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PERTURBATION SERIES OF THE EULER HYDRODINAMIC EQUATIONS AT SMALL FROUD'S NUMBER

Mario Bone
Institute of Oceanography and Fisheries
P.O. Box 500, Split, 21000, CROATIA
e-mail: bone@izor.hr

Abstract: Motions of the ocean and atmosphere are characterized by the large Reynolds and small Froud numbers. In order to describe these motions the Euler equations of ideal fluid are considered and the expansion in perturbation series is obtained using the dimensionless form depending on the Froud number. It is shown that expanding the dimensionless Euler momentum equation in the perturbation series it is defined only for the fluid in motion. The perturbation is singular and should include the zero order velocities in the perturbation series. In the C. Eckart notation motions of the atmosphere and oceans were considered as first or higher order perturbation terms which complicates definition of the first order energy equation. Taking into account singularity of the expansion the first order energy equation follows clearly from the applied perturbation method. The obtained equation has the form accepted by C. Eckart, although that in the perturbation series the first order velocities are of the zero order due to the singularity of the series.

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1. Introduction

The ocean and atmosphere motions are at high Reynolds number and C. Eckart [2] considered the first order perturbations of an ideal stratified hydrostatic fluid in order to analyze these motions. In this way he summarized results

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of the classical hydrodynamics of ocean and atmosphere. His approach to the perturbation method is heuristic due that he did not started considering the dimensionless equations depending on some small number, as it is usual in the perturbation theory [1]. In the C. Eckart [2] notation velocities are the first or higher order terms in the perturbation series which complicates definition of the first order energy equation The kinetic energy of the first order velocities obviously is of the second order and it is accepted in the first order equations by C. Eckart [2] as a compromise. Here, the perturbation series of the dimensionless hydrodynamic equations of an ideal fluid depending on the small Froud number was considered. It is demonstrated that the perturbation series is defined only for the fluid in motion, i.e. the perturbation series is singular and the equations should include velocities of the zero order. In this case the obtained perturbation equations have the form accepted by C. Eckart [2], apart that the first order velocities are of the zero order and the first order energy equation is clearly defined according to the applied perturbation method.

2. Dimensionless Equations

Let us consider the Euler hydrodynamic equation for momentum

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\omega \times \mathbf{v} = -\alpha \nabla p - \nabla \Phi \,. \tag{1}$$

Scaling velocity $\mathbf{v} = \overline{V}\mathbf{v}^*$, time $t = Tt^*$, distance $\mathbf{r} = H\mathbf{r}^*$, specific volume $\alpha = \overline{\alpha}\alpha^*$, pressure $p = \frac{gH}{\overline{\alpha}}p^*$ and potential $\Phi = gH\Phi^*$ where $\overline{V}$ is the characteristic velocity, g earth acceleration, H characteristic fluid depth and introducing the constrain for the time scale

$$T = \frac{H}{\sqrt{gH}} \tag{2}$$

(which is not the time scale of a hydrodynamic process) as well as taking into account that the square root of the Froud number ϵ is $\overline{V} = \epsilon\sqrt{gH}$, the dimensionless momentum equation can be written in suitable form in order to apply perturbation method at small Froud's number

$$\alpha^* \nabla^* p^* + \epsilon[\frac{\partial \mathbf{v}^*}{\partial t^*} + 2\sqrt{\frac{H}{g}}\omega \times \mathbf{v}^*] + \epsilon^2[\mathbf{v}^* \cdot \nabla^* \mathbf{v}^*] = -\nabla^* \Phi^* \,. \tag{3}$$

The close set of equations, including momentum, continuity and thermodynamic equations, is the quasi-linear system [3]. In the case of sea and atmosphere the restriction to the real analytic Cauchy data and equations [3] is realistic. According to the Cauchy-Kowaleski Theorem [3] in this case there exists unique real analytic solution for the considered system. Let us concer the dimensionless equations and the class of the nontrivial solutions depending on ϵ in some physical region and time interval. For two different characteristic velocities $\overline{V}_0$ and $\overline{V}$, where in both cases the characteristic depth H is the same, at each point $\mathbf{r}_0 = H\mathbf{r}_0^*$ of the considered physical region and the time $t_0 = \sqrt{\frac{H}{g}}t_0^*$ in the considered time interval can be written

$$\mathbf{v}(\mathbf{r}_0, t_0) = \overline{V}_0\mathbf{v}^*(\mathbf{r}_0^*, t_0^*, \epsilon_0) = \overline{V}\mathbf{v}^*(\mathbf{r}_0^*, t_0^*, \epsilon)\,, \tag{4}$$

and taking into account that the characteristic depth is the same in both cases, can be written

$$\frac{\mid \mathbf{v}^*(\epsilon) \mid}{\mid \mathbf{v}^*(\epsilon_0) \mid} = \frac{\epsilon_0}{\epsilon}\,. \tag{5}$$

It follows that $\mid \mathbf{v}^*(\epsilon) \mid < \infty$ is continuous functions of ϵ for $\epsilon > 0$, i.e. the ocean and atmosphere per definition in motion and the perturbation series are singular.

The continuity of the scalar variables in ϵ follows considering the characteristic fluxes, e.g. $\overline{V}_0\overline{\alpha}_0 = \overline{V}\,\overline{\alpha}$, as well as for the other variables.

In this case, according to Weierstrass Theorem [4] on the degree to which polynomials can approximate a continuous function, there exists a polynomial $\mathbf{B}_n^*(\epsilon)$ such that $\mid \mathbf{v}^*(\epsilon) - \mathbf{B}_n^*(\epsilon) \mid \leq \eta$, where $\eta \to 0$ when $n \to \infty$. Wendroff [4] gave the proof of the theorem using the Bernstein polynomial. It can be written

$$\begin{aligned}
\mathbf{v}^* &= \mathbf{v}_0^*(\epsilon_0) + \epsilon\mathbf{v}_1^*(\epsilon_0) + \epsilon^2\mathbf{v}_2^*(\epsilon_0) + \cdots\,, \\
\alpha^* &= \alpha_0^*(\epsilon_0) + \epsilon\alpha_1^*(\epsilon_0) + \epsilon^2\alpha_2^*(\epsilon_0) + \cdots\,, \\
p^* &= p_0^*(\epsilon_0) + \epsilon p_1^*(\epsilon_0) + \epsilon^2 p_2^*(\epsilon_0) + \cdots\,, \\
&\vdots
\end{aligned}$$

and the close nonlinear system of the hydrodynamic equations can be expanded in the perturbation series of ϵ.

3. Note on the C. Eckart Perturbation Series

The C. Eckart [2] approach to the oceans and atmosphere hydrodynamics is the groundwork on the application of perturbation method on the considered subject. In his book C. Eckart [2] summarized the results of the classical sea and air hydrodynamics, as noted above. This is the reason to discuss the results presented here in the context of the C. Eckart [2] book.

Let us for the non-homogeneous term the zero order potential be defined by geopotential and the first order potential by the tidal potential. The zero and first orders close system of the linear equations equal to the system of equations considered by C. Eckart [2] can be written although that the first order velocities should be of the zero order due to the singularity of the perturbation series. The zero and first perturbation equations of momentum are

$$\alpha_0^* \nabla^* p_0^* = -\nabla^* \Phi_0^* \,, \tag{6}$$

$$\alpha_0^* \nabla^* p_1^* + \alpha_1^* \nabla^* p_0^* + \frac{\partial \mathbf{v}_0^*}{\partial t^*} + 2\sqrt{\frac{H}{g}} \omega \times \mathbf{v}_0^* = -\nabla^* \Phi_1^* \,, \tag{7}$$

and it can be seen that the zero and first order equations, with the resulting first order energy equation, differ from the equations accepted by C. Eckart [2] only in the notation of the velocity order in the perturbation series, as it is noted.

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