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A CHARACTERIZATION OF CURVES WITH ARITHMETIC GENUS ZERO USING TORSION FREE SHEAVES

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we give a few characterizations in terms of torsion free sheaves of reduced and connected curves with arithmetic genus zero.

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1. Introduction

Let X be a reduced and connected projective curve defined over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. Let $\mathcal{B}(X)$ denote the set of the irreducible components of X. For any union A of some the irreducible components of X let $A^{[c]}$ denote the union of all $T \in \mathcal{B}(X)\backslash A$. Let $f : C \to X$ be the normalization map. Hence C is a smooth curve and C is connected if and only if X is irreducible. For all integers $n > r > 0$ let $G(r, n)$ denote the Grassmannian of all $(n - r)$-dimensional linear subspaces of $\mathbb{K}^{\oplus r}$. For any a pure rank r vector bundle E on X and any n-dimensional linear subspace $V \subseteq H^0(X, E)$ spanning E let $h_{E,V} : X \to G(r, n)$ denote the morphism associated to the pair (E, V). Hence $\det(E) \cong h_{E,V}(\mathcal{O}_{G(r,n)}(1))$. Set $h_E := h_{E,H^0(X,E)}$. Let F be a torsion free sheaf on X. F has *pure rank* r if $F|X_{reg}$ is a rank r locally free sheaf. Assume that F has pure rank. Set $\widetilde{F} := f^*(F)/\mathrm{Tors}(f^*(F))$. Since C is a smooth curve, every torsion free sheaf on C is locally free. Thus $\widetilde{F}$ is a rank r vector bundle. F is said to be *strongly positive* (resp. *totally nef*) if $\widetilde{F}$ is ample (resp. its restriction to each connected component of C is nef). Hence a

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strongly positive pure rank sheaf is totally nef.

Here we prove the following results concerning the case $p_a(X) = 0$.

Proposition 1. *Let X be a reduced and connected projective curve. Assume $p_a(X) = 0$. Let E be an ample vector bundle on X. Then $h^1(X, E) = 0$, E is spanned and h_E is an embedding. Let F be a strongly positive torsion free sheaf F with pure rank. Then the ample vector bundle $\widetilde{F}$ is spanned and $\mathcal{O}_{\mathbb{P}(\widetilde{F})}(1)$ is very ample.*

Proposition 2. *Let X be a reduced and connected projective curve. $p_a(X) = 0$ if and only if $h^1(X, F) = 0$ for every a totally nef torsion free sheaf F on X with pure rank r.*

Proposition 3. *Assume $p_a(X) = 0$. Let F be a strongly positive torsion free sheaf on X with pure rank r. Then F is spanned.*

2. The Proofs

Remark 1. Let X be a reduced projective curve. Then $\deg(\omega_X) = -2\chi(\mathcal{O}_X)$ (use twice duality or see, [1], [3], Proposition 3.1.6, part 2, for $\mathcal{F} := \mathcal{O}_X$).

Remark 2. Let F be a torsion free sheaf on X with pure rank r. We have $\chi(F) = \deg(F) + r(1 - g)$. Fix $P \in X$. Let $\ell F, P$ be the minimal integers $\dim_{\mathbb{K}}(M/F_P)$, where F_P is the germ of F at P and M is a rank r free $\mathcal{O}_{X,P}$. Hence $\ell(F, P)$ is a non-negative integer and $\ell(F, P) = 0$ if F is locally free at P. Hence $\ell(F, P) = 0$ for any F if $P \in X_{reg}$. Set $\ell(F) := \sum_{P \in \mathrm{Sing}(X)} \ell(F, P)$. Hence $\ell(F)$ is a non-negative integer and $\ell(F) = 0$ if and only if F is locally free. It is easy to check that $\deg(\widetilde{F}) = \deg(F) - \ell(F)$ ([3], Proposition 3.2.4).

Lemma 1. *Let F, G be pure rank 1 torsion free sheaves on X such that $F \subseteq G$ and $h^1(X, F) = 0$. Then $h^1(X, G) = 0$ and $h^0(X, G) = h^0(X, F) + \mathrm{length}(G/F)$. If F is spanned, then G is spanned. If the base locus of F is finite, then the base locus of G is finite.*

Proof. Look at the exact sequence on X:

$$0 \to F \to G \to G/F \to 0 \tag{1}$$

Since G/F has finite support, $h^1(X, G/F) = 0$. Since $h^1(X, F) = 0$, (1) gives $h^1(X, G) = 0$ and $h^0(X, G) = h^0(X, F) + \mathrm{length}(G/F)$. Now assume that F is spanned. Let G' be the subsheaf of G spanned by $H^0(X, G)$. Thus G' is spanned and $h^0(X, G') = h^0(X, G)$. Since F is spanned, G' contains F. Hence the first part of the lemma applied to the inclusion $F \hookrightarrow G'$ gives $h^1(X, G') = 0$ and $h^0(X, G') = h^0(X, F) + \mathrm{length}(G'/F)$. Since $h^0(X, G') = h^0(X, G)$ and G/G' has finite support, we get $G' = G$, proving the spannedness of G. In the same way we get the last assertion. □

Lemma 2. *Fix an integer $b \geq 2$ and let $Y \subset \mathbb{P}^b$ a general union of b lines through a point, i.e. any union Y of b lines through P spanning $\mathbb{P}^b$. Fix any such line D. Then* $\deg((\omega_Y|D)/\mathit{Tors}(\omega_Y|D)) = -1$.

Proof. Let $f : C \to Y$ be the normalization map. Set $B := f^{-1}(D)$ and $L := f^*(\omega_Y)/\mathrm{Tors}(f^*(\omega_Y))$. We need to prove that $\deg(L|B) = -1$. Since the permutation group S_b of $\mathcal{B}(Y)$ is a subgroup of $\mathrm{Aut}(Y)$, it is equivalent to prove $\deg(L) = -b$. Remark 1 gives $\deg(\omega_Y) = -2$. Remark 2 shows that we need to prove that $\ell(\omega_Y, P) = b - 2$. This is well-known and checked by a local computation by induction on b. □

Lemma 3. *Let T be an integral projective curve. The following conditions are equivalent:*

(a) Every ample vector bundle on T is spanned.

(b) Every ample vector bundle E on T is spanned and the associated morphism h_E is an embedding.

(c) $T \cong \mathbb{P}^1$.

Proof. Obviously, (b) implies (a). If $T \cong \mathbb{P}^1$, then (b) holds, because (b) holds for line bundle and any ample vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles of positive degree. Now assume $T \neq \mathbb{P}_1$, i.e. $p_a(T) > 0$. Fix $P \in T_{reg}$. Then P is a base point of the ample line bundle $\mathcal{O}_T(P)$. Hence (a) fails. □

Remark 3. Let X be a reduced and connected projective curve. We recall that $p_a(X) = 0$ if and only if $T \cong \mathbb{P}^1$ for all $T \in \mathcal{B}(X)$, the singularities of X are given by smooth branches with independent tangents (the seminormal one-dimensional singularities) and the associated graph of X is contractible ([2], Proposition 1.8).

We recall that there are nodal connected curve X such that $p_a(X) = 0$ and X has an indecomposable rank 2 vector bundle ([4], Remark at p. 390).

Proof of Proposition 1. Set $s := \sharp(\mathcal{B}(X))$. The result is well-known and very easy is X irreducible. Hence we may assume $s \geq 2$ and that the result is true for all connected curves with at most $s - 1$ irreducible components. Since $p_a(X) = 0$, Remark 3 shows the existence of an ordering $T_1, \dots, T_s$ of the irreducible components of X such that for all $i \in \{1, \dots, s-1\}$ the scheme $(T_1 \cup \cdots \cup T_i) \cap T_{i+1}$ has length 1. Set $Y := T_1 \cup \cdots \cup T_{s-1}$, and $D := T_s$. For any vector bundle E on X we have an exact sequence

$$0 \to E \to E|Y \otimes E|D \to E|(Y \cap D) \to 0 \tag{2}$$

Let E be an ample vector bundle on X. By the inductive assumption $E|Y$ is spanned, $h^1(Y, E|Y) = 0$, and $h_{E|Y}$ is an embedding of Y. Since $D \cong \mathbb{P}^1$ and $E|D$ is ample, $E|D$ is a direct sum of line bundles of degree ≥ 1. Since $Y \cap D$ is scheme-theoretically a point, the restriction map $H^0(D, E|D) \to H^0(D \cap Y, E|(Y \cap D))$ is surjective. Using (2) we easily get $h^1(X, E) = 0$ and that E is spanned. Since each connected component of C is smooth and rational, the statements concerning $\widetilde{F}$ follow from the classification of vector bundles on $\mathbb{P}^1$. □

Proof of Proposition 2. The " only if " part is obvious, because $\mathcal{O}_X$ is totally nef. Assume $p_a(X) = 0$. Since the result is obvious if $X \cong \mathbb{P}^1$ by the classification of all vector bundles on $\mathbb{P}^1$, we may assume $s := \sharp(\mathcal{B}(X)) \geq 2$ and that the result is true for all connected curves with arithmetic genus 0 and at most $s-1$ irreducible components. The duality for locally Cohen-Macaulay projective schemes gives $h^1(X, F) = h^0(X, Hom(F, \omega_X))$ ([1], Theorem VIII.1.15). Assume $h^0(X, Hom(F, \omega_X)) > 0$. Fix $\sigma \in H^0(X, Hom(F, \omega_X)) \backslash \{0\}$. By [1], Lemma I.2.9, to get a contradiction it is sufficient to show that $\sigma|(X_{reg} \cap T) = 0$ for all $T \in \mathcal{B}(X)$. Since the dual graph of X is contractible and $s \geq 2$, there is $T \in \mathcal{B}(X)$ such that $\sharp(T \cap T^{[c]}) = 1$. The shape of the dual graph of X shows that $T^{[c]}$ is connected and $p_a(T^{[c]}) = 0$. Set $F' := (F|T^{[c]})/\mathrm{Tors}(F|T^{[c]})$. Since $f^{-1}(T^{[c]})$ is the union of $s-1$ of the connected components of C and $f^*(F')/\mathrm{Tors}(f^*(F'))$ is the restriction of $\widetilde{F}$ to these $s-1$ connected components, F' is a strongly positive torsion free sheaf on $T^{[c]}$ with pure rank r. The inductive assumption gives $h^0(T^{[c]}, Hom(F', \omega_{T^{[c]}})) = 0$. Hence $\sigma|(T^{[c]} \cap X_{reg}) = 0$. Lemma 2 gives $\deg((\omega_X|T)/\mathrm{Tors}(\omega_X|T)) = -1$. Since F is totally nef, part (a) shows that σ vanishes on $X_{reg} \cap T$. □

Proof of Proposition 3. Fix $Q \in X_{reg}$. Since $F(-Q)$ is totally nef, Proposition 2 implies $h^1(X, F(-Q)) = 0$. Thus F is spanned at Q. Fix $P \in \mathrm{Sing}(X)$. To prove that F is spanned at P it is sufficient to show that $h^0(X, F) > h^0(X, A)$ for all subsheaves A of F such that F/A is the length 1 skyscraper sheaf $\mathbb{K}_P$ supported by P. Lemma 1 shows that it is sufficient to prove $h^1(X, A)$. Since F is torsion free with pure rank r, $A \subset F$ and F/A has finite support, A is torsion free with pure rank r. By Proposition 2 it is sufficient to prove that A is totally nef. Since the tensor product functor is right exact, the inclusion $A \hookrightarrow F$ induces a map $j : \widetilde{A} \to \widetilde{F}$ whose cokernel is finite and with length one.

Since j is an isorphism outside $f^{-1}(P)$ and $\widetilde{A}$ is torsion free, j is injectiive. The ampleness of $\widetilde{F}$ implies the nefness of $\widetilde{A}$. □

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