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ON SOME GENERALIZATIONS OF PREINVEX FUNCTIONS

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Abstract: We give some insights on two generalizations of preinvex functions due to R. Pini [11]. Moreover, we take into consideration a modification of the class of semi-preinvex functions introduced by Yang and Chen [16] in order to obtain equivalence between this (modified) class and the class of invex functions. Finally we correct and generalize some results of Weir [15] on semilocally convex functions.

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1. Introduction

Since the classical notion of invex function [2, 7] requires a differentiability assumption, the class of *preinvex functions*, not necessarily differentiable, has been introduced in [13, 14]. Let f be a real-valued function defined on a subset of $\mathbb{R}^n$ and $\eta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We say that a subset X of $\mathbb{R}^n$ is $\eta-$*invex* if for every $x, y \in X$ the segment $[y, y + \eta(x, y)]$ is contained in X.

Let f be a real-valued function defined on an $\eta-$invex set X; f is *preinvex with respect to* η if the following inequality holds:

$$f\left[y + \lambda\eta(x, y)\right] \leq \lambda f(x) + (1 - \lambda)f(y),\ \forall x, y \in X,\ \forall\lambda \in [0, 1]\ . \tag{1}$$

A differentiable function satisfying (1) is also invex and this is the reason why functions satisfying (1) are called preinvex.

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Preinvex functions have been studied by various authors, also in view of applications to optimization problems: see, e.g. [2, 6, 9, 10, 13, 14]. Recently in [6] the present author has clarified some relationships between preinvex functions and other classes of generalized convex functions.

2. Two Generalizations of Preinvex Functions

R. Pini [11] introduced the following generalizations of preinvex functions.

Definition 1. Let $f : X \longrightarrow \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is an η-invex set; f is pre-pseudoinvex on X if there exists a function $\eta(x, y) : X \times X \longrightarrow \mathbb{R}^n$ and a positive function b such that

$$f(x) < f(y) \Longrightarrow f(y + \lambda\eta(x, y)) \leq f(y) + (\lambda - 1)\lambda b(x, y),$$
$$\forall x, y \in X, \forall \lambda \in (0, 1).$$

Obviously the class of pseudoconvex functions (defined without assuming differentiability: see, e.g. [1]) is a subset of the class of pre-pseudoinvex functions. In [11] it is proved that if f is preinvex with respect to η, then f is also pre-pseudoinvex with respect to the same η. Mohan and Neogy [10] remark (without proof) that if $f : X \longrightarrow \mathbb{R}$ is pre-pseudoinvex on the η-invex set $X \subseteq \mathbb{R}^n$, then every local minimum point of f is also a global minimum point. The proof is easy: absurdly suppose that y is a local minimum point, but not global. Therefore there exists $x \in X$ such that $f(x) < f(y)$. From the definition of pre-pseudoinvexity it follows that there exists $b(x, y) > 0$ such that,$\forall \lambda \in (0, 1)$ we have $f(y + \lambda\eta(x, y)) \leq f(y) + (\lambda - 1)\lambda b(x, y) < f(y)$, from which $f(y) > f(y + \lambda\eta(x, y))$, $\forall \lambda \in (0, 1)$, in contradiction with the assumption.

The present author has shown in [6] the existence of differentiable pseudoconvex functions (therefore pre-pseudoinvex also) which are not preinvex. So we can conclude that the class of preinvex functions is strictly contained in the class of pre-pseudoinvex functions, contrary to what happens for the classes of invex and pseudoinvex functions which coincide.

Definition 2. Let $f : X \longrightarrow \mathbb{R}$, with $X \subseteq \mathbb{R}^n$ an η-invex set; f is pre-quasiinvex on X if there exists a function $\eta(x, y) : X \times X \longrightarrow \mathbb{R}^n$ such that

$$f(y + \lambda\eta(x, y)) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in X, \quad \forall \lambda \in (0, 1).$$

It is quite immediate to prove that if f is preinvex on X, with respect to η, then it is also pre-quasiinvex with respect to the same η. We note also that there always exists at least a function η for which a function f is pre-quasiinvex:

simply take the function $\eta(x, y) = 0$.

Mohan and Neogy [10] assert (without proof) that a pre-pseudoinvex function is also pre-quasiinvex with respect to the same function η. This result is not correct. Consider, e.g., the function

$$f(x) = \begin{cases} 0, & x \neq 1\,, \\ 1, & x = 1\,, \end{cases}$$

which is pseudoconvex, but not quasiconvex. This function is therefore also pre-pseudoinvex with respect to $\eta(x, y) = x - y$, but not pre-quasiinvex with respect to this choice of η.

As for what concerns the differentiable case, we can state (similarly to preinvex functions) the following results.

Theorem 3. *Let $f : X \longrightarrow \mathbb{R}$, with $X \subseteq \mathbb{R}^n$ open η-invex set; let f be differentiable and pre-pseudoinvex with respect to η. Then f is also pseudoinvex with respect to the same η.*

Proof. Take $x, y \in X$, with $f(y) \geq f(x)$. Being f pre-pseudoinvex with respect to η, we have

$$[f(y + \lambda\eta(x, y)) - f(y)]\,(1/\lambda) \leq (\lambda - 1)b(x, y) < 0.$$

Taking the limit for $\lambda \to 0^+$ we obtain $\nabla f(y)\eta(x, y) < 0$. □

Theorem 4. *Let $f : X \longrightarrow \mathbb{R}$, with $X \subseteq \mathbb{R}^n$ open η-invex set; let f be differentiable and pre-quasiinvex with respect to η. Then f is also quasiinvex with respect to the same η.*

Proof. Similar to the proof of Theorem 3. □

3. On Modified Semi-Preinvex Functions

It is well known that a differentiable function satisfying (1) is also invex [2] and this is the reason why functions satisfying (1) are called preinvex in [14]. However, the converse is not generally true, in the sense that if f is invex with respect to a certain η, it is not necessarily preinvex with respect to that function η. For example, $f(x) = e^x$, $x \in \mathbb{R}$, is invex with respect to $\eta = -1$, but it is not preinvex with respect to the *same* function η. Some conditions, assuring that a differentiable function which is invex on an $\eta-$invex set X is also preinvex on X, with respect to the same η, are given by Mohan and Neogy [10]. Mititelu [9] states that for the case of an *open* set X, the two classes of (differentiable) invex functions and differentiable preinvex functions coincide. This claim is disproved in [6]. Now we go further in this type of inestigations.Yang and Chen [16] present the following generalization of the class of preinvex functions.

Definition 5. A set X in $\mathbb{R}^n$ is said to satisfy the “semi-connected” property, if for any $x, y \in X$ and $\lambda \in [0,1]$, there exists a vector $\tau(y,x,\lambda) \in \mathbb{R}^n$ such that $x + \lambda\tau(y,x,\lambda) \in X$.

Definition 6. Let X be a set in $\mathbb{R}^n$ having the “semi-connected” property with respect to $\tau(y,x,\lambda) : X \times X \times [0,1] \longrightarrow \mathbb{R}^n$ and $f(x)$ be a real function on X. Then f is called *semi-preinvex* with respect to $\tau(y,x,\lambda)$, if for all $x, y \in X$ and $\lambda \in [0,1]$

$$f(y + \lambda\tau(y,x,\lambda)) \leq \lambda f(y) + (1-\lambda)f(x)$$

and $\lim_{\lambda\downarrow 0} \lambda\tau(y,x,\lambda) = 0$.

We now consider the following slight modification of Definition 6.

Definition 7. Let $f : X \longrightarrow \mathbb{R}$, with $X \subseteq \mathbb{R}^n$. Then f is said to be modified semi-preinvex or m-semi-preinvex if there exists a function $\xi : X \times X \times [0,1] \longrightarrow \mathbb{R}^n$ such that, for all $x, u \in X$:

(1) $u + \xi(x,u,\lambda) \in X$, $\forall\lambda \in [0,1]$.

(2) $\lim_{\lambda\downarrow 0} \xi(x,u,\lambda) = 0$; $\lim_{\lambda\downarrow 0} \frac{\xi(x,u,\lambda)}{\lambda}$ exists finite.

(3) $f(u + \xi(x,u,\lambda)) \leq \lambda f(x) + (1-\lambda)f(u), \forall\lambda \in [0,1]$.

The modified definition of semi-preinvexity enables us to state that, under differentiability assumptions, the classes of m-semi-preinvex functions and of invex functions coincide.

Theorem 8. *Let $f : X \longrightarrow \mathbb{R}$ be differentiable. Then f is invex if and only if it is m-semi-preinvex.*

Proof. Sufficiency. Assume f is m-semi-preinvex. Let $x, u \in X$. Then, for each $\lambda \in (0,1]$:

$$f(x) - f(u) \geq \frac{f(u + \xi(x,u,\lambda)) - f(u)}{\lambda}.$$

By differentiability of f we have

$$f(u + \xi(x,u,\lambda)) = f(u) + \xi(x,u,\lambda)^T \nabla f(u) + o(\lambda).$$

Therefore,

$$f(x) - f(u) \geq \frac{\xi(x,u,\lambda)^T \nabla f(u) + o(\lambda)}{\lambda}, \forall\lambda \in (0,1].$$

Taking limits as $\lambda \downarrow 0$ and using 2) of Definition 7, we have $f(x) - f(u) \geq \eta(x,u)^T \nabla f(u)$, where $\eta(x,u) = \lim_{\lambda\downarrow 0} \frac{\xi(x,u,\lambda)}{\lambda}$. Defining $\eta : X \times X \longrightarrow \mathbb{R}^n$ in this fashion, $\forall x, u \in X$, it follows that f is invex with respect to η.

Necessity. Assume f is invex. If $\nabla f(u) \neq 0$, then an appropriate η must be defined by

$$\eta(x,u) = \frac{(f(x) - f(u))\nabla f(u)}{\nabla f(u)^T \nabla f(u)} + v\,,$$

where $v^T \nabla f(u) \leq 0$. Taking $v = -\nabla f(u)$ gives $f(x) - f(u) > \eta(x,u)^T \nabla f(u)$, $\forall x \in X$. Thus, for each $x \in X$ there exists $\bar{\lambda}(x,u) \in (0,1]$ such that

$$f(u + \lambda\eta(x,u)) \leq \lambda f(x) + (1-\lambda) f(u), \forall \lambda \in \left[0, \bar{\lambda}(x,u)\right].$$

If, for $\lambda \in \left[0, \bar{\lambda}(x,u)\right]$ we let $\xi(x,u,\lambda) = \lambda\eta(x,u)$, then $\lim_{\lambda\downarrow 0} \xi(x,u,\lambda) = 0$ and $\lim_{\lambda\downarrow 0} \frac{\xi(x,u,\lambda)}{\lambda} = \eta(x,u)$. Now, for $\lambda \in \left(\bar{\lambda}(x,u), 1\right]$, define $\xi(x,u,\lambda)$ by

$$\xi(x,u,\lambda) = \begin{cases} 0, \text{if } f(u) \leq f(x) \\ x - u, \text{if } f(u) > f(x). \end{cases}$$

Then $f(u + \xi(x,u,\lambda) \leq \lambda f(x) + (1-\lambda) f(u), \forall \lambda \in \left(\bar{\lambda}(x,u), 1\right]$. If $\nabla f(u) = 0$, then as f is invex, u is a global minimizer, so

$$f(u) \leq \lambda f(x) + (1-\lambda) f(u), \forall x \in X, \forall \lambda \in [0,1]\,.$$

Thus, we can take $\xi(x,u,\lambda) = 0$, $\forall x \in X, \forall \lambda \in [0,1]$. So f is m-semi-preinvex. □

4. Another Class of Generalized Preinvex Functions

Another generalization of convexity, known as *semilocal convexity*, was introduced by Ewing [4] and extended by Kaul and Kaur [8]. We recall that a subset C of $\mathbb{R}^n$ is *locally starshaped* at $y \in C$ if there exists a maximal positive number $a(x,y) \leq 1$ such that $y + \lambda(x-y) \in C, \forall \lambda \in [0, \epsilon(x,y)]$. C is locally starshaped if it is locally starshaped at each of its points. So, we can conclude that each open set in $\mathbb{R}^n$ is a locally stashaped set. A scalar function $f : C \longrightarrow \mathbb{R}$, where C is locally starshaped, is called *semilocally convex on* C if for any $x, y \in C$ there exists a positive number $d(x,y) \leq a(x,y)$ such that

$$f(y + \lambda(x-y)) \leq \lambda f(x) + (1-\lambda) f(y), \forall \lambda \in [0, d(x,y)]\,.$$

Semilocally convex functions have also been applied by Weir [15] in obtaining a theorem of the alternative and optimality and duality results for a (nondifferentiable) programming problem. However, the results of Weir are based on the following premises (Lemma 3.2 in [15]): Let C be a locally starshaped set in $\mathbb{R}^n$; then $cl(C)$ is convex. The lemma is false, as put in evidence by X.M. Yang

[17]. Indeed, there exist both locally starshaped sets whose closure is convex and locally starshaped sets whose closure is not convex.

Example. Consider the starshaped sets $A = [1, 2) \cup (2, 3]$ and $B = [1, 2) \cup (4, 5]$. We have that $cl(A) = [1, 3]$ is convex but $cl(B) = [1, 2] \cup [4, 5]$ is not convex.

Due to the above error, also Lemma 3.3 and Theorem 3.5 in [15] are incorrect. On the grounds of what previously recalled, it is possible (see also [12]) to extend to the preinvex case the notion of semilocally convex functions and to obtain, assuming explicitly the convexity of the closure of a locally starshaped set, results more general than the ones obtained (with incorrect proofs) by Weir.

Definition 9. A set $C \subseteq \mathbb{R}^n$ is η-locally starshaped at y if there exists a maximal positive number $\epsilon(x, y) \leq 1$ and a vector function η such that $y + \lambda\eta(x, y) \in C, \forall\lambda \in [0, \epsilon(x, y))$.

Definition 10. Let C be an η-locally starshaped set in $\mathbb{R}^n$ and $f : C \longrightarrow \mathbb{R}$. f is semilocally preinvex on C if there exists, for any $x, y \in C$, a positive number $d(x, y) \leq \epsilon(x, y)$ such that

$$f(y + \lambda\eta(x - y)) \leq \lambda f(x) + (1 - \lambda)f(y), \forall\lambda \in [0, d(x, y)]$$

.

We consider now the following minimization problem.

$$\begin{cases} \text{minimize } f(x)\,, \\ \text{s.t. } g_i(x) \leq 0, \quad i = 1, ..., m\,. \end{cases} \qquad (P)$$

where f and each $g_i, i = 1, ..., m$, are defined on $X \subseteq \mathbb{R}^n$. We introduce the notion of *conic extension* of the image set $F(X) = \{(f(x) - f(x^*), g(x)), x \in X\}$.

Definition 11. Let x^* be a solution for problem (P). The set $Z = F(X)+\mathbb{R}^{m+1} = \left\{(f(x) - f(x^*), g(x)) + v, x \in X, v \in \mathbb{R}^{m+1}_+\right\}$ is called the conic extension of the image set of (P).

We have the following result which is a generalization of a classical result for a convex problem (P) (see, e.g., Cambini [3], Giannessi [5]).

Theorem 12. *Let x^* be a solution of (P). If the conic extension Z is convex and a suitable constraint qualification holds (e.g. there exists $\bar{x} \in X$ such that $g_i(\bar{x}) < 0, \forall i = 1, ..., m$), then there exists a vector $\lambda^* \geq 0$ such that (x^*, λ^*) is a saddle point for the Lagrangian function $L(x, \lambda) = f(x) + \lambda g(x)$.*

A sufficient condition for the convexity of Z is the classical condition: $X \subseteq \mathbb{R}^n$ is a convex set, f and every $g_i, i = 1, ..., m$, are convex functions on X.

Another, more general sufficient condition for the convexity of Z has been provided by Weir and Mond [14]: if f and every $g_i, i = 1, ..., m$, are preinvex, with respect to η, on the η-invex set $X \subseteq \mathbb{R}^n$, then Z is convex. The following separation theorem corrects the corresponding separation result contained in Lemma 3.3 of [15].

Theorem 13. *Let $S \subseteq \mathbb{R}^n$be a locally starshaped set such that $cl(S)$ is convex, and let $T \subseteq \mathbb{R}^n$ be a convex set, with $int(T) \neq \emptyset$. If S and T are disjoint, then there exists a separating hyperplane between S and T, i.e. a nonzero vector $p \in \mathbb{R}^n$ such that*

$$p^T x \geq \alpha, \quad \forall x \in S\,,$$
$$p^T x \leq \alpha, \quad \forall x \in T\,.$$

Proof. The proof is performed on the same lines of [15]. □

Taking Theorem 14 into account, it is not difficult to prove, on the same lines of [15], the following resultwhich generalizes Theorem 13.

Theorem 14. *Let x^* be an optimal solution for (P). If the conic extension Z is a locally starshaped set such that $cl(Z)$ is convex and a suitable constraint qualification holds, then there exists a vector $\lambda^* \geq 0$ such that (x^*, λ^*) is a saddle point of the Lagrangian function.*

This result corrects a similar result of Weir [15]. Sufficient conditions such that Z is locally starshaped have been given by Weir [15].

Theorem 15. *Let $X \subseteq \mathbb{R}^n$ be a locally starshaped set, f and every g_i, $i = 1, \ldots, m$, semilocally convex functions. Then $Z = \left\{F(X) + \mathbb{R}_+^{m+1}\right\}$ is locally starshaped.*

A more general sufficient condition such that Z is locally starshaped is contained in the following result.

Theorem 16. *Let $X \subseteq \mathbb{R}^n$ be $\eta-$locally starshaped, f and every $g_i, i = 1, ...m$, semilocally preinvex with respect to η. Then $Z = \left\{F(X) + \mathbb{R}_+^{m+1}\right\}$ is locally starshaped.*

Proof. Let $z^1, z^2 \in Z$. We have to prove that $\lambda z^1 + (1-\lambda)z^2 \in Z, \forall \lambda \in \left[0, a(z^1, z^2)\right)$, i.e. there exists $x^3 \in X$ and $r^3 \in \mathbb{R}_+^{m+1}$ such that $\lambda z^1 + (1-\lambda)z^2 = F(x^3) + r^3$. We note that $z^1 \in Z$ implies the existence of $x^1 \in X$ and $r^1 \in \mathbb{R}_+^{m+1}$ such that $z^1 = F(x^1) + r^1$; in a similar way there exist $x^2 \in X$ and $r^2 \in \mathbb{R}_+^{m+1}$ such that $z^2 = F(x^2) + r^2$. If we put $x^3 = x^2 + \lambda\eta(x^1, x^2)$, since F is locally preinvex, we have $F(x^3) \leq \lambda F(x^1) + (1-\lambda)F(x^2), \forall \lambda \in \left[0, d(x^1, x^2)\right)$, i.e. there exists $r \in \mathbb{R}_+^{m+1}$ such that $F(x^3) + r = \lambda F(x^1) + (1-\lambda)F(x^2)$. So we have $\lambda z^1 + (1-\lambda)z^2 = \lambda F(x^1) + \lambda r^1 + (1-\lambda)F(x^2) + (1-\lambda)r^2 = F(x^3) + r + \lambda r^1 + (1-\lambda)r^2$.

Being $\mathbb{R}_+^{m+1}$ a convex cone, we have $r + \lambda r^1 + (1-\lambda)r^2 \in \mathbb{R}_+^{m+1}$, and if we put $r^3 = r + \lambda r^1 + (1-\lambda)r^2$ and $a(z^1, z^2) = d(x^1, x^2)$ we get the thesis. □

So, on the grounds of what previously expounded, following the same lines of Weir [15], we obtain the next generalization of the results of this author.

Theorem 17. *Consider the problem (P), where f and every $g_i, i = 1, ..., m$, are $\eta-$semilocally preinvex functions on the $\eta-$locally starshaped set of $\mathbb{R}^n$. Let $cl(Z)$ be a convex set and let x^* be an optimal solution for (P) for which a suitable constraint qualification holds. Then there exists $\lambda^* \geq 0$ such that (x^*, λ^*) is a saddle point for the Lagrangian function.*

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