - \partial_{x_j}(C^d(u^k))u^{k+1}, \qquad x \in \partial\Omega, \quad t \in \Re. \tag{12}$$

Since $\sup_{t\in[0,T]} \|u^k\|_{H^s(\Omega)} \leq \delta$ for every $k \leq m$ and $s > d/2+2$, by Sobolev embedding, there exists a positive constant μ, depending on δ such that $\|u^k\|_{L^\infty(\Omega\times(0,T))} \leq \mu$, $\|\nabla_x u^k\|_{L^\infty(\Omega\times(0,T))} \leq \mu$, $\|\partial_\alpha\partial_\beta u^k\|_{L^\infty(\Omega\times(0,T))} \leq \mu$, for α, $\beta = 1, \cdots, d$.

Let us consider system (13) and introduce the notation

$$F_j^{k+1} = -\sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \partial_{x_j}(B^{\alpha,\beta}(u^k))\partial_\alpha\partial_\beta u^{k+1}$$
$$- \sum_{\alpha=1}^{d} \partial_{x_j}(C^{\alpha}(u^k))\partial_\alpha u^{k+1} + \partial_{x_j}(D(u^k))u^{k+1}. \quad (13)$$

Thanks to Moser estimates (see reference [1]), for $\partial_{u_l}B^{\alpha,\beta} \in C^\infty(R^d)$, $\partial_{u_l}B^{\alpha,\beta}(0) = 0$, and $u^k(\cdot,t) \in H^s(\Omega)\cap L^\infty(\Omega)$, the function $\partial_{u_l}B^{\alpha,\beta}(u^k)$ belongs to $H^s(\Omega)$ and there exists a constant C, which depends on s and $\|u^k\|_{L^\infty(\Omega)}$, such that $\|\partial_{u_l}B^{\alpha,\beta}(u^k)\|_{H^s(\Omega)} \le C(\|u^k\|_{L^\infty(\Omega)}, s)\|u^k\|_{H^s(\Omega)}$.

Since for every $j, i = 1, \cdots, d$ and $k \le m$, $\partial_{x_j}u_i^k(\cdot,t) \in H^{s-1}(\Omega)\cap L^\infty(\Omega)$, the application of Moser results (see [1]) yields the following estimates

$$\|\partial_{u_i}B^{\alpha,\beta}(u^k)\partial_{x_j}u_i^k\|_{H^{s-1}(\Omega)} \le c_1(s)\times$$
$$\Big(\|\partial_{u_i}B^{\alpha,\beta}(u^k)\|_{L^\infty(\Omega)}\,\|\partial_{x_j}u_i^k\|_{H^{s-1}(\Omega)} + \|\partial_{x_j}u_i^k\|_{L^\infty(\Omega)}\,\|\partial_{u_i}B^{\alpha,\beta}(u^k)\|_{H^{s-1}(\Omega)}\Big)$$
$$\le c_1(s)\left(\mu\|\partial_{u_i}B^{\alpha,\beta}(u^k)\|_{L^\infty(\Omega)} + \mu^2 C\right). \quad (14)$$

Due to Moser estimates, we obtain, because the functions $\partial_\alpha\partial_\beta u^{k+1}(\cdot,t) \in H^{s-2}(\Omega)\cap L^\infty(\Omega)$,

$$\|\partial_{x_j}(B^{\alpha,\beta}(u^k))\partial_\alpha\partial_\beta u^{k+1}\|_{H^{s-2}(\Omega)}$$
$$\le c_2(s)\Bigg((\sum_{i=1}^{d} \|\partial_{u_i}B^{\alpha,\beta}(u^k)\|_{L^\infty(\Omega)}\mu + \mu^2 dC)\|\partial_\alpha\partial_\beta u^{k+1}\|_{L^\infty(\Omega)}$$
$$+ \|u^{k+1}\|_{H^s(\Omega)}\|\,\|\partial_{x_j}(B^{\alpha,\beta}(u^k))\|_{L^\infty(\Omega)}\Bigg), \quad (15)$$

whence, by Sobolev embedding,

$$\sup_{t\in[0,T]} \|\partial_{x_j}(B^{\alpha,\beta}(u^k))\partial_\alpha\partial_\beta u^{k+1}\|_{H^{s-2}(\Omega)} \le c_3(s,\mu)\sup_{t\in[0,T]}\|u^{k+1}\|_{H^s(\Omega)}. \quad (16)$$

By means of similar computations, we get

$$\sup_{t\in[0,T]} \|\partial_{x_j}(C^{\alpha}(u^k))\partial_\alpha u^{k+1}\|_{H^{s-2}(\Omega)} \le c_4(s,\mu)\sup_{t\in[0,T]}\|u^{k+1}\|_{H^s(\Omega)}; \quad (17)$$

and

$$\sup_{t\in[0,T]} \|\partial_{x_j}(D(u^k))u^{k+1}\|_{H^{s-2}(\Omega)} \le c_5(s,\mu)\sup_{t\in[0,T]}\|u^{k+1}\|_{H^s(\Omega)}. \quad (18)$$

Hence,

$$\sup_{t\in[0,T]} \|F_j^{k+1}\|_{H^{s-2}(\Omega)} \le \text{const.}(s,\mu)\sup_{t\in[0,T]}\|u^{k+1}\|_{H^s(\Omega)}. \quad (19)$$

As a consequence of (19),

$$\sup_{t\in[0,T]} \|L_{k+1}\omega_j^{k+1}\|_{H^{s-2}(\Omega)} \leq \text{const.}(s,\mu) \sup_{t\in[0,T]} \|u^{k+1}\|_{H^s(\Omega)}. \tag{20}$$

Thanks to condition (iii),

$$\|\omega_j^{k+1}(t)\|^2_{H^{s-1}(\Omega)} \leq \Gamma T(\sup_{t\in[0,T]} \|L_k\omega_j^{k+1}\|^2_{H^{s-2}(\Omega)}) + \|\partial_t\omega_j^{k+1}(0)\|^2_{H^{s-1}(\Omega)}. \tag{21}$$

Because of (20),

$$\|\omega_j^{k+1}(t)\|^2_{H^{s-1}(\Omega)} \leq \Gamma T(\text{const.}(s,\mu) \sup_{t\in[0,T]} \|u^{k+1}\|^2_{H^s(\Omega)}) + \|\partial_{x_j} g^{k+1}\|^2_{H^{s-1}(\Omega)}; \tag{22}$$

whence

$$\sup_{t\in[0,T]} \|u^{k+1}\|^2_{H^s(\Omega)} \leq \Gamma T(\text{const.}(s,\mu) \sup_{t\in[0,T]} \|u^{k+1}\|^2_{H^s(\Omega)}) + \|g^{k+1}\|^2_{H^s(\Omega)}; \tag{23}$$

Since $\|g\|_{H^s(\Omega)} \leq \delta$, we obtain for all $k \in \mathbb{N}$, $\|g^k\|_{H^s(\Omega)} \leq \delta$. If T is small enough, then

$$\sup_{t\in[0,T]} \|u^{k+1}\|^2_{H^s(\Omega)} \leq \delta^2. \tag{24}$$

By induction process, we deduce that $\sup_{t\in[0,T]} \|u^{k+1}\|_{H^s(\Omega)} \leq \delta$, for all $k \in \mathbb{N}$.

We prove now that the sequence $(u^k)_{k\in\mathbb{N}}$ satisfies a contraction property in $C([0,T], L^2(\Omega)^d)$. Let us denote by W^k the difference $u^k - u^{k-1}$. The functions W^k turn out to be solutions of the following system

$$L_{k-1}W^k = \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \Big(B^{\alpha,\beta}(u^{k-2}) - B^{\alpha,\beta}(u^{k-1})\Big)\,\partial_\alpha\partial_\beta u^{k-1}$$
$$+ \sum_{\alpha=1}^{d} \Big(C^{\alpha}(u^{k-2}) - C^{\alpha}(u^{k-1})\Big)\,\partial_\alpha u^{k-1} - \Big(D(u^{k-2}) - D(u^{k-1})\Big)\,u^{k-1}. \tag{25}$$

Since the L^∞-norms of the first-order derivatives of the functions $B^{\alpha,\beta}$, C^α, D, are bounded, by the Mean-Value Theorem,

$$\|\Big(B^{\alpha,\beta}(u^{k-2}) - B^{\alpha,\beta}(u^{k-1})\Big)\,\partial_\alpha\partial_\beta u^{k-1}\|_{L^2(\Omega)}$$
$$\leq \|(DB^{\alpha,\beta}(w))\cdot(u^{k-2} - u^{k-1})\|_{L^2(\Omega)}\|\partial_\alpha\partial_\beta u^{k-1}\|_{L^\infty(\Omega)}$$
$$\leq \mu\,\text{const.}\,\|W^{k-1}\|_{L^2(\Omega)}. \tag{26}$$

Similarly, it holds true

$$\begin{aligned}&\| \left(C^\alpha(u^{k-2}) - C^\alpha(u^{k-1})\right) \partial_\alpha u^{k-1}\|_{L^2(\Omega)} \leq \mu \,\text{const.}\, \|W^{k-1}\|_{L^2(\Omega)};\\ &\text{and}\\ &\| \left(D(u^{k-2}) - D(u^{k-1})\right) u^{k-1}\|_{L^2(\Omega)} \leq \mu \,\text{const.}\, \|W^{k-1}\|_{L^2(\Omega)}.\end{aligned} \tag{27}$$

Therefore,

$$\|L_{k-1}W^k\|_{C([0,T];L^2(\Omega))} \leq \text{const.}\, \|W^{k-1}\|_{C([0,T];L^2(\Omega))}. \tag{28}$$

Thanks to (iii), $\sup_{t\in[0,T]} \|W^k\|^2_{L^2(\Omega)} \leq \Gamma\, T(\sup_{t\in[0,T]} \|L_{k-1}W^k\|^2_{L^2(\Omega)}) + \|g^k - g^{k-1}\|^2_{H^1(\Omega)}$; whence, because of (28),

$$\sup_{t\in[0,T]} \|W^k\|^2_{L^2(\Omega)} \leq \Gamma\, T(\text{const.} \sup_{t\in[0,T]} \|W^{k-1}\|^2_{L^2(\Omega)}) + \|g^k - g^{k-1}\|^2_{H^1(\Omega)}. \tag{29}$$

Provided that T is small enough, (e.g., $2T\Gamma$ const. < 1), we obtain

$$\sup_{t\in[0,T]} \|W^k\|^2_{L^2(\Omega)} \leq \frac{1}{2} \sup_{t\in[0,T]} \|W^{k-1}\|^2_{L^2(\Omega)} + \|g^k - g^{k-1}\|^2_{H^1(\Omega)} \tag{30}$$

By means of a suitable choice of the sequence $(\epsilon_k)_k$, the infinite series $\sum \|g^k - g^{k-1}\|^2_{H^1(\Omega)}$ turns out to be convergent. As a consequence of this result, we obtain that the sequence of functions $(u^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];L^2(\Omega))$. Let us denote by u the limit of $(u^k)_{k\in\mathbb{N}}$. As proved above, the sequence $(u^k(t))_{k\in\mathbb{N}}$ is bounded in $H^s(\Omega)$. Hence, $(u^k(t))_{k\in\mathbb{N}}$ turns out to be weakly convergent in $H^s(\Omega)$; by uniqueness of the weak limit, $u(\cdot,t) \in H^s(\Omega)$, for all $t \in [0,T]$. In addition, by interpolation, $(u^k)_{k\in\mathbb{N}}$ is convergent in $C([0,T], H^{s'}(\Omega))$ for every $s' \in (0,s)$.

Passing to the limit in the iteration scheme, let us study the convergence of the sequence $(L_{k-1}u^k)_{k\in\mathbb{N}}$. For the sake of convenience, we write $L_{k-1}u^k = \partial_t^2 u^k + Q_{k-1}u^k$ and $\tilde{L}u = \partial_t^2 u + \tilde{Q}u$. Since $B^{\alpha,\beta}$, C^α, D, are bounded in $\Omega\times[0,T]$, the sequence $(Q_{k-1}u^k)_{k\in\mathbb{N}}$ converges to $\tilde{Q}u$, as $k\to\infty$, in $C([0,T];L^2(\Omega))$. As a matter of fact, if $s' \geq 2 + d/2$, by interpolation and Sobolev embeddings, the convergence holds true in $C([0,T];C_b(\Omega))$, where $C_b(\Omega)$ is the space of continuous and bounded functions defined in Ω. As far as the convergence of the iteration scheme (11) is concerned, the sequence $(\partial_t^2 u^k)_k$ is convergent to $-\tilde{Q}u$, in $C([0,T], C_b(\Omega))$. Furthermore, the function u admits a second-order derivative with respect to t, and it holds true that $\partial_t^2 u = -\tilde{Q}u$, in $\Omega\times[0,T]$. Thus, $\tilde{L}u = 0$; moreover, passing to the limit, the function u turns out to satisfy the boundary condition and the initial datum.

The solution to the initial boundary-value problem (1) is unique in the space $C([0,T];H^2(\Omega))$. Denoting by w the difference $u_1 - u_2$, by means of the same computations carried out to prove the estimate (30), we obtain indeed $\sup_{t\in[0,T]} \|w\|^2_{L^2(\Omega)} \leq \Gamma\, T\, \text{const.} \sup_{t\in[0,T]} \|w\|^2_{L^2(\Omega)}$. If T is sufficiently small, by the previous estimate, we get $\sup_{t\in[0,T]} \|w\|^2_{L^2(\Omega)} = 0$. Hence, $u_1 = u_2$.

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