International Journal of Pure and Applied Mathematics
Volume 49 No. 1 2008, 135-146

# A COUPLED STOCHASTIC DIFFERENTIAL MODEL IN FINANCE UNDER LOCAL LIPSCHITZ NONLINEARITY

T.E. Govindan[1], Carlos Ibarra-Valdez[2], J. Ruiz de Chávez[3 §]

[1]Department of Mathematics, ESFM
Instituto Politécnico Nacional
México D.F., 07738, MEXICO
e-mail: tegovindan@yahoo.com

[2,3]Departamento de Matematicas
Universidad Autonoma Metropolitana – UAM
Unidad Iztapalapa
Av. San Rafael Atlixco No. 186, Col Vicentina
A.P. 55-534, Iztapalapa, Mexico D.F., 09340, MEXICO
[2]e-mail: ibvc@xanum.uam.mx
[3]e-mail: jrch@xanum.uam.mx

**Abstract:** A local Lipschitz condition for a dynamical stochastic coupled model of financial markets is considered. Existence and uniqueness of solutions and their continuous dependence with respect to the initial conditions are established. This generalization of the usual Lipschitz assumption allows to include as examples, markets with time-varying interest rates and volatilities, among others.

**AMS Subject Classification:** 60H10, 62P05
**Key Words:** stochastic coupled model, trading, Itô equation, local Lipschitz condition, existence and uniqueness of a solution, continuous dependence

## 1. Introduction

In [3], we presented a dynamical model, which is a coupling of a deterministic

Received: October 25, 2008

§Correspondence author

dynamics for trading, with a stochastic dynamics for securities prices evolution. In order to establish existence, uniqueness and stability results, we assumed Lipschitz conditions on both vector fields: the one giving the deterministic dynamics and the other two, representing the stochastic dynamics.

On the other hand, the Lipschitz hypothesis for models describing financial markets dynamics is rather restrictive, since several standard and important models in economics and finance involve non-Lipschitz coefficients, such as those developed for modeling interest rate dynamics (a very well-known example is the CIR model [5]).

In our model, consisting in asset price evolution and trading dynamics, the necessity of considering non Lipschitz assumptions appears in both components: in the asset prices (for example, bonds which are to be represented by interest rates) and also in the trading, since some stylized trader behaviors can be modeled by Cobb-Douglas type functions.

There are several ways of relaxing the Lipschitz condition, nevertheless, to the best of our knowledge, no systematic and generally accepted approach has been found. Some of the well-known approaches are represented by the works of Yamada [7], Zhang and Fang [8], Deelstra and Delbaen [2], but, none of these can be directly applied to our case. Therefore, the objective of this work is to use the local Lipschitz condition to obtain existence, uniqueness and continuous dependence of solutions for our dynamical coupled model, in a similar fashion as in Wissel [6] where such conditions were introduced for stochastic differential equations with applications to interest rate models. It may not be an ideal relaxation, but, it at least allows one to apply the results established in the case of time-varying interest rates and volatility coefficients, see [5], p. 254.

In Section 2 we introduce the stochastically coupled model, together with its mathematical formulation; in Section 3 we state the locally Lipschitz condition, and establish the main results. In Section 4 some financial applications are given.

## 2. The Dynamical Stochastic Coupled Model

As in [3], we shall consider a market with $n$ assets (securities, shares, options, etc.) and $m$ market participants (agents, traders) who exchange inside a trading floor. Let $S_i(t)$ be the spot unitary price of security $i$ at time $t$, and $S(t) = (S_1(t), ..., S_n(t))$. Now let $q_i^j(t)$ for $(1 \leq i \leq n,\ 1 \leq j \leq m)$, be the amount of security $i$ held by trader $j$ at time $t$. Denote by $q^j = (q_1^j, \ldots, q_n^j)$ the portfolio held by participant $j$, and by $Q = \left[q_i^j\right]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ the state of the market at time $t$.

We now make the following assumptions:

(1) Trading activities are modeled as *deterministic* since traders take decisions that follow pre-determined styles depending on price levels, trends and flows of information.

(2) Price dynamics is definitively random since it results from the interaction between two complex forces: the trading dynamics and a multitude of factors such as information, external shocks, rumors, etc.

(3) Assets can be traded continuously and divided infinitesimally.

(4) Assets pay no dividends. There are also no taxes or transaction costs in the market. See [3] for a more detailed account of the model and pertaining remarks.

The precise mathematical formulation goes as follows:

Let $\mathbb{R}^n$ be an $n$-dimensional Euclidean space with the norm defined as $|x|^2 = \sum_{i=1}^{n} |x_i|^2$, where $x = (x_2, x_2, \dots, x_n)$ is a vector in $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote the space of $n \times m$ matrices with the norm defined as $\|A\|^2 = \sum_i \sum_j |a_{ij}|^2 = \mathrm{tr} AA^T$, where $A = [a_{ij}]$ is a matrix and the transpose of a matrix $A$ is denoted by $A^T$ .

By a filtered probability space with increasing family of Borel fields which is denoted as $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ we mean a probability space $(\Omega, \mathcal{F}, P)$ with a system $\{\mathcal{F}_t\}_{t\geq 0}$ of sub-Borel fields of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$, if $s < t$. It is also called a 'stochastic basis'.

We denote by $\mathcal{C} = \mathcal{C}[0, T]$ the Banach space of $\mathbb{R}^n$-valued continuous functions defined on [0,T], and equipped with the usual supremum norm, i.e., $\|v\|_{\mathcal{C}} = \sup_{0\leq t\leq T} |v(t)|$; and by $\mathcal{C}^r = \mathcal{C}^r[0, T]$, for $r \in \mathbb{N}$, the vector space of $r$-fold continuously differentiable functions.

Consider the dynamics of the trading floor modeled by the following system of deterministic differential equations

$$\frac{dq_i^j(t)}{dt} = f_{ij}\big(t, Q(t), S(t)\big), \tag{2.1}$$

where $f_{ij} : \mathbb{R}_+ \times \mathbb{R}^{n\times m} \times \mathbb{R}^n \to \mathbb{R}$ $(1 \leq i \leq n,\ 1 \leq j \leq m,\ \mathbb{R}_+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty))$ are continuous, or in the matrix form as

$$\frac{dQ(t)}{dt} = F(t, Q(t), S(t)), \qquad t > 0, \tag{2.2}$$

$$Q(0) = Q_0.$$

where $F = [f_{ij}]$ and $Q$ are mappings $F : \mathbb{R}_+ \times \mathbb{R}^{n\times m} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $Q : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ and $\|Q_0\| < \infty$. The price dynamics is generated by the system of stochastic differential equations

$$dS_i(t) = a_i(t, Q(t), S(t))dt + \sum_{k=1}^{K} b_{ik}\big(t, Q(t), S(t)\big)dW_k(t), \tag{2.3}$$

where

$$a_i : \mathbb{R}_+ \times \mathbb{R}^{n\times m} \times \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, n,$$

$$b_{ik} : \mathbb{R}_+ \times \mathbb{R}^{n\times m} \times \mathbb{R}^n \to \mathbb{R};\ k = 1, 2, \ldots, K,$$

and $W_k(t)$ is a standard one-dimensional Wiener process. Let the continuous mappings $A : \mathbb{R}_+ \times \mathbb{R}^{n\times m} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R}_+ \times \mathbb{R}^{n\times m} \times \mathbb{R}^n \to \mathbb{R}^{n\times K}$ be defined by $A(t, Q, s) = \big(a_1(t, Q, s), ..., a_n(t, Q, s)\big)$ and $B(t, Q, S) = [b_{ik}(t, Q, S)]_{\substack{1\le i\le n\\ 1\le k\le K}}$. Finally, $W(t) = \big(W(t), \ldots, W_K(t)\big)$ is a $K$-dimensional Wiener process with respect to the stochastic basis. Now the above componentwise system (2.3) can be expressed as the following Itô's SDE:

$$\begin{aligned} dS(t) &= A\big(t, Q(t), S(t)\big)dt + B\big(t, Q(t), S(t)\big)dW(t), \quad t > 0, \\ S(0) &= S_0, \end{aligned} \tag{2.4}$$

where $S_0$ is a random variable independent of $\{W(t), t \ge 0\}$ such that $E|S_0|^2 < \infty$.

We now introduce the solution concept for the coupled system (2.2) and (2.4).

**Definition 1.** By a solution of the coupled system (2.2) and (2.4), we mean a joint stochastic process $\{S(t), Q(t); t \in [0, T]\}$ for some $0 < T < \infty$, defined on the stochastic basis $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\ge 0})$ such that:

(i) $Q(t)$ and $S(t)$ are $\mathcal{F}_t$-adapted for each $t \in [0, T]$ and continuous in $t$ almost surely (a.s.).

(ii) $E|S(t)|^2 < \infty$ and $E\|Q(t)\| < \infty$, for each $t \in [0, T]$.

(iii) $S(t)$ satisfies the stochastic integral equation:

$$S(t) = S_0 + \int_0^t A\big(u, Q(u), S(u)\big)du + \int_0^t B\big(u, Q(u), S(u)\big)dW(u), \quad \text{a.s.} \tag{2.5}$$

for each $t \in [0, T]$, where the second integral is understood in the sense of Itô's stochastic integral (see [1], [5]), and

(iv) $Q(t)$ satisfies the integral equation

$$Q(t) = Q_0 + \int_0^t F\big(u, Q(u), S(u)\big)du, \quad t > 0, \quad \text{a.s.}. \tag{2.6}$$

## 3. Existence, Uniqueness and Continuous Dependence of Solutions

In this section we consider our main results. We refer to Wissel [6] for details on the assumptions made here and to Govindan [4] for the ideas of the proofs.

The following conditions hold a.s.:

**Assumptions.** There exist positive constants $C_i(T)$, $i = 1, 2$ such that

$(A_1)$ $\big\|F(t, Q_1, s') - F(t, Q_2, s'')\big\| \leq C_1(T)\big[\|Q_1 - Q_2\| + |s' - s''|\big]$.

$(A_2)$ $|A(t, Q_1, s') - A(t, Q_2, s'')|^2 + \|B(t, Q_1, s') - B(t, Q_2, s'')\|^2$
$$\leq C_2(T)\big[\|Q_1 - Q_2\|^2 + |s' - s''|^2\big].$$

**Theorem 1.** *Suppose that the assumptions $(A_1)$-$(A_2)$ are satisfied. Then there exists a time $0 < \theta_m = \theta_{\max} \leq \infty$ such that the coupled system (2.2) and (2.4) has a unique solution in this interval. Further, if $\theta_m < \infty$, then* $\lim\limits_{t \uparrow \theta_m} E|X(t)|^2 = \infty$.

To prove this theorem, we need some preparation, see [4].

Let $\Gamma^1_{r_1}$ be the closed subspace of $\mathcal{C}_1 = \mathcal{C}_1\big([0, T], L_2(\Omega, \mathbb{R}^n)\big)$ $(0 < T < \infty)$ consisting of measurable and $\mathcal{F}_t$ -adapted processes $\{S(t), t \in [0, T]\}$ such that the norm

$$\|S\|^2_{\mathcal{C}_1} := \sup_{0 \leq t \leq T} E|S(t) - S(0)|^2 \leq r_1 \quad (r_1 > 0).$$

Similarly, $\Gamma^2_{r_2}$ is the closed subspace of $\mathcal{C}_2 = \mathcal{C}_2\big([0, T], L_2(\Omega, \mathbb{R}^{n \times m})\big)$ consisting of measurable and $\mathcal{F}_t$ -adapted processes $\{Q(t), t \in [0, T]\}$ such that the norm

$$\|Q\|^2_{\mathcal{C}_2} := \sup_{0 \leq t \leq T} E\|Q(t) - Q(0)\|^2 \leq r_2 \quad (r_2 > 0).$$

Define the maps $G_1$ and $G_2$ on $\Gamma^i_{r_i}$, $i = 1, 2$:

$$(G_1 S)(t) \;=\; S_0 + \int_0^t A\big(u, Q(u), S(u)\big)du + \int_0^t B\big(u, Q(u), S(u)\big)dW(u)$$

and

$$(G_2 Q)(t) = Q_0 + \int_0^t F\big(u, Q(u), S(u)\big)du.$$

**Lemma 1.** *a) The Lipschitz and linear growth conditions in* $(A_2)$ *are uniform in* $Q$.

*b) The local Lipchitz and linear growth conditions in* $(A_1)$ *are uniform in* $S$.

*Proof.* a) Let $C(T) = \max\{C_i(T),\ i = 1,2\}$. Consider

$$Q_1(t) - Q_2(t) = \int_0^t \left[F\big(u, Q_1(u), S_1(u)\big) - F\big(u, Q_2(u), S_2(u)\big)\right] du.$$

By exploiting assumption $(A_1)$, we get

$$\|Q_1(t) - Q_2(t)\| \le C(T)\int_0^t |S_1(u) - S_2(u)| du + C(T)\int_0^t \|Q_1(u) - Q_2(u)\| du\,, \text{ a.s.}$$

An application of Gronwall's Lemma [1] then yields

$$\begin{aligned} \|Q_1(t) - Q_2(t)\| &\le C(T)e^{C(t)t}\int_0^t |S_1(u) - S_2(u)| du \\ &\le tC(T)e^{C(T)t}\|S_1 - S_2\|_{\mathcal{C}_1}, \quad \text{a.s. } t \in [0,T]. \end{aligned}$$

Hence,

$$\begin{aligned} &|A(t,Q_1,S_1) - A(t,Q_2,S_2)|^2 + \|B(t,Q_1,S_1) - B(t,Q_2,S_2)\|^2 \\ &\qquad \le C(T)\left[t^2C^2(T)e^{2C(T)t}\|S_1 - S_2\|^2_{\mathcal{C}_1} + \|S_1 - S_2\|^2_{\mathcal{C}_1}\right] \\ &\qquad \le H_1(T)\|S_1 - S_2\|^2_{\mathcal{C}_1}, \quad \text{a.s.}, \qquad (3.1) \end{aligned}$$

where $H_1(T) = C(T)\big[T^2C^2(T)e^{2TC(T)} + 1\big]$.

Next, using assumption $(A_1)$, we have

$$\begin{aligned} \|Q(t)\| &\le \|Q_0\| + \int_0^t \|F\big(t, Q(u), S(u)\big)\| du \\ &\le \|Q_0\| + C(T)t + C(T)\int_0^t \|Q(u)\| du + C(T)\int_0^t \|S(u)\| du, \quad \text{a.s.} \end{aligned}$$

Letting $h(t) = \|Q_0\| + C(T)t + C(T)\int_0^t \|S(u)\| du$, we have

$$\|Q(t)\| \le h(t) + C(T)\int_0^t \|Q(u)\| du, \quad \text{a.s.}$$

Another application of Gronwall's Lemma yields

$$\|Q(t)\| \le h(t)e^{C(T)t}, \quad \text{a.s.} \qquad t \in [0,T].$$

Hence,

$$|A(t,Q,S)|^2 + \|B(t,Q,S)\|^2 \le C(T)\left[h^2(T)e^{2C(T)t} + |S|^2 + 1\right]$$
$$\le C^2(T)\left[3\|Q_0\|^2 + 3C^2(T)t^2 + 1 + (3C^2(T)t+1)\|S\|^2_{\mathcal{C}_1}\right]$$
$$\le H_2(T)\left[1 + \|S\|^2_{\mathcal{C}_1}\right], \quad \text{a.s.}$$

where $H_2(T) = C^2(T)\max\{3\|Q_0\|^2 + 3C^2(T)t^2 + 1, 3C^2(T)t+1\}$.

b) The proof follows as in Part (a) and we obtain

$$\|F(t,Q_1,S_1) - F(t,Q_2,S_2)\| \le H_3(T)\|Q_1 - Q_2\|_{\mathcal{C}_2}, \quad \text{a.s.}$$

and

$$\|F(t,Q,S)\| \le H_4(T)\|Q\|_{\mathcal{C}_2}. \quad \text{a.s.}$$

□

**Lemma 2.** *For arbitrary $Q \in \Gamma^2_{r_2}$, $(G_2Q)(\cdot)$ is continuous in $L_2(\Omega,\mathbb{R}^{n\times m})$.*

*Proof.* Take $t,h>0$, such that $t,t+h \in [0,T]$. Consider

$$\big(G_2Q\big)(t+h) - \big(G_2Q\big)(t) = \int_t^{t+h} F(u,Q(u),S(u))du.$$

Thus, from Lemma 1, we have

$$E\|(G_2Q)(t+h) - (G_2Q)(t)\|^2 = E\left\|\int_t^{t+h} F(u,Q(u),S(u))du\right\|^2$$
$$\le h\int_t^{t+h} E\|F(u,Q(u),S(u)\|^2du \le h^2H_4(T)\left[1 + \|Q\|_{\mathcal{C}_2}\right].$$

Hence,

$$E\|(G_2Q)(t+h) - (G_2Q(t)\|^2 \to 0 \qquad \text{as} \qquad h \to 0.$$

Similarly, one can show that

$$E\|(G_2Q)(t-h) - (G_2Q)(t)\|^2 \to 0 \qquad \text{as} \qquad h \to 0,$$

proving the claim. □

**Lemma 3.** *For arbitrary $S \in \Gamma^1_{r_1}$, $(G_1S)(\cdot)$ is continuous in $L_2(\Omega,\mathbb{R}^n)$.*

*Proof.* Let $t,h>0$ such that $t$, $t+h \in [0,T]$. Consider

$$(G_1S)(t+h) - (G_1S)(t) = \int_t^{t+h} A(u,Q(u),S(u))du + \int_t^{t+h} B(u,Q(u),S(u))dW(u). \quad (3.2)$$

The first integral on the right hand side of (3.2) is treated exactly as before. So, let us only look at the stochastic integral $I_2(t) = \int_0^t B(u, Q(u), S(u))dW(u)$ :

$$\begin{aligned} E|I_2(t+h) - I_2(t)|^2 &= E\left|\int_t^{t+h} B(u, Q(u)), S(u))dW(u)\right|^2 \\ &= E\int_t^{t+h} |B(u, Q(u), S(u))|^2 du \\ &\leq h^2 H_2(T)\left[1 + \|S\|^2_{\mathcal{C}_1})\right], \end{aligned}$$

which goes to zero as $h \to 0$.

Similarly, one can show that

$$E|(G_1S)(t-h) - (G_1S)(t)|^2 \longrightarrow 0, \qquad \text{as} \qquad h \to 0. \qquad \square$$

**Lemma 4.** *$G_2$ maps $\Gamma^2_{r_2}$ into itself, that is $G_2(\Gamma^2_{r_2}) \subset \Gamma^2_{r_2}$.*

*Proof.* For any $Q \in \Gamma^2_{r_2}$, we have

$$\begin{aligned} E\|(G_2Q)(t) - Q(0)\|^2 &= E\left\|\int_0^t F(u, Q(u), S(u))du\right\|^2 \\ &\leq t\int_0^t E\|F(u, Q(u), S(u))\|^2 du\,. \end{aligned}$$

By picking a $t_1 \in [0, T]$ sufficiently small, it follows that

$$E\|(G_2S)(t) - Q_0\|^2 \leq r_2,$$

for all $t \in [0, t_1]$. $\square$

**Lemma 5.** *$G_1(\Gamma^1_{r_1}) \subset \Gamma^1_{r_1}$.*

*Proof.* For any $S \in \Gamma^1_{r_1}$, consider

$$(G_1S)(t) - S_0 = \int_0^t A(u, Q(u), S(u))du + \int_0^t B(u, Q(u), S(u))dW(u).$$

Now

$$\begin{aligned} &E|(G_1S)(t) - S(0)|^2 \\ &\quad = E\left|\int_0^t A(u, Q(u), S(u))du + \int_0^t B(u, Q(u), S(u))dW(u)\right|^2 \\ &\quad \leq 2t\int_0^t E|A(u, Q(u), S(u))|^2 du + 2\int_0^t E|B(u, Q(u), S(u))|^2 du \\ &\qquad\qquad \leq 2(t+1)H_2(T)[1 + \|S\|^2_{\mathcal{C}_1}] \leq r_1, \end{aligned}$$

for all $t \in [0, t_2]$, by choosing $0 < t_2 \leq t_1$ smaller, if necessary. $\square$

*Proof of Theorem 1.* Let $S_1, S_2 \in \Gamma^1_{r_1}$. Then for $t \in [0, t_2]$, we have

$$\begin{aligned}(G_1S_1)(t) - (G_1S_2)(t) &= \int_0^t [A(u, Q_1(u), S_1(u)) - A(u, Q_2(u), S_2(u))]du \\ &+ \int_0^t [B(u, Q_1(u), S_1(u)) - B(u, Q_2(u), S_2(u))]dW(u).\end{aligned}$$

Hence,

$$\sup_{0\le t\le t_2} E|(G_1S_1)(t) - (G_1S_2)(t)|^2 \le 2(t+1)H_1(T)\|S_1 - S_2\|^2_{\mathcal{C}_1}.$$

Choose $t_3 > 0$ still smaller, if necessary, so that $G_1$ is a strict contraction on $\Gamma^1_{r_1}$ in the norm of $C([0, T]; L_2(\Omega, \mathbb{R}^n))$. Therefore $G_1$ has a unique fixed point $S \in \Gamma^1_{r_1}$ and this fixed point is the unique solution of stochastic equation (4) on $[0, t_3]$. Next, we continue the solution for $t \ge t_3 = \theta_1$, say for $t \in [\theta_1, \theta_2]$. We say that a function $\widehat{S}(t)$ is a continuation of $S(t)$ to the interval $[\theta_1, \theta_2]$ if:

(a) $\widehat{S} \in \mathcal{C}([0, \theta_2], L_2(\Omega, \mathbb{R}^n))$, and

(b) $\widehat{S}(t) = \widehat{S}(t - \theta_1) + \int_{\theta_1}^t A(u, Q(u), \widehat{S}(u))du + \int_{\theta_1}^t B(u, Q(u), \widehat{S}(u))dW(u)$, a.s.

The terminology 'continuation' applied to $\widehat{S}(t)$ is justified by the observation that if we define a new function $v(t)$ on $[0, \theta_2]$ by setting

$$v(t) = \begin{cases} S(t) & \text{if} \quad 0 \le t \le \theta_1 \\ \widehat{S}(t) & \text{if} \quad \theta_1 \le t \le \theta_2. \end{cases}$$

then $v(t)$ is a solution of the stochastic equation (4) on $[0, \theta_2]$. The existence and uniqueness of the continuation $\widehat{S}(t)$ is demonstrated exactly as above with some minor changes. Repeating this procedure, one continues the solution till the time $\theta_m = \theta_{\max}$ where $[0, \theta_m]$ is the maximum interval of the existence and uniqueness of a solution. For $\theta_m$ finite, $\lim E|S(t)|^2 = +\infty$ as $t \uparrow +\infty$. If not, then there exist a sequence $\{\tau_n\}$ converging to $\theta_m$ and a finite positive number $\delta$ such that $E|S(\tau_n)|^2 < \delta$, for all $n$. Taking $n$ sufficiently large so that $\tau_n$ is infinitesimally close to $\theta_m$, one can use the previous arguments to extend the solution beyond $\theta_m$, which is a contradiction. It can be shown similarly that $G_2$ has a unique fixed point in $\Gamma^2_{r_2}$. This completes the proof.

We now study the local continuous dependence of the solution $S(t)$ of equation (4) with respect to the initial condition $S_0$. Define this relation as a map:

$$\phi : S_0 \to S(t).$$

**Theorem 2.** *Under assumptions* $(A_1)$*-*$(A_2)$*,* $\phi$ *is continuous.*

*Proof.* Consider

$$S^n(t) - S(t) = S_0^n - S_0 + \int_0^t [A(u, Q^n(u), S^n(u)) - A(u, Q(u), S(u))]\, du + \int_0^t [B(u, Q^n(u), S^n(u)) - B(u, Q(u), S(u))]\, dW(u).$$

Then

$$\begin{aligned} &E|S^n(t) - S(t)|^2 \\ &\le 3E|S_0^n - S_0|^2 + 3(t+1)\int_0^t E\left[|A(u, Q^n(u), S^n(u)) - A(u, Q(u), S(u))|^2 \right. \\ &\qquad \left. + |B(u, Q^n(u), S^n(u)) - B(u, Q(u), S(u))|^2\right] du \\ &\le 3E|S_0^n - S_0|^2 + 3(t+1)\int_0^t H_1(T)\|S^n - S\|_{\mathcal{C}_1}^2\, du \\ &\le 3E|S_0^n - S_0|^2 + 3(T+1)tH_1(T)\|S^n - S\|_{\mathcal{C}_1}^2. \end{aligned}$$

By Gronwall's Lemma,

$$\|S^n - S\|_{\mathcal{C}_1}^2 \le 3E|S_0^n - S_0|^2 e^{3(T+1)H_1(T)t^2}, \quad t \in [0, T],$$

which establishes the result. □

## 4. An Application to a Coupled Model with Time Varying Interest Rates and Volatility Assets

Following Shreve [5], Problem 5.8, p. 254, we have that every strictly positive asset is a generalized geometric Brownian motion with volatility coefficient $\sigma$ possibly random. Hence, the example that follows is rather, general, excepting that we assume a deterministic and time-varying volatility.

We shall consider a market consisting of two assets: (1) a bond, with spot unitary price denoted by $S_1(t)$, which has a time varying interest rate and obeys a deterministic dynamics; (2) an equity with spot price $S_2(t)$ which has time varying return rate and volatility, and obeys a (generalized) geometric Brownian motion. The dynamical equations are thus given by

$$dS_1(t) = r(t)\, S_1(t)dt, \qquad dS_2(t) = \mu(t)S_2(t)dt + \sigma(t)S_2(t)dW(t), \tag{4.1}$$

where $r, \mu, \sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous or at least locally bounded. This is a sort of generalized Black Scholes market, for which the standard european options and some other financial derivatives written on the equity can be valued by the standard generalizations of the Black Scholes formula, see [5]. There are two traders: the first one can be identified with an investor or group of investors who have as targets to obtain certain, no less, no more, amounts of assets, denoted respectively by $q_1^*$, $q_2^*$. Hence they behave like the Walras' auctioner, with quantities instead of prices, and equilibria prices substituted by the set points $q_1^*$, $q_2^*$. The dynamics for the first trader is then

$$\frac{dq_1^1}{dt} = a_1(q_1^1(t) - q_1^*), \qquad \frac{dq_2^1}{dt} = a_2(q_2^1(t) - q_2^*). \tag{4.2}$$

The second trader could be identified with a central bank, or with any authority or investor interested in stabilizing prices about the benchmarks $S_1^*, S_2^*$. Thus he/she behaves like the Walras' auctioner with a mixing among prices and quantities

$$\frac{dq_1^2}{dt} = b_1(S_1(t) - S_1^*), \qquad \frac{dq_2^2}{dt} = b_2(S_2(t) - S_2^*). \tag{4.3}$$

Note that the constants $a_1, a_2, b_1$ and $b_2$ are all positive; and that the solutions to price equations are positive, and hence make financial sense, since they are essentially exponential functions or processes:

$$S_1(t) = S_1(0)\exp\left(\int_0^t r(u)du\right), \qquad S_2(t) = S_2(0)\exp(\Sigma\, W(t) + (R - \Sigma^2/2)t),$$

where

$$R = \frac{1}{t}\int_0^t r(u)du, \qquad \Sigma = \sqrt{\frac{1}{t}\int_0^t \sigma^2(u)du}.$$

The above assertion is due in part, to the fact that there is no feedback from trading (the demand/supply dynamics) to prices evolution. Of course, in a more realistic example, such an influence must be considered, in spite of the fact that many technical and financial questions could have no clear answer. On the other hand, We have here feedback from prices evolution to trading dynamics. So there is a coupling yet it has only one directionality.

The benchmarks $q_1^*$, $q_2^*$, and $S_1^*$, $S_2^*$ could be set as time dependent, which is more realistic, but then in the first case (portfolio dynamics of the first trader) we would not have exactly a 'Walras Tattônement', and in the second one (portfolio dynamics of the second trader), there would be no 'mixed 'Tattônement'. These dynamics are more difficult to interpret. Also, in this connection, if one allows feedback from trading to prices dynamics, by considering, say:

$$dS_2(t) = \mu(t)(S_2(t) - q_2^2(t))dt + \sigma(t)S_2(t)dW(t),$$

then we would have a sort of 'time-dependent mean reversion', which is harder to analyze.

Defining

$$Q(t) = \begin{bmatrix} q_1^1(t) & q_1^2(t) \\ q_2^1(t) & q_2^2(t) \end{bmatrix}, \qquad F(t,Q,S) = \begin{bmatrix} a_1(q_1^1 - q_1^*) & b_1(S_1 - S_1^*) \\ a_2(q_2^1 - q_2^*) & b_2(S_2 - S_2^*) \end{bmatrix},$$

and

$$A(t,Q,S) = \begin{bmatrix} r(t)S_1 \\ \mu(t)S_2 \end{bmatrix}, \qquad B(t,Q,S) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma(t)S_2 \end{bmatrix},$$

one can express equations (4.1)-(4.3) as the coupled model (2.2)-(2.4). It is straightforward to check that these coefficients satisfy hypothesis $(A_1)$ and $(A_2)$. Therefore, Theorems 1 and 2 are applicable in this situation.

## References


[1] L. Arnold, *Stochastic Differential Equations: Theory and Applications,* Wiley-Interscience, John Wiley and Sons, New York-London-Sydney (1974).

[2] G. Deelstra, F. Delbaen, *Existence of Solutions of Stochastic Differential Equations Related to the Bessel Process,* Selected Papers, Homepage of Prof. Freddy Delbaen: http://www.math.ethz.ch/~delbaen/ftp/preprints/existentie_rev.pdf.

[3] T.E. Govindan, C. Ibarra Valdez, J. Ruiz de Chavez, A dynamical stochastic coupled model for financial markets, *Physica A*, **381** (2007), 317-328.

[4] T.E. Govindan, Autonomous semilinear stochastic Volterra integrodifferential equations in Hilbert spaces, *Dynamic Systems Appl.*, **3** (1994), 51-74.

[5] S.E. Shreve, *Stochastic Calculus for Finance II Continuous-Time Models,* Springer Finance, New York (2004).

[6] J. Wissel, Some results on strong solutions of SDE with applications to interest rate models, *Stochastic Proc. Appl.*, **117** (2007), 720-741.

[7] T. Yamada, On the successive approximation of solutions of stochastic differential equations, *J. Math. Kyoto Univ.*, **21**, No. 3 (1981), 501-515.

[8] S. Zhang, T. Fang, A study of a class of stochastic differential equations with non-Lipschitzian coefficients, *Probab. Theory Related Fields*, **132** (2005), 356-390.