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TRANSIENT THERMOELASTIC ANALYSIS IN COMPOSITES WITH LAYERS OF FUNCTIONALLY GRADED MATERIALS

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Abstract: This paper is concerned with the theoretical treatment of thermoelastic problem in multilayer functionally graded composites subjected to the generation of heat in the body as well as at the interfaces with imperfect thermal contact under arbitrary initial temperature distribution. The closed-form solution is obtained for transient temperature distributions by establishing Sturm-Liouville integral transform considering series expansion function in terms of eigenfunction of Sturm-Liouville boundary value problem for composite one-dimensional region consisting of multilayered composites.

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1. Introduction

Functionally Graded Materials (FGMs) that decrease thermal stresses have been developed for wide range of thermal and structural applications, including thermal gradient structures, wear and corrosion resistant coating, and joining

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of metal and ceramic for making structural components with improved performance. Transient thermoelastic analysis in a compositionally graded structure consisting of several layers in contact has numerous applications in engineering. Multilayered structures are normally used to separate a region of high temperature from that of the low temperature. To maintain the required mechanical integrity, high temperature side is predominately ceramic, owing to its better thermal resistance, while low temperature side would be metallic to produce desirable mechanical characteristics. It offers many challenging problems for theoretical and experimental studies.

Agarwal et al [1] examined the analytical solution of transient temperature distribution and developed a general algorithm for studying different sublayers in an FGM system having discretely assigned properties corresponding to its volume fractions of the constituents. Awaji et al [2] obtained one-dimensional temperature/stress distributions in a stress-relief-type plate of functionally graded materials under thermal shock, in relation to both the temperature dependent thermal properties and continuous and gradual variation of thermomechanical properties of the FGM. Sugano et al [4] investigated the thermoelastic solution of FGM rotating disk under intermittent heat supply using modified Vodicka's method.

In all aforementioned investigations, it is concluded that no authors have considered any thermoelastic problem, in which sources are generated, which satisfies the time-dependent heat conduction equation. The success of this novel research mainly lies with the new mathematical procedures with a much simpler approach for optimization for the design in terms of material usage and performance in engineering problem, particularly in the determination of thermoelastic behaviour of multilayered structures.

2. The Transformation and its Essential Property

Consider a system of equation for multilayered FGM having 'k' number of parallel layers with constant properties in each layer. Thus, in this generalized model, gradation is not continuous, which usually is the case in most of the FGM applications. The mathematical formulation of this heat conduction problem is given as

$$\hat{L}x_{i,n}(r) = \beta_i \lambda_n s(r) x_{i,n}(r) \quad \text{in } \lambda_n > 0, r_i \le r \le r_{i+1} \text{ for } i = 1, 2...k\,, \qquad (1)$$

where

$$\hat{L} = \frac{d}{dr}\left(\alpha_i P(r)\frac{d}{dr}\right) + Q(r) \tag{2}$$

and each layer is considered homogenous, isotropic, and having the thermal properties that are constant within the layer and different from those of the adjacent layers, subjected to the following interfacial and boundary conditions

$$\begin{aligned}\frac{d}{dr}\left(\alpha_i x_{i,n}(r)\right)\Big|_{r=r_{i+1}} &= \frac{d}{dr}\left(\alpha_{i+1} x_{i+1,n}(r)\right)\Big|_{r=r_{i+1}}\\ &= \frac{1}{R_i}[x_{i+1,n}(r) - x_{i,n}(r)]\Big|_{r=r_{i+1}}, \quad i = 1, 2...(k-1)\,,\end{aligned} \tag{3}$$

$$-\frac{d}{dr}\left(\alpha_1 x_{1,n}(r)\right) + h_0 x_{1,n}(r)\Big|_{r=r_1} = 0, \qquad h_0 \geq 0\,, \tag{4}$$

$$\frac{d}{dr}\left(\alpha_k x_{k,n}(r)\right) + h_k x_{k,n}(r)\Big|_{r=r_{k+1}} = 0, \quad h_k \geq 0\,, \tag{5}$$

where h_0 as surface co-efficient at $r = r_1$, h_k is the surface co-efficient at $r = r_{k+1}$, eigenvalues of the problem is denoted as λ_n, the characteristics of the i-th layer are α_i and β_i, the characteristics of the coordinate used in the separation of variable of differential equation are $P(r)$, $Q(r)$ and $s(r)$.

The equations (1)-(5) constitute the mathematical formulation of Sturm-Liouville boundary value problem for layers of FGM composite region consisting of k layers.

3. Solution of Sturm-Liouville Problem

The solution of the nonhomogenous problem of the heat conduction in a composite medium consisting of multilayer is obtained based on the orthogonal expansion techniques as suggested by Wankhede [5] for linear homogenous boundary value problems for composites region. For generality, we assume the general solution of the equation (1) as

$$x_{i,n}(r) = A_{i,n}\phi_{i,n}(r) + B_{i,n}\psi_{i,n}(r)\,, \tag{6}$$

where $\phi_{i,n}(r)$ and $\psi_{i,n}(r)$ are two linearly independent solution of equation (1), $A_{i,n}$ and $B_{i,n}$ are the arbitrary constants, by substituting equation (6) into equation (1), we obtain $2k$ simultaneous equations from which we can calculate $A_{i,n}$ and $B_{i,n}$. Also eliminating $A_{i,n}$ and $B_{i,n}$ form $2k$ simultaneous equations one obtains the frequency equation. After substituting the value of

$A_{i,n}$ and $B_{i,n}$, we get the required solution of the Sturm-Liouville transform for functionally graded composite consisting of k layers subjected to interfacial and boundary conditions.

3.1. Orthogonality of Eigenfunction

For testing the orthogonality of eigenfunction $x_{i,n}(r)$ of equation (1), it is assumed that $x_{i,n}(r)$ and $x_{i,m}(r)$ are the two solutions of equation (1). We multiply equation (1) by $x_{i,m}(r)$ and the other equation $\hat{L}x_{i,m}(r) = \beta_i \lambda_m s(r) x_{i,m}(r)$ by $x_{i,n}(r)$ and subtracting. Then integrating with respect to r in the interval $r_i \le r \le r_{i+1}$, one obtains

$$\sum_{i=1}^{k} \int_{r_i}^{r_{i+1}} \left[x_{i,m}(r) \frac{d}{dr}\left(\alpha_i P(r) \frac{d}{dr} x_{i,n}(r) \right) - x_{i,n}(r) \frac{d}{dr}\left(\alpha_i P(r) \frac{d}{dr} x_{i,m}(r) \right) \right] dr$$
$$= (\lambda_m - \lambda_n) \sum_{i=1}^{k} \beta_i \int_{r_i}^{r_{i+1}} s(r) x_{i,n}(r) x_{i,m}(r) dr \,. \quad (7)$$

On integrating by part, the left hand side group of equation (7) vanishes after using the boundary and interfacial conditions for the eigenfunctions $x_{i,n}(r)$ and $x_{i,m}(r)$, and the right hand side group can be equated to Kronecker delta as

$$N_{mn} \sum_{i=1}^{k} \beta_i \int_{r_i}^{r_{i+1}} s(r) x_{i,n}(r) x_{i,m}(r) dr = \delta_{mn} \,. \quad (8)$$

Here $\sqrt{N_{mn}}$ is the normalizing constant and Kronecker delta δ_{mn} takes the value zero if $n \ne m$ and unity for $n = m$.

3.2. Definition of Sturm-Liouville Integral Transform

Thus from equation (8), we establish the Sturm-Liouville integral transform for functionally graded composites one-dimensional region consisting of k layers as

$$u_i^*(n) = \beta_i \int_{r_i}^{r_{i+1}} s(r) x_{i,n}(r) u_i(r) dr \,, \qquad i = 1, 2 ... k \,, \quad (9)$$

where $u_i^*(n)$ Sturm-Liouville transform for functionally graded composites of $u_i(r)$ with respect to the kernel $x_{i,n}(r)$ and weight function $s(r)$.

3.3. Completeness Relation

In view of completeness of eigenfunction expansion and the orthogonal property (8), the inverse transform of equation (9) is readily obtained, providing that $u_i(r)$ $(i = 1, 2...k)$ are continuous and have piecewise continuous first and second derivatives in the interval $r_i \leq r \leq r_{i+1}$, and satisfies boundary and interfacial conditions [3] of the eigenvalue problem as

$$u_i(r) = \sum_n A(n) x_{i,n}(r) \sum_{i=1}^{k} u_i^*(n), r_i \leq r \leq r_{i+1} \text{ for } i = 1, 2...k, n = 1, 2.. \quad (10)$$

where the coefficients $A(n)$ are obtained from equation (8). It is also noted that the right hand side series of equation (10) is an absolutely and uniformly convergent series [3].

3.4. Properties of Sturm-Liouville Integral Transform

To solve the problem stated above, it is sufficient to find only the effect of the transformation defined in equation (9) on the expression

$$\frac{1}{\beta_i s(r)} \frac{d}{dr}\left(\alpha_i P(r) \frac{d}{dr} u_i(r)\right) + \frac{\alpha_i}{\beta_i} \frac{Q(r)}{s(r)} u_i(r), r_i \leq r \leq r_{i+1} \text{ for } i = 1, 2...k. \quad (11)$$

Integrating by parts twice, we obtain:

$$\sum_{i=1}^{k} \int_{r_i}^{r_{i+1}} x_{i,n}(r) \frac{d}{dr}\left(\alpha_i P(r) \frac{d}{dr} u_i(r)\right) dr = \sum_{i=1}^{k} \left\{ \left[\alpha_i x_{i,n}(r) P(r) \frac{d}{dr} u_i(r) \right.\right.$$
$$\left.\left. - \alpha_i \frac{d}{dr} x_{i,n}(r) P(r) u_i(r) \right]_{r_i}^{r_{i+1}} + \int_{r_i}^{r_{i+1}} u_i(r) \frac{d}{dr}\left(\alpha_i P(r) \frac{d}{dr} x_{i,n}(r)\right) dr \right\}. \quad (12)$$

But $x_{i,n}(r)$ satisfies the equation (1), hence

$$\sum_{i=1}^{k} \int_{r_i}^{r_{i+1}} \beta_i s(r) x_{i,n}(r) \left[\frac{1}{\beta_i s(r)} \frac{d}{dr}\left(\alpha_i P(r) \frac{d}{dr} u_i(r)\right) - \frac{\alpha_i}{\beta_i} \frac{Q(r)}{s(r)} u_i(r) \right] dr$$
$$= \sum_{i=1}^{k} \alpha_i P(r) \left[x_{i,n}(r) \frac{d}{dr} u_i(r) - u_i(r) \frac{d}{dr} x_{i,n}(r) \right]_{r_i}^{r_{i+1}} - \lambda_n \sum_{i=1}^{k} u_i^*(n) \quad (13)$$

$$= \sum_{i=1}^{k} \alpha_i P(r_{i+1}) \left[x_{i,n}(r_{i+1}) \left[\frac{d}{dr} u_i(r) \right]_{r_{i+1}} - u_i(r_{i+1}) \left[\frac{d}{dr} x_{i,n}(r) \right]_{r_{i+1}} \right]$$

$$-\alpha_i P(r_i)\left[x_{i,n}(r_i)\left[\frac{d}{dr}u_i(r)\right]_{r_i} - u_i(r_i)\left[\frac{d}{dr}x_{i,n}(r)\right]_{r_i}\right] - \lambda_n \sum_{i=1}^{k} u_i^*(n) \qquad (14)$$

$$= x_{k,n}(r_{k+1})P(r_{k+1})\left[\frac{d}{dr}u_k(r) - \frac{1}{x_{k,n}(r_{k+1})}\frac{d}{dr}x_{k,n}(r)u_k(r)\right]_{r_{k+1}}$$
$$+\sum_{i=1}^{k-1}\left\{\alpha_i P(r_{i+1})\left[x_{i,n}(r_{i+1})\left[\frac{d}{dr}u_i(r)\right]_{r_{i+1}} - u_i(r_{i+1})\left[\frac{d}{dr}x_{i,n}(r)\right]_{r_{i+1}}\right]\right.$$
$$-\alpha_{i+1}P(r_{i+1})\left[x_{i+1,n}(r_{i+1})\left[\frac{d}{dr}u_{i+1}(r)\right]_{r_{i+1}} - u_{i+1}(r_{i+1})\left[\frac{d}{dr}x_{i+1,n}(r)\right]_{r_{i+1}}\right]$$
$$\left.-\alpha_1 x_{1,n}(r_1)P(r_1)\left[\frac{d}{dr}u_1(r) - \frac{1}{x_{1,n}(r)}\frac{d}{dr}x_{1,n}(r)u_1(r)\right]_{r_1}\right\} - \lambda_n \sum_{i=1}^{k} u_i^*(n) \qquad (15)$$

Using the boundary and interfacial conditions (3), (4) and (5), we obtain

$$\sum_{i=1}^{k}\int_{r_i}^{r_{i+1}} \beta_i s(r)x_{i,n}(r)\left[\frac{1}{\beta_i x_{i,n}(r)}\frac{d}{dr}\left(\alpha_i P(r)\frac{d}{dr}u_i(r)\right) - \frac{1}{\beta_i}\frac{Q(r)}{s(r)}u_i(r)\right]dr$$
$$= x_{k,n}(r_{k+1})P(r_{k+1})\left[\alpha_k\frac{d}{dr}u_k(r) + h_k u_k(r)\right]_{r_{k+1}}$$
$$+\sum_{i=1}^{k-1}\left[\alpha_i x_{i,n}(r_{i+1})P(r_{i+1})\left[\frac{d}{dr}u_i(r)\right]_{r_{i+1}}\right.$$
$$-P(r_{i+1})u_i(r_{i+1})\frac{1}{R_i}\left[x_{i+1,n}(r_{i+1}) - x_{i,n}(r_{i+1})\right]$$
$$-\alpha_{i+1}x_{i+1,n}(r_{i+1})P(r_{i+1})\left[\frac{d}{dr}u_{i+1}(r)\right]_{r=r_{i+1}}$$
$$+u_{i+1}(r_{i+1})P(r_{i+1})\frac{1}{R_i}\left[x_{i+1,n}(r_{i+1}) - x_{i,n}(r_{i+1})\right]$$
$$-x_{1,n}(r_1)P(r_1)\left[\alpha_1\frac{d}{dr}u_1(r) + h_0 u_1(r)\right]_{r=r_1} - \lambda_n\sum_{i=1}^{k}u_i^*(r). \qquad (16)$$

Rearranging the terms in the above expression, we get the required property of the transform as

$$\sum_{i=1}^{k}\int_{r_i}^{r_{i+1}} \beta_i s(r)x_{i,n}(r)\left[\frac{1}{\beta_i x_{i,n}(r)}\frac{d}{dr}\left(\alpha_i P(r)\frac{d}{dr}u_i(r)\right) - \frac{1}{\beta_i}\frac{Q(r)}{s(r)}u_i(r)\right]dr$$

$$= x_{k,n}(r_{k+1})P(r_{k+1})\left[\alpha_k \frac{d}{dr}u_k(r) + h_k u_k(r)\right]_{r_{k+1}}$$
$$+\sum_{i=1}^{k-1} P(r_{i+1})\Big\{x_{i,n}(r_{i+1})\left[\alpha_i \frac{d}{dr}u_i(r) - \frac{1}{R_i}\left[u_{i+1}(r) - u_i(r)\right]\right]_{r_{i+1}}$$
$$-x_{i+1,n}(r_{i+1})\left[\alpha_{i+1}\frac{d}{dr}u_{i+1}(r) - \frac{1}{R_1}\left[u_{i+1}(r) - u_i(r)\right]\right]_{r_{i+1}}\Big\}$$
$$-x_{1,n}(r_1)P(r_1)\left[\alpha_1 \frac{d}{dr}u_1(r) + h_0 u_1(r)\right]_{r_1} - \lambda_n \sum_{i=1}^{k} u_i^*(n). \quad (17)$$

Hence equation (9) is the fundamental property of Sturm-Liouville transform for functionally graded composite consisting of k layers defined in equation (9), which removes group of terms quoted in equation (11).

3.5. Discussion

Sturm-Liouville transform for functionally graded composite consisting of k layers defined in equation (9) is nothing but the extension of the integral transforms given in [5]-[6]. The integral transforms given in [5]-[6] can be deduced from the integral transform defined in (9) by particularizing the coefficients and parameters in equation (9).

4. Statement of the Problem

Now we consider composites with layers of functionally graded annular disk defined by $r_i \le r \le r_{i+1}$ $(i = 1, 2...k)$. We assume that internal heat source and the boundary heat source and interfacial heat source are arbitrary but integrable functions of time. The differential equation governing transient temperature distribution $T_i(r,t)$ of a thin one-dimensional functionally graded disk with homogeneities along radial direction is mathematically determined in the i-th layer as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda_i \frac{\partial T_i(r,t)}{\partial r}\right) + Q_i(r,t) = (\rho C)_i \frac{\partial T_i(r,t)}{\partial t}$$
$$\text{in } \lambda_i > 0,\ r_i \le r \le r_{i+1} \ \text{ for } \ i = 1, 2...k\,, \quad (18)$$

where $Q_i(r,t)$ denotes distributed source, λ_i for thermal conductivity and heat capacity per unit volume $(\rho C)_i$ with ρ_i for density and C_i as specific heat respectively for the i-th layer. Here it is assumed that the coefficients of thermal conductivity and the other thermal properties in each layer is considered homogenous, isotropic, that are constant within the layer and different from those of the adjacent layers, which in view of the condition of ideal thermal contact and the systems are subjected to the following initial, interfacial and boundary conditions

$$T_i(r,t)\Big|_{t=0} = V_i(r), \quad \text{in } r_i \le r \le r_{i+1}, i = 1, 2...k, \tag{19}$$

$$\lambda_i \frac{\partial T_i(r,t)}{\partial r} = \lambda_{i+1} \frac{\partial T_{i+1}(r,t)}{\partial r}$$
$$= h_i[T_{i+1}(r_{i+1},t) - T_i(r_{i+1},t)] + \bar{Q}_i(t), i = 1, 2...(k-1) \text{ for } t > 0 \tag{20}$$

$$-\lambda_1 \frac{\partial T_1(r,t)}{\partial r} + h_0 T_1(r,t)\Big|_{r=r_1} = f_1(t), \qquad h_0 \ge 0 \tag{21}$$

and

$$\lambda_k \frac{\partial T_k(r,t)}{\partial r} + h_k T_k(r,t)\Big|_{r=r_{k+1}} = f_2(t), \quad h_k \ge 0 \tag{22}$$

when $h_{i+1} \longrightarrow \infty$, the boundary condition reduce to

$$T_i(r,t) = T_{i+1}(r,t), \quad \text{at } r = r_{i+1}, i = 1, 2...(k-1). \tag{23}$$

Here $h_0 \ge 0$ and $h_k \ge 0$ are respectively, given surface coefficient linearly related to the heat transfer coefficients at $r = r_1$ and $r = r_{k+1}$, and the corresponding surface sources are represented by the arbitrary function $f_1(t)$ and $f_2(t)$. Equation (20) expresses the discontinuity of the temperature at the interfaces and $\overline{Q}_i(t)$ $(i = 1, 2..k-1)$ denotes the source at the interfaces. In case of the preface interfacial thermal contact, we let the interfacial conductance $h_i(i = 1, 2..k-1) \longrightarrow \infty$, in which case the equation express the continuity across the interfaces. Equations (19)-(23) constitute the mathematical formulation of the problem under consideration consisting of k layers.

4.1. Solution of the Problem

First we apply with respect to r the transform defined in equation (9) to equations (18) and (19). For this particular case, we consider the setting of values based on composite Hankel transform as

$$P(r) = r, \quad Q(r) = 0, \quad s(r) = r, \quad \lambda_n = \mu_n^2,$$
$$\phi_{i,n}(r) = J_0\left(\frac{\mu_n r}{a_i}\right), \quad \psi_{i,n}(r) = Y_0\left(\frac{\mu_n r}{a_i}\right). \tag{24}$$

On account of the property (24), we obtain

$$\left(\frac{d}{dt}+\mu_n^2\right)\sum_{i=1}^{k}T_i^*(n,t)=G(n,t) \tag{25}$$

subject to

$$T_i^*(n,0)=V_i^*(n)\,, \tag{26}$$

where

$$G(n,t)=\sum_{i=1}^{k}\int_{r_i}^{r_{i+1}} r x_{i,n}(r)Q_i(r,t)dr+x_{k,n}(r_{k+1})r_{k+1}f_2(t)$$
$$+x_{1,n}(r_1)r_1f_1(t)+\sum_{i=1}^{k-1}\left[x_{i,n}(r_{i+1})-x_{i+1,n}(r_{i+1})\right]r_{k+1}\overline{Q}_i(t). \tag{27}$$

The differential equation (25) can be solved by applying Laplace transform and taking its inverse. We obtain the solution as

$$\sum_{i=1}^{k}T_i^*(n,t)=\exp\left(-\mu_n^2 t\right)\sum_{i=1}^{k}V_i^*(n)+\int_0^t\exp\left[-\mu_n^2(t-t')\right]G(n,t')dt'\,. \tag{28}$$

Finally applying the inverse transform defined in (10), we obtain the temperature distribution $T_i(r,t)$ as

$$T_i(r,t)=\sum_n A(n)x_{i,n}(r)\sum_{i=1}^{k}T_i^*(n,t)\,,$$
$$r_i<r<r_{i+1},\qquad i=1,2,...k,\quad n=1,2,...\,, \tag{29}$$

We now put the general solution (29) in more explicit form as

$$T_i(r,t)=\sum_n A(n)x_{i,n}(r)\sum_{i=1}^{k}exp\left(-\mu_n^2 t\right)\Bigg\{\sum_{i=1}^{k}\rho_i c_{pi}\int_{r_i}^{r_{i+1}} r x_{i,n}(r)V_i^*(n)dr$$
$$+\int_0^t exp\left(-\mu_n^2 t'\right)\Bigg[\sum_{i=1}^{k}\int_{r_i}^{r_{i+1}} r x_{i,n}(r)Q_i(r,t')dr+x_{k,n}(r_{k+1})r_{k+1}f_2(t')$$
$$+x_{1,n}(r_1)r_1f_1(t')+\sum_{i=1}^{k-1}\left[x_{i,n}(r_{i+1})-x_{i+1,n}(r_{i+1})\right]r_{k+1}\overline{Q}_i(t')\Bigg]dt'\Bigg\},$$
$$r_i<r<r_{i+1},\quad i=1,2,...k,\quad n=1,2,...\,. \tag{30}$$

4.2. Remark on Solution

The general solution given in (30) depends on the thermal properties and the source function in the composite body. The initial source $V_i(r)$ gives rise to transient terms only, which die out exponentially with time t. The volume sources $Q_i(r,t)$, interfacial sources $\bar{Q}_i(t)$, and surface sources $f_1(t)$ and $f_2(t)$ even when independent of time, give rise to both transient terms (which die out exponentially with time t) and steady state terms.

Finally we remark that the general solution (30) satisfies the differential equation (18) and the initial conditions (19), but however the boundary and interfacial conditions (20)-(23) are not satisfied unless $\bar{Q}_i(t) = f_1(t) = f_2(t) = 0$. This is due to the non convergence of the series (30) when, $f_1(t) \neq 0$, $f_2(t) \neq 0$ and/or $\bar{Q}_i(t) \neq 0$; except when the time dependent source function $Q_i(r,t)$, $\bar{Q}_i(t)$, $f_1(t)$ and $f_2(t)$ are Dirac delta functions in time.

4.3. The Associated Thermal Stresses

Firstly, Hooke's law for a plane stress distribution components σ_{rr}^i and $\sigma_{\theta\theta}^i$ of the FGM annular disk are expressed in terms of radial displacement u_r^i as follows:

$$\sigma_{rr}^i = \frac{E_i}{1-\upsilon^2}\left(\frac{\partial u_r^i}{\partial r} + \upsilon \frac{u_r^i}{r}\right) - \frac{E_i}{1-\upsilon}\,\alpha_i T_i(r,t)\,, \tag{31}$$

$$\sigma_{\theta\theta}^i = \frac{E_i}{1-\upsilon^2}\left(\upsilon\frac{\partial u_r^i}{\partial r} + \frac{u_r^i}{r}\right) - \frac{E_i}{1-\upsilon}\,\alpha_i T_i(r,t)\,, \tag{32}$$

where $T_i(r,t)$ denotes the temperature change, E_i and α_i are Young's modulus and the coefficient of linear thermal expansion, respectively for i-th layer, while Poisson's ratio υ is assumed to be constant value for each layer. Substituting equations (31)-(32) into the equilibrium equation, disregarding the body force and the inertia term, the differential equation for the radial displacement in the i-th layer u_r^i satisfying the following Euler's differential is obtained as

$$\frac{\partial^2 u_r^i}{\partial r^2} + \frac{1}{r}\frac{\partial u_r^i}{\partial r} - \frac{1}{r^2}u_r^i = (1+\upsilon)\alpha_i E_i \frac{\partial T_i(r,t)}{\partial r}\,. \tag{33}$$

It is obvious that the homogeneous solution to equation (33) can be obtained by assuming

$$u_r^i = C r^\beta\,, \tag{34}$$

where C is an arbitrary constant. Substituting equation (34) into equation (33) and omitting the right-hand side, we obtain auxiliary equation as $\beta^2 - 1 = 0$.

The particular solution to equation (33) is easy to obtain, and the complete solution to it is as follows:

$$u_r^i = C_1 r + C_2 r^{-1} + (1+\upsilon)\alpha_i E_i r^{-1}\left(\int_{r_1}^{r} rT_i(r,t)dr\right), \tag{35}$$

where C_1 and C_2 is an arbitrary constants, and

Finally, the desired expressions for the stress components can be obtained by substituting equation (35) into equations (31) and (32). For the considered problem, boundary condition and continuity condition for the stress field are expressed as follows:

$$\overline{u}_r^1 = 0 \ (r = r_1, r_k), \ \sigma_{rr}^i = 0 \ (r = r_1, r_k)\,. \tag{36}$$

The unknown coefficients in equation (35) are determined from equation (36).

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