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SOME RESULTS ON SURVIVAL EXPONENTIAL ENTROPY

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Abstract: The notion of entropy is of fundamental importance in the different areas such as probability and statistics, communication theory, economics and physics. In this paper, we propose two new broad classes of measures of uncertainty for a random variable X based on the survival exponential entropy and derive explicit expressions of the proposed measures for one and two parametric exponential distributions. Also its particular cases have been studied.

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1. Introduction

Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fileds. The concept of entropy is of particular importance in the field of information theory and it was introduced by Shannon [8].

If X is a random varible with an absolutely continuous distribution with probability density function $f(x)$, then the Shannon entropy is defined by

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$$H_{sh}(X) = -\int_{-\infty}^{\infty} f(x) \log f(x)\, dx\,. \tag{1.1}$$

Renyi [7] generilize Shannon entropy and is given by

$$H_r(X) = \frac{1}{1-\alpha} \log \int_{-\infty}^{\infty} f^{\alpha}(x)\, dx,\ \alpha > 0 \text{ and } \alpha \neq 1\,. \tag{1.2}$$

$H_{sh}(X)$ and $H_r(X)$ are the particular cases of the exponentail entropy of order α and is defined by

$$B_{\alpha}(X) = \left(\int_{-\infty}^{\infty} f^{\alpha}(x)\ dx \right)^{\frac{1}{1-\alpha}},\ \alpha > 0, \quad \alpha \neq 1\,.$$

In particular

$$H_{sh}(X) = \lim_{\alpha \to 1} (\log B_{\alpha}(X)) \quad \text{and} \quad H_r(x) = \log B_{\alpha}(X)\ .$$

Exponential entropy in (1.3), has been defined and studied by Campbell [1] and generalized by Koski and Persson [4].

2. Survival Exponential Entropty

The survival function of a random variable X is defined as

$$R_X(x) = P(X > x) = \int_x^{\infty} f(x)\ dx\,. \tag{2.1}$$

Based on the notation, Rao et al [5] and Wang et al [6] defined the cumulative residual entropy as

$$\varepsilon(X) = -\int_{-\infty}^{\infty} R_X(x) \log R_X(x)\ dx\,. \tag{2.2}$$

Based on the notation introduced in (2.1), survival and the generalized survival exponential entropies can be defined as

Definition. For a random variable X, the survival exponential entropy of order α is

$$M_{\alpha}(X) = \left(\int_{-\infty}^{\infty} R_X^{\alpha}(x)\ dx \right)^{\frac{1}{1-\alpha}}, \quad \alpha \geq 0\,. \tag{2.3}$$

Definition. For a random variable X, the generalized survival exponential entropy of order (α, β) is

$$S_{\alpha,\beta}(X) = \left(\frac{\int\limits_{-\infty}^{\infty} R_X^{\alpha}(x)\ dx}{\int\limits_{-\infty}^{\infty} R_X^{\beta}(x)\ dx} \right)^{\frac{1}{\beta-\alpha}}, \ \alpha, \beta \geq 0, \alpha \neq \beta\,, \tag{2.4}$$

also,

$$\operatorname*{Lim}_{\beta\to\alpha} S_{\alpha,\beta}(X) = \ \exp\left(-\frac{\int\limits_{-\infty}^{\infty} R_X^{\alpha}(x) \log R_X(x)\ dx}{\int\limits_{-\infty}^{\infty} R_X^{\alpha}(x)\, dx} \right). \tag{2.5}$$

3. Survival Exponential Entropy of One Parametric Exponential Distribution

Consider a random variable X from the exponential distribution with parameter λ, then the density function is

$$f(x) = \lambda \exp(-\lambda x)\,, \quad x \geq 0,\ \lambda > 0\,.$$

The survival function is

$$R_X(x) = P(X > x) = \int\limits_{x}^{\infty} \lambda\ \exp(-\lambda x)\ dx = \exp(-\lambda x)\ , \quad x \geq 0\,.$$

Therefore

$$\int\limits_{0}^{\infty} R_X^{\alpha}(x)\ dx = \int\limits_{0}^{\infty} \exp(-\lambda\alpha x)\ dx = \frac{1}{\lambda\alpha}\,.$$

Survival exponential entropy of order α is

$$M_\alpha(X) = \left(\int\limits_{-\infty}^{\infty} R_X^{\alpha}(x)\ dx \right)^{\frac{1}{1-\alpha}} = \left(\frac{1}{\lambda\alpha}\right)^{\frac{1}{1-\alpha}} \tag{3.1q}$$

$(\alpha \geq 0)$.

Generilized survival exponential entropy of order (α, β) is

$$S_{\alpha,\beta}(X) = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\beta-\alpha}}\ , \ \ \alpha \neq \beta \tag{3.2}$$

and

$$\underset{\beta\to\alpha}{\text{Lim}} S_{\alpha,\beta}\left(X\right) = \exp\left(\frac{1}{\alpha}\right). \tag{3.3}$$

Cumulative residual entropy is

$$\varepsilon(X) = \frac{1}{\lambda}. \tag{3.4}$$

Theorem 3.1. *If $X_1, X_2, ..., X_n$ are independent random variables having an exponential distribution with parametres $\lambda_i, i = 1, 2, ..., n$, then the exponential entropy of $Z = \min\left(X_1, X_2, ..., X_n\right)$ of order α is $M_\alpha\left(Z\right) = \left(\frac{1}{\alpha\sum_{i=1}^{n}\lambda_i}\right)^{\frac{1}{1-\alpha}}$ and the generalized survival exponential entropy of order α, β is independent of the parameters λ_i.*

Proof. Since $Z = \min\left(X_1, X_2, ...X_n\right)$, therefore the c.d.f. of Z is

$$F_Z\left(z\right) = P\left(Z \leq z\right) = 1 - \prod_{i=1}^{n}\left(\exp\left(-\lambda_i z\right)\right) = 1 - \exp\left(-z\sum_{i=1}^{n}\lambda_i\right).$$

Surivaval function of Z is

$$R_Z\left(z\right) = 1 - F_Z\left(z\right) = \exp\left(-z\sum_{i=1}^{n}\lambda_i\right).$$

Survival exponential entropy of Z of order α is

$$M_\alpha\left(Z\right) = \left(\frac{1}{\alpha\sum_{i=1}^{n}\lambda_i}\right)^{\frac{1}{1-\alpha}}, \ \alpha \geq 0. \tag{3.5}$$

Generalized survival exponential entropy of Z of order (α, β) is

$$S_{\alpha,\beta}\left(Z\right) = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\beta-\alpha}}, \ \alpha, \beta \geq 0, \alpha \neq \beta. \tag{3.6}$$

Clearly equation (3.6) is independent of the prameters λ_i.

$$\underset{\beta\to\alpha}{\text{Lim}} S_{\alpha,\beta}\left(Z\right) = \exp\left(\frac{1}{\alpha}\right). \tag{3.7}$$

Cumulative residual entropy of Z is

$$\varepsilon\left(Z\right) = \frac{1}{\sum_{i=1}^{n}\lambda_i}. \tag{3.8}$$

Corollery 3.1. *If $X_1, X_2, ...X_n$ are i.i.d. random variables, then:*

$$M_\alpha(Z) = \left(\frac{1}{n\alpha\lambda}\right)^{\frac{1}{1-\alpha}} \tag{3.9}$$

and

$$\varepsilon(Z) = \frac{1}{n\lambda}\,. \tag{3.10}$$

4. Survival Exponential Entropy of the Two Parametric Exponential Distribution

Consider a random variable X having a two parametric exponential distribution with a density function

$$f_X(x) = \lambda \exp\left(-\lambda(x-\theta)\right),\ x \geq \theta,\ \lambda > 0\,.$$

The survival function is given by

$$R_{(X)}(x) = \exp\left(-\lambda(x-\theta)\right)\,.$$

Therefore,

$$\int_0^\infty R_{(X)}^\alpha(x)\ dx = \int_0^\infty \exp\left(-\lambda\alpha(x-\theta)\right)dx = \frac{1}{\lambda\alpha}\exp\left(\lambda\alpha\theta\right)\,,$$

Survival exponential entropy of X is

$$M_\alpha(X) = \left(\frac{1}{\lambda\alpha}\exp\left(\lambda\alpha\theta\right)\right)^{\frac{1}{1-\alpha}}\,. \tag{4.1}$$

Generalized survival exponential entropy of X is given by

$$S_{\alpha,\beta}(X) = \left(\frac{\beta}{\alpha}\exp\left(\lambda\theta(\alpha-\beta)\right)\right)^{\frac{1}{\beta-\alpha}}\,, \tag{4.2}$$

$$\underset{\beta\to\alpha}{Lim}\ S_{\alpha,\beta}(X) = \exp\left\{-\left(\frac{\lambda\alpha\theta\ -\ 1}{\alpha}\right)\right\}\,, \tag{4.3}$$

$$\varepsilon(X) = \left(\exp\left(\lambda\theta\right)\right)\left(\frac{1+\lambda\theta}{\lambda}\right)\,. \tag{4.4}$$

Corellary 4.1. *For $\theta = 0$, equation (4.1), (4.2), (4.3) and (4.4), reduce to equation (3.1), (3.2), (3.3) and (3.4) respectively.*

Theorem 4.1. *If X_i, $i = 1, 2, ..., n$ are independent random variables, having two parametric exponential distribution with parameters λ_i, θ, then the* *survival exponential entropy of* $Z = \min(X_1, X_2, ..., X_n)$ *is given by*

$$M_\alpha(Z) = \left(\frac{1}{\alpha \sum\limits_{i=1}^{n} \lambda_i} \exp\left(\theta\alpha \sum_{i=1}^{n} \lambda_i \right) \right)^{\frac{1}{1-\alpha}}$$

and the generlized survival exponential entropy of order α, β *is given by*

$$S_{\alpha,\beta}(Z) = \left(\frac{\beta}{\alpha} \exp\left(\theta \sum_{i=1}^{n} \lambda_i (\alpha - \beta) \right) \right)^{\frac{1}{\beta-\alpha}}, \quad \alpha, \beta \geq 0, \ \alpha \neq \beta_0 .$$

Proof. Since, $Z = \min(X_1, X_2, ..., X_n)$, therefore, c.d.f. of Z is

$$F_Z(z) = P(Z \leq z) = 1 - \exp\left(- \ (z-\theta) \sum_{i=1}^{n} \lambda_i \right) .$$

Survival function is

$$R_Z(z) = \exp\left\{ -(z-\theta) \sum_{i=1}^{n} \lambda_i \right\} .$$

Therefore,

$$\int_0^\infty R_Z(z) \ dz = \frac{1}{\alpha \sum\limits_{i=1}^{n} \lambda_i} \exp\left(\alpha \ \theta \sum_{i=1}^{n} \lambda_i \right) .$$

Survival exponential entropy of order α is

$$M_\alpha(Z) = \left\{ \frac{1}{\alpha \sum\limits_{i=1}^{n} \lambda_i} \exp\left(\alpha \ \theta \sum_{i=1}^{n} \lambda_i \right) \right\}^{\frac{1}{1-\alpha}}, \quad \alpha \geq 0 . \tag{4.5}$$

The generlized survival exponential entropy of order α, β is

$$S_{\alpha,\beta}(Z) = \left(\frac{\beta}{\alpha} \exp\left(\theta \sum_{i=1}^{n} \lambda_i (\alpha - \beta) \right) \right)^{\frac{1}{\beta-\alpha}}, \quad \alpha, \beta \geq 0, \ \alpha \neq \beta , \tag{4.6}$$

$$\lim_{\beta \to \alpha} S_{\alpha,\beta}(Z) = \exp\left(- \frac{\alpha\theta \sum\limits_{i=1}^{n} \lambda_i \ - 1}{\alpha} \right) , \tag{4.7}$$

$$\varepsilon(Z) = \left(\frac{1+\theta \sum\limits_{i=1}^{n} \lambda_i}{\sum\limits_{i=1}^{n} \lambda_i} \right) \left(\exp \left(\theta \sum_{i=1}^{n} \lambda_i \right) \right) . \tag{4.8}$$

Special Case. For $\theta = 0$ equations (4.5), (4.6), (4.7) and (4.8), reduce to equation (3.5), (3.6), (3.7) and (3.8) respectively.

5. Conclusion

We have introduced entropy type measures and referred to them as the survival exponential and generalized survival exponential entropies. These measures are parallel to well known entropies introduced and used in several disciplines and contexts in the scientific literature. The relation between these measures and one and two parametric exponential distributions have been obtained. Their order statistics have been discussed in the form of theorems. The use of the distribution function (or equivalently of the survival function) for the definition of the measures of the entropy, seems natural because of the distribution function is more regular than the density functions as it is defined in an integral form unlike the density function.

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