International Journal of Pure and Applied Mathematics
Volume 49 No. 1 2008, 21-27

ON THE PROBABILITY IN SUGENO MEASURE SPACE BASED ON $\sigma - \lambda$ RULES

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Abstract: This paper provides characteristic theorem of Sugeno measures based on $\sigma - \lambda$ rules. Then, definitions and properties of multidimensional random variables and their distribution function on Sugeno measures space are given.

AMS Subject Classification: 15A22, 68Q40
Key Words: Sugeno measures, random variables, distribution function

1. Introduction

It is well-known that Sugeno measure is a generally non-additivity measure. It is an important extension of probability measures. The probability on probability space is a non-negative and additive set function. However, in real analysis and probability theory, people more and more pay their attentions to the con-

Received: July 24, 2008

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dition of countable additivity of outer measures, because many mathematicians have found out that the condition of countable additivity is too strong to master enough in some practicable applications. On the other hand, although the countable additivity is good at describing measure problem which under the ideal conditions or null-defect situations, the people have found that the defect cannot be avoided in many conditions. In addition, some measures which concern subjective judgements or non-repeated experiments are non-additive in nature. So we try to use the more weaker $\sigma - \lambda$ rules to replace the countable additivity. The most straight method is using the outer measures to replace the measures in order to discuss some problem. In this paper, we provide characteristic theorem of Sugeno measures based on $\sigma - \lambda$ rules, at the same time, the definition and properties of multidimensional random variables and their distribution functions in Sugeno measures space are given.

2. Sugeno Measure Based on $\sigma - \lambda$ Rules

Definition 2.1. Let X be a nonempty set and ζ be a non empty class of subsets of X. A non-negative set function μ on ζ satisfies $\sigma - \lambda$ rules if and only if when $\lambda \in \left(-\frac{1}{\sup \mu}, \infty\right) \cup \{0\}$ the following equation

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \begin{cases} \dfrac{1}{\lambda}\left\{\prod_{i=1}^{\infty}[1+\lambda\mu(A_i)] - 1\right\}, & \lambda \neq 0, \\ \sum_{i=1}^{\infty} \mu(A_i), & \lambda = 0, \end{cases}$$

holds. where A_i is a disjoint sequence of ζ and $\bigcup_{i=1}^{\infty} A_i \in \zeta$.

Particularly, if $\lambda = 0$, then $\sigma - \lambda$ rules is $\sigma-$ additivity. Correspondingly, μ is called measure on ζ.

Definition 2.2. Let $\mathcal{F}$ be a $\sigma-$algebra of subsets of X. μ is called Sugeno measure based on $\sigma - \lambda$ rules on $\mathcal{F}$ if and only if μ satisfies $\sigma - \lambda$ rules and $\mu(X) = 1$. For brevity, we write g_λ.

Remark 2.1. In Definition 2.1, if $\lambda = 0$, then g_λ is classical probability measure. If we select suitable parameter λ, it could be described generally non-additivity measures (see [2, 3, 4]).

Remark 2.2. In Definition 2.2, let $n = 2$, then

$$\mu\left(A \bigcup B\right) = \begin{cases} \mu(A) + \mu(B) + \lambda\mu(A)\mu(B), & \lambda \neq 0, \\ \mu(A) + \mu(B), & \lambda = 0. \end{cases}$$

Definition 2.3. Let m be a additivity measure and let us introduce the transformation t of this measure, i.e. a function $t : [0, \infty) \to [0, 1]$ with the following properties:

(1) $t(0) = 0, t(m(X)) = 1$;

(2) t is monotonically nondecreasing;

(3) t is continuous.

Theorem 2.1. (see [1]) *Let $m : \zeta \to [0, \infty)$ be a additivity measure, then $g_\lambda = t \circ m$ is Sugeno measure based on $\sigma - \lambda$ rules if and only if the function $t : [0, \infty) \to [0, \infty)$ has the following equation: $t(x) = (c^x - 1)/\lambda$, with $c \in (0, 1), \lambda \in \left(-\frac{1}{\sup g_\lambda}, 0\right]$.*

Theorem 2.2. (see [2]) For every Sugeno measure g_λ based on $\sigma - \lambda$ rules, there exists a $T-$function makes set function $\theta_T \circ g_\lambda$ to be a probability measure and $T-$function is

$$\theta_T(x) = \begin{cases} \dfrac{ln(1 + \lambda x)}{ln(1 + \lambda)}, & \lambda \neq 0, \\ x, & \lambda = 0, \end{cases}$$

where $c \in [0, 1]$.

Remark 2.3. Let $c = 1 + \lambda$, then $t(x) = (c^x - 1)/\lambda$ in Theorem 2.1 and $T-$function in Theorem 2.2 are reversible function.

We can get the following theorem by Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Let $m : \zeta \to [0, \infty)$ be a additivity measure. Then there exists a $T-$function such that the set function $\theta_T \circ (\theta_T^{-1} \circ m)$ is a probability measure.*

3. Multidimensional $g_\lambda-$Random Variables and its Distribution

The triple $(X, \mathcal{F}, g_\lambda)$ is called Sugeno measure space, for brevity, we write $g_\lambda-$space. The following discussions are all in $g_\lambda-$space.

Definition 3.1. Let ξ and η be real-valued function in $g_\lambda-$ space $(X, \mathcal{F}, g_\lambda)$ respectively. Then vector (ξ, η) is called 2-dimensional $g_\lambda-$random variables or

2-dimensional $g_\lambda-$random vector.

Definition 3.2. Let (ξ, η) be a 2-dimensional $g_\lambda-$random variables. Then its distribution function is defined by

$$F_{g_\lambda}(x, y) = g_\lambda \{(\xi \le x) \cap (\eta \le y)\} = g_\lambda(\xi \le x, \eta \le y).$$

This distribution function is also called joint distribution function of $g_\lambda-$random variables ξ and η.

Definition 3.3. If all possible value of 2-dimensional $g_\lambda-$random variables (ξ, η) are (x_i, y_j), $i, j = 1, 2, \cdots$, then (ξ, η) is called 2-dimensional discrete $g_\lambda-$random variables. We write $g_{\lambda i,j} = g_\lambda \{\xi = x_i, \eta = y_j\} (i, j = 1, 2, \cdots)$, then $g_{\lambda i,j}$ $i, j = 1, 2, \cdots$, is called distribution rule of 2-dimensional discrete $g_\lambda-$random variables (ξ, η) or joint distribution rules of $g_\lambda-$random variables ξ and η.

Lemma 3.1. $g_\lambda \{x_1 < x \le x_2\} = \frac{F_{g_\lambda}(x_2) - F_{g_\lambda}(x_1)}{1+\lambda F_{g_\lambda}(x_1)}$.

Proof. For every set $A, B \in (X, \mathcal{F}, g_\lambda)$ and $B \subset A$, we obtain $g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$. $A = A - B \cup B$ when $B \subset A$, then

$$g_\lambda(A - B) = \frac{g_\lambda(A) - g_\lambda(B)}{1 + \lambda g_\lambda(B)}.$$

Similarly,

$$g_\lambda \{x_1 < x \le x_2\} = g_\lambda \{(x \le x_2) - (x \le x_1)\} = \frac{g_\lambda \{x \le x_2\} - g_\lambda \{x \le x_1\}}{1 + \lambda g_\lambda \{x \le x_2\}}$$

i.e.,

$$g_\lambda \{x_1 < x \le x_2\} = \frac{F_{g_\lambda}(x_2) - F_{g_\lambda}(x_1)}{1 + \lambda F_{g_\lambda}(x_1)}.$$

Theorem 3.1. Let $F_{g_\lambda}(x, y)$ be a joint distribution function of 2-dimensional $g_\lambda-$random variables (ξ, η). Then:

(1) $F_{g_\lambda}(x, y)$ is a monotonically nondecreasing function for x or y. That is, for y, if $x_2 > x_1$, then $F_{g_\lambda}(x_2, y) \ge F_{g_\lambda}(x_1, y)$. For x, if $y_2 > y_1$, then $F(x, y_2) \ge F(x, y_1)$.

In fact, if $\lambda \in \left(-\frac{1}{\sup g_\lambda}, 0\right)$, we have $0 \le 1 + \lambda \cdot g_\lambda$, for every fixed variables y, when $x_2 > x_1$, by Lemma 3.1 we obtain

$$F_{g_\lambda}(x_2, y) - F_{g_\lambda}(x_1, y) = g_\lambda \{x_1 < \xi \le x_2, \eta \le y\} \cdot (1 + \lambda F_{g_\lambda}(x_1, y)) \ge 0.$$

Similarly, for every fixed variables x, if $y_2 > y_1$, then $F_{g_\lambda}(x, y_2) \ge F_{g_\lambda}(x, y_1)$.

(2) $F_{g_\lambda}(x, y)$ is a right continuous separately about ξ and η. That is

$$F_{g_\lambda}(x, y) = F_{g_\lambda}(x + 0, y), \qquad F_{g_\lambda}(x, y) = F_{g_\lambda}(x, y + 0).$$

Theorem 3.2.

$$g_\lambda(x_1 < \xi \le x_2, y_1 < \eta \le y_2) = \frac{F(I) - \lambda \begin{vmatrix} F_{g_\lambda}(x_1, y_2) & F_{g_\lambda}(x_2, y_2) \\ F_{g_\lambda}(x_1, y_1) & F_{g_\lambda}(x_2, y_1) \end{vmatrix}}{[1 + \lambda F_{g_\lambda}(x_2, y_1)][1 + \lambda F_{g_\lambda}(x_1, y_2)]}.$$

where $F(I) = F_{g_\lambda}(x_2, y_2) - F_{g_\lambda}(x_2, y_1) - F_{g_\lambda}(x_1, y_2) + F_{g_\lambda}(x_1, y_1)$.

Proof. Let $A = (\xi \le x_2, \eta \le y_2), B = (\xi \le x_2, \eta \le y_1), C = (\xi \le x_1, \eta \le y_1), D = (\xi \le x_1, \eta \le y_2)$. It is clearly that

$$\begin{aligned} & (x_1 < \xi \le x_2, y_1 < \eta \le y_2) \\ = & \{[(\xi \le x_2, \eta \le y_2) - (\xi \le x_2, \eta \le y_1)] \cup (\xi \le x_1, \eta \le y_1)\} \\ & -(\xi \le x_1, \eta \le y_2). \end{aligned}$$

Therefore

$$\begin{aligned} & g_\lambda(x_1 < \xi \le x_2, y_1 < \eta \le y_2) = g_\lambda \{[(A - B) \cup C] - D\} \\ = & \frac{g_\lambda[(A - B) \cup C] - g_\lambda(D)}{1 + \lambda g_\lambda(D)} \\ = & \frac{g_\lambda(A - B) + g_\lambda(C) + \lambda g_\lambda(A - B) g_\lambda(C) - g_\lambda(D)}{1 + \lambda g_\lambda(D)} \\ = & \frac{g_\lambda(A - B)[1 + \lambda g_\lambda(C)] + g_\lambda(C) - g_\lambda(D)}{1 + \lambda g_\lambda(D)} \\ = & \frac{\frac{g_\lambda(A) - g_\lambda(B)}{1 + \lambda g_\lambda(B)}[1 + \lambda g_\lambda(C)] + g_\lambda(C) - g_\lambda(D)}{1 + \lambda g_\lambda(D)} \\ = & \frac{F(I) - \lambda \begin{vmatrix} F_{g_\lambda}(x_1, y_2) & F_{g_\lambda}(x_2, y_2) \\ F_{g_\lambda}(x_1, y_1) & F_{g_\lambda}(x_2, y_1) \end{vmatrix}}{[1 + \lambda F_{g_\lambda}(x_2, y_1)][1 + \lambda F_{g_\lambda}(x_1, y_2)]} \end{aligned}$$

The proof is completed. □

Remark 3.1. By two theorems above, we can obtain that there exist some same properties of distribution function between 2-dimensional $g_\lambda-$random variables and 2-dimensional random variables. At the same time, there are some different properties of that.

Definition 3.4. For distribution function $F_{g_\lambda}(x, y)$ of 2-dimensional $g_\lambda-$random variables(ξ, η), if there exists non-negative function $f_{g_\lambda}(x, y)$, for every x, y such that

$$F_{g_\lambda}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{g_\lambda}(u, v) du dv,$$

then (ξ, η) is called a continuous 2-dimensional $g_\lambda-$random variable and function $f_{g_\lambda}(x, y)$ is called a probability density of 2-dimensional $g_\lambda-$random variable or joint probability density of $g_\lambda-$random variables ξ and η.

The above discussion about 2-dimensional $g_\lambda-$random variables can be extended to n-dimensional $(n > 2)$ $g_\lambda-$random variables easily.

Definition 3.5. Let $\xi_1, \xi_2, \cdots, \xi_n$ be a real-valued function on $(X, \mathcal{F}, g_\lambda)$, respectively. If $(X, \mathcal{F}, g_\lambda)$ is a $g_\lambda-$space, then $n-$dimensional vector $(\xi_1, \xi_2, \cdots, \xi_n)$ is called $n-$dimensional $g_\lambda-$random variables or $n-$dimensional $g_\lambda-$random vector.

Definition 3.6. For arbitrary real-valued numbers $x_1, x_2, \cdots, x_n$, multivariate function

$$F_{g_\lambda}(x_1, x_2, \cdots, x_n) = g_\lambda\left\{\xi_1 \leq x_1, \xi_2 \leq x_2, \cdots, \xi_n \leq x_n\right\}$$

is called distribution function of $n-$dimensional $g_\lambda-$random variables $(\xi_1, \xi_2, \cdots, \xi_n)$ or joint distribution function of $g_\lambda-$random variables $\xi_1, \xi_2, \cdots, \xi_n$.

Definition 3.7. If we view 2-dimensional $g_\lambda-$random variables (ξ, η) as a whole, then it has a distribution function $F_{g_\lambda}(x, y)$. On the other hand, ξ and η are all $g_\lambda-$random variables, so they have their own distribution function and denoted as $F_{g_\lambda\xi}(x), F_{g_\lambda\eta}(y)$, respectively. They are said as the order to be marginal distribution function of $g_\lambda-$random variables (ξ, η) on ξ and on η.

Theorem 3.3. Marginal distribution function can be determined by distribution function $F_{g_\lambda}(x, y)$ of (ξ, η). In fact,

$$F_{g_\lambda\xi}(x) = g_\lambda\left\{\xi \leq x\right\} = g_\lambda\left\{\xi \leq x, \eta < \infty\right\} = F_{g_\lambda}(x, \infty).$$

That is to say, as long as let $y \to \infty$ in the function $F_{g_\lambda}(x, y)$, we can obtain $F_{g_\lambda\xi}(x)$. Similarly $F_{g_\lambda\eta}(y) = F_{g_\lambda}(\infty, y)$.

Definition 3.8. We write

$$g_{\lambda_{i.}} = \sum_{j=1}^{\infty} g_{\lambda_{i,j}} = g_\lambda\left\{\xi = x_i\right\} (i = 1, 2, \cdots),$$

$$g_{\lambda_{.j}} = \sum_{i=1}^{\infty} g_{\lambda_{i,j}} = g_\lambda\left\{\eta = y_j\right\} (j = 1, 2, \cdots).$$

Then $g_{\lambda_{i.}}(i = 1, 2, ...)$ and $g_{\lambda_{.j}}(j = 1, 2, ...)$ are said to be marginal distribution rules of $g_\lambda-$random variables (ξ, η) on ξ and on η.

Definition 3.9. Let ξ and η be $g_\lambda-$random variables and $f_{g_\lambda}(x, y)$ be a joint probability density function of (ξ, η), we write

$$f_{g_\lambda\xi}(x) = \int_{-\infty}^{\infty} f_{g_\lambda}(x, y)dy, \qquad f_{g_\lambda\eta}(y) = \int_{-\infty}^{\infty} f_{g_\lambda}(x, y)dx.$$

Then $f_{g_\lambda\xi}(x)$ and $f_{g_\lambda\eta}(y)$ are said to be marginal probability density function on $\xi$and on η respectively .

Theorem 3.4. Let $F_{g_\lambda}(x,y)$, $F_{g_\lambda\xi}(x)$ and $F_{g_\lambda\eta}(y)$ be distribution function of 2-dimensional $g_\lambda-$random variables (ξ,η) respectively, if for any x,y, we get

$$g_\lambda(\xi\le x,\eta\le y)=\theta_T^{-1}\left\{\theta_T[g_\lambda(\xi\le x)]\cdot\theta_T[g_\lambda(\eta\le y)]\right\},$$

i.e.,

$$F_{g_\lambda}(x,y)=\theta_T^{-1}\left\{\theta_T[F_{g_\lambda\xi}(x)]\cdot\theta_T[F_{g_\lambda\eta}(y)]\right\}.$$

Then $g_\lambda-$random variables ξ and η are called mutual independent. Here $\theta_T(x)=\frac{ln(1+\lambda x)}{ln(1+\lambda)},\theta_T^{-1}(x)=\frac{(1+\lambda)^x-1}{\lambda}$.

Acknowledgements

This work is supported by the National Science Fund of P.R. China (10771171) and the Natural Scientific Fund of Gansu Education Office (0601-20).

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