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# CONCLUDING REMARKS TO PAPER PROPERTIES OF FUNCTION SPACES GENERATED BY THE AVERAGED MODULI OF SMOOTHNESS

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**Abstract:** This paper contains the material of Section 6 of [4] which has not been included in the original publication [4], due to an editorial error during the pre-print preparation of the journal issue containing the original submission (see the Editorial Note at the end of the presend text). The present paper consists of two sections: Section 1 provides the necessary preliminary orientation, while Section 2 *coincides exactly with the unpublished Section 6 in the original submission*. In the present paper, the list of references contains six publications (more precisely, [9, 10, 8, 6, 7, 3]) which appeared only in Section 6 of the original submission, and have consequently *not* been included in the list of references in [4]. The present paper is *an integral part* of [4], and the two papers should be considered only together, *as a single complete entity*. The readers can also find the whole original material (as it was intended to appear in [4]) in the R&D Report [5].

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## 1. Orientation

This paper contains the material of Section 6 of [4] which has not been included in the original publication [4], due to an error of the editor during the journal's process of reviewing and revision of the original submission (see the Editorial Note at the end of the presend text). The next Section *coincides exactly with the unpublished Section 6 in the original submission*. In the present paper, the list of references contains the publications [9, 10, 8, 6, 7, 3] which appeared only in Section 6 of the original submission, and have consequently *not* been included in the list of references in [4]. References [4, 5] are relevant only to the afore-mentioned editorial error, and, as such, they appear only in the present paper.

This article is *an integral part* of [4], and the two articles should be considered only together, *as a single complete entity*. The readers can also find the whole original material (as it was intended to appear in [4]) in the R&D Report [5].

All the necessary preliminaries needed for reading [4] and the present article can be found in Section 2 of [4]. To make this paper self-contained, here we only recall from [4] the definition of Peetre's $K$-functional

$$K(t, f; A, B) = K(t, f) = \inf_{f=f_0+f_1} (\|f_0\|_A + t\|f_1\|_B),$$

where $A$, $B$ are normed spaces (or, most generally, quasi-seminormed abelian groups).

## 2. Concluding Remarks to [4]

In this section, we make a concise analysis of the results, obtained in [1] and presented briefly in [2] and in detail here *(author's remark: i. e., in [4])*, in the context of the development of the theory of function spaces and approximation theory, from the period of writing [1] (1983–1986) to the present day (2007). Based on this analysis, we propose some model topics for further research.

The concept of Wiener amalgam spaces was introduced by H. G. Feichtinger (see, e. g. [9] and the references there), and a study of their properties was initiated by Feichtinger and an increasing number of other authors. It turns out that this general concept unites in itself many previous particular constructions used in different contexts for different purposes without previous connection among these analogous constructions. The results of [1] presented in this paper *(author's remark: i. e., in [4])* show that the spaces generated by the averaged moduli of smoothness are important for numerical analysis particular cases of Wiener amalgam spaces. The results on the equivalence between the averaged moduli and $K$-functionals obtained in [1] show that it is very important to try to extend the theory of $K$-functionals to the case when the spaces $A$, $B$ in the $K$-functional $K(t,.;A,B)$ *depend on the step* $t$. Based on the negative results obtained in the previous section *(author's remark: i. e., in Section 5 of [4])*, in [2] we conjectured that the $A$-spaces induced by the averaged moduli (and, more generally, space scales obtained via the real interpolation functor applied on a $K$-functional between Wiener amalgam spaces with quasi-seminorms depending on the step of the $K$-functional) are, in general, **not** closed with respect to real or complex interpolation (which is a major difference with the classical interpolation spaces induced by the integral moduli). Our conjecture has recently been proved to be correct (see [10]). In contrast, the interpolation techniques developed in [1] and presented in the previous section of this paper, work for both classical interpolation spaces and Wiener amalgam spaces. The interpolation techniques presented here *(author's remark: i. e., in [4])* for the $A_{p,t}$-spaces and other special Wiener amalgam spaces like the $\dot{W}^k_{p,t}$-spaces can be extended for general Wiener amalgam spaces depending on the step $t$ of the $K$-functional.

New results have been obtained about Marchaud-type inequalities, some of which involve spaces which are particular Wiener amalgams (see, e. g., [8] and the references therein). We have not discussed here *(author's remark: i. e., in [4])* the progress in this direction because we intend to consider this topic in detail in the forthcoming paper [6] where related types of inequalities will be considered in greater generality, in the context of Wiener amalgam spaces and even more general types of spaces and $K$-functionals, in which the spaces depend on the step of the $K$-functional.

A close relationship was established between the theory of $K$-functionals and optimal smoothing techniques based on a small-penalty approach to Tikhonov regularization of ill-posed inverse problems (see [7] and the references therein). In this case, of interest are only very small and very large values of the step $t : 0 < t < \infty$, i. e., only values of $t$ and $1/t$ close to 0. We note that, while intermediate values of $t$ (i.e., $t : \varepsilon < t < 1/\varepsilon$ for any $\varepsilon > 0$) are 'responsible' for the bulk of the norm of a real interpolation space, it is only the range of values of $t$ and $1/t$ close to 0 that determines whether a function is, or is not, element of the space in the set-theoretic sense. Because of this, we propose to develop a new approach to the computation of $K$-functionals between spaces which depend on the step $t$ of the $K$-functional (and for Wiener amalgam spaces, in particular), based on the following propositions:

1. Instead of trying to obtain embedding results (quantitative theory of interpolation spaces), try to obtain only set-theoretic inclusions (qualitative theory of interpolation spaces).

2. The advantage of pursuing the more modest objective of qualitative theory is that the $K$-functionals have to be computed only for values of $t$ and/or $1/t$ close to 0.

3. As a main general new tool for computation of general classes of $K$-functionals

$$K(t, .) = K^{(t)}(t, .) = K(t, .; A(t), B(t))$$

between spaces which may depend on the step of the $K$-functional, we propose to use standard *singular perturbation expansion techniques* starting from the known 'unperturbed' cases corresponding to $t = 0$ or $t = \infty$. Another modification of this idea is to use as 'unperturbed' starting point of the perturbation expansion *$K$-functionals which have already been successfully computed.* For example, using the isometric quasilinearization for $K$-functionals between Hilbert spaces (see [3]), to obtain singular perturbation expansions of the $K$-functionals between Banach spaces which are 'nearly' Hilbert. While in the former of these variants the expansion is in powers of $t$ or $1/t$, in the latter case the expansion is in powers of the small parameter measuring the proximity between the target (perturbed) Banach space and the original (unperturbed) Hilbert space (e. g., the parameter $1/p - 1/2$ for $p$ near 2). Under the assumptions in items 1 and 2 there are also other prospective ways to use the 'method of small parameter' for computing $K$-functionals.

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## Editorial Note

The paper "Properties of function spaces generated by the averaged moduli of smoothness", *International Journal of Pure and Applied Mathematics*, **41**, No. 9 (2007) was accepted for publication in 2007. Section 6 of the same paper has not been included in the original printed publication because of mistake during the pre-print preparation of the issue. The Managing Editor of the journal is deeply sorry for the mistake.

This is to certify that the present addendum has been included in the original submission by Professor Dr. L.T. Dechevsky, on August 13, 2007.