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LAGUERRE PLANES AND THE 8-POINT BUNDLE FORMS

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Abstract: The author proved in *The bundle form* $\mathfrak{B}1^2$ *characterizes ovoidal Laguerre planes* that $\mathfrak{B}1^2$ implies Kahn's Full Bundle Theorem, and thus characterizes ovoidal Laguerre planes. In this paper, we show that ovoidal Laguerre planes are also characterized by each of the 8-point bundle forms $\mathfrak{B}1^1$ and $\mathfrak{B}0$, but not by $\mathfrak{B}2$.

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1. Introduction

Kahn demonstrated in [2] that ovoidal Laguerre planes are characterized by the Full Bundle Theorem, a compilation of all singular and nonsingular bundle forms. In [4], the author showed that the singular bundle forms hold in any Laguerre plane and that the nonsingular 8-point bundle form $\mathfrak{B}1^2$ implies the other 25 nonsingular bundle forms. It follows that $\mathfrak{B}1^2$ is equivalent to the Full Bundle Theorem, and hence characterizes ovoidal Laguerre planes.

In this article we prove the following more general result:

Theorem 1. *Each of the following 8-point bundle forms characterizes ovoidal Laguerre planes.*

$\mathfrak{B}1^2, \qquad \mathfrak{B}1^1, \qquad \mathfrak{B}0.$

We also explain why $\mathfrak{B}2$, the only other nonsingular bundle form with eight distinct points, cannot characterize ovoidal Laguerre planes.

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The methods used in this paper are in the same vein as those used by the author in [4], where the bundle forms are stated in terms of pencils. For easy reference, we will repeat here the necessary axioms, definitions, and properties.

The fundamental objects of a Laguerre plane are traditionally called cycles and spears. We retain this nomenclature, rather than adopting the more common terms "circles" and "points", respectively, since it is more natural in the context of bundle forms to view cycles as points. Thus an "8-point bundle form" is a bundle form involving eight distinct spears.

Classically, spears are represented by oriented lines and cycles by oriented circles and unoriented points. These are related by the property touch. Two objects are said to touch if they are tangent and their orientations agree at the point of tangency, or, in the case one is a point, if they are incident. Two spears are parallel if their underlying lines are parallel and their orientations agree. This classic model is the one introduced by Laguerre in [5].

Since we are taking a synthetic approach, *spears* and *cycles* are undefined terms, related by a property *touch*, whose characteristics are determined by the axioms of the geometry. We say two spears are *parallel* if they are the same spear or there is no cycle touching both spears. If two spears are not parallel, they are *nonparallel*. Two cycles *touch* if they are the same cycle or there is exactly one spear touching both. In the latter case we say the two cycles *touch in the spear*.

2. Axioms of General Laguerre Planes

The axioms for a general Laguerre plane are as follows. This system is equivalent to that given by van der Waerden and Smid in [8].

1. Touch is symmetric.

2. There is a spear and a cycle not touching the spear.

3. There is a cycle that touches three distinct spears.

4. Given three spears, no two of which are parallel, there is a unique cycle that touches all three spears.

5. Given a cycle and a spear, there is a unique spear touching the cycle and parallel to the given spear.

6. Given two non-parallel spears and a cycle touching one spear but not the other, there is a unique cycle touching all three.

3. Useful Properties of Spears and Cycles

The following basic properties of spears and cycles are easily proved from the axioms.

1. There are at most two spears touching each pair of distinct cycles.

2. If two spears are parallel to a third spear, they are parallel to each other.

3. If two distinct cycles touch a third cycle in a spear, they touch each other in that same spear.

4. Pencils

Our approach relies heavily on the properties of pencils. The *null pencil* determined by a cycle and a spear touching the cycle is the set of all cycles touching the given cycle in the spear. The *positive pencil* determined by a pair of non-parallel spears is the set of all cycles touching both spears. We will use the general term *pencil* to refer to positive and null pencils.[1]

We say that a spear *touches* a pencil if it touches each cycle of the pencil. In this situation, we also say that the pencil *touches* the spear. Two pencils *intersect* in a cycle if the cycle is contained in both pencils. Two pencils are *parallel* if each spear touching one of the pencils is parallel to a spear touching the other pencil. A positive pencil cannot be parallel to a null pencil.

5. Useful Properties of Pencils

The proofs for the following properties are routine and are thus omitted.

1. Each null pencil touches a unique spear. It is uniquely determined by this spear and and any cycle contained in the pencil. The cycles of the pencil touch each other in the spear.

2. A spear touches a positive pencil if, and only if, it is one of the spears that determine the pencil.

3. If two distinct cycles touch a common spear, then they are contained in a positive or null pencil, but not both. Moreover, the pair of cycles is contained in exactly one such pencil.

[1] Positive and null pencils are often referred to as elliptic and parabolic pencils, respectively.

4. Two distinct pencils intersect in at most one cycle.

5. If two cycles of a pencil touch a spear, then all cycles in the pencil do.

6. Distinct parallel pencils do not intersect.

7. Parallelism is a transitive relation on the set of pencils.

8. Given any cycle and any pencil, there is a unique pencil containing the cycle and parallel to the pencil.

9. If a given spear is not parallel to any spear touching a pencil, then there is a unique cycle in the pencil touching the given spear.

Pencils are denoted by greek letters, spears by upper case roman letters, and cycles by lower case roman letters. The preceding properties allow us to use the alternative notation xy, for example, for the pencil containing the distinct cycles x and y, and ST for the positive pencil of cycles touching the distinct spears S and T.

6. Bundle Forms

Three pencils are called *concurrent* if they intersect in a common cycle. A *bundle configuration* is a set of four pencils, no three of which are concurrent or parallel, and with the property that at least five of the six pairs of pencils intersect or are parallel. We denote the bundle configuration containing α, β, γ, δ, for example, by $\alpha\beta\gamma\delta$. A bundle configuration is *singular* if any two pencils in the configuration touch a common spear and *nonsingular* otherwise. It is said to *close* if all six pairs consist of intersecting or parallel pencils. A *bundle form* is an implication that a bundle configuration closes. A *nonsingular bundle form* is a bundle form whose bundle configuration is nonsingular. As previously mentioned, an 8-*point bundle form* is a bundle form whose configuration involves eight distinct spears, i.e., in which all four pencils are positive. For explicit statements of the 26 nonsingular bundle forms, see [4].

The nonsingular bundle forms we will be using are illustrated in Figure 1, where pencils are represented by lines and cycles by points. For each bundle form, solid points represent cycles of intersection in the hypothesis and a hollow point represents the cycle of intersection in the conclusion. For $\mathfrak{B}0$ and $\mathfrak{B}1^1$, the conclusion includes the possibility that $\delta \parallel \gamma$. Pencils represented by lines that appear to be parallel are actually parallel. The pencil δ is null in $B\mathfrak{B}0$ and $B\mathfrak{B}1^2$. All other pencils are positive.

In general Laguerre planes, the existence of a pencil containing two given

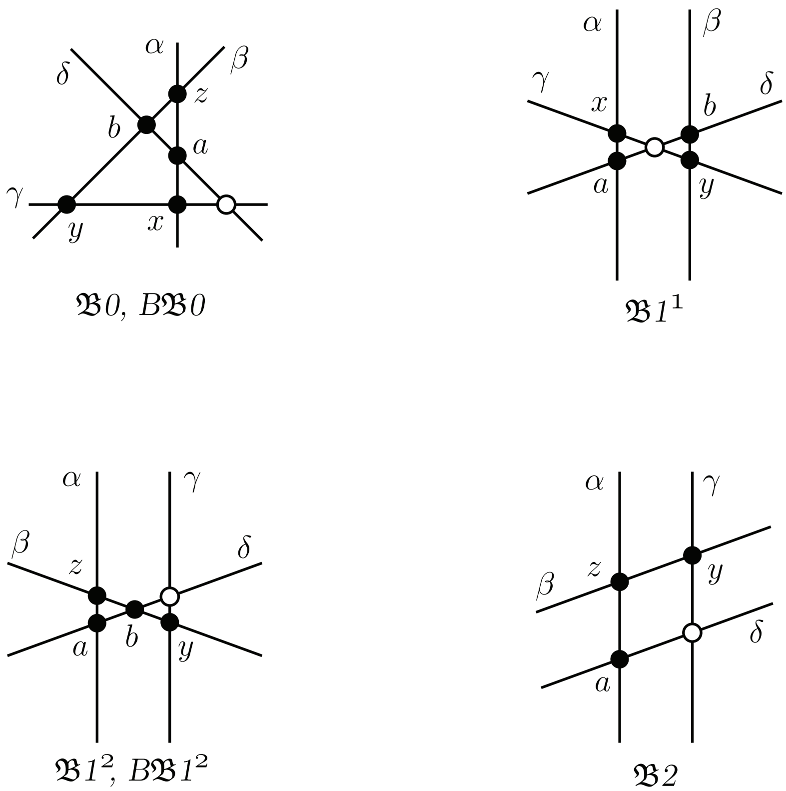

Figure 1: Some nonsingular bundle forms. δ is null in $B\mathfrak{B}0$ and $B\mathfrak{B}1^2$

cycles is only guaranteed when the two cycles touch a common spear, which is not necessarily true for all pairs of cycles. Thus the set of points (cycles) and lines (pencils) forms a 3-dimensional near-linear space: an incidence geometry satisfying the conditions 1) every line contains at least two points, and 2) there is at most one line containing any two given points (see, e.g., [1]). The bundle forms are variants of the affine Veblin-Young axiom applied within this near-linear space.

7. Five Lemmas

We adopt the following conventions. The four pencils of the bundle form in the consequence for each lemma are denoted by α, β, γ, and δ, as in Figure 1. One of the spears touching α, β, γ, δ is A, B, C, D, respectively. If a pencil is positive, and thus touches a second spear, we denote this second spear by appending a prime to the letter representing the first spear touching the pencil. Hence if α is positive, for example, then $\alpha = AA'$. When it is assumed that two pencils intersect, the cycle of intersection will be denoted as in Figure 1.

Notice that for $\mathfrak{B}1^2_1$, $B\mathfrak{B}1^2_1$, and $\mathfrak{B}2$, δ intersects α and $\alpha \parallel \gamma$. Since $\alpha\beta\gamma\delta$ is nonsingular, it follows that that no spear touching δ is parallel to any spear touching γ in these three cases.

Lemma 1. $\mathfrak{B}1^1 \Longrightarrow B\mathfrak{B}1^2$.

Proof. Since $D \nparallel C, C'$, there is a cycle u in γ that touches D. Since $\alpha \neq \gamma$, we know $u \neq a$. Thus $\varepsilon = au$ is a positive or null pencil touching D.

Suppose ε is positive. Then $\mathfrak{B}1^1$ implies that ε intersects β in a cycle v. But then $v, b \in \beta$ and both touch D, so $v = b$. Hence $b \in \varepsilon$, so $\delta = ab = av = au = \varepsilon$. This is a contradiction, since ε is positive and δ is null.

Thus ε is null. Then both ε and δ are null and touch D, so $\varepsilon \parallel \delta$. But a is in both pencils, so $\varepsilon = \delta$. Hence δ intersects γ in u. □

Lemma 2. $\mathfrak{B}1^1$ *and* $B\mathfrak{B}1^2 \Longrightarrow \mathfrak{B}2$.

Proof. This was proved in [4]. □

Lemma 3. $\mathfrak{B}1^1$ *and* $\mathfrak{B}2 \Longrightarrow \mathfrak{B}1^2$.

Proof. Since α, β, δ are not concurrent, $a \neq z$. Let ε be the pencil parallel to β and containing a. Since $a \notin \beta$, we know $\varepsilon \neq \beta$. Then $\mathfrak{B}2$ tells us ε intersects γ in a cycle u. Applying $\mathfrak{B}1^1$ to $\beta\gamma\delta\varepsilon$, we conclude that either $\delta \parallel \gamma$ or δ intersects γ in a cycle. But $\delta \nparallel \gamma$, since $\gamma \parallel \alpha$ and $\delta \nparallel \alpha$. □

Lemma 4. $\mathfrak{B}0 \Longrightarrow B\mathfrak{B}0$.

Proof. The proof is the same as for Lemma 1, except that $u = a$ implies directly that δ intersects γ in u. Also, $D \nparallel C, C'$ is assumed. □

Lemma 5. $\mathfrak{B}0$ *and* $B\mathfrak{B}0 \Longrightarrow \mathfrak{B}1^2$.

Proof. Since $C \nparallel D, D'$, there is a cycle u in δ that touches C. If $u = y$, then δ intersects γ in y. Otherwise, $\varepsilon = uy$ is a positive or null pencil touching

C.

Suppose $\varepsilon \nparallel \alpha$. Appling $\mathfrak{B}0$ or $B\mathfrak{B}0$, according as ε is positive or null, we find that ε intersects α in a cycle v. Then v touches C and A. Since $C \parallel A$, we deduce that $C = A$. But $\alpha\beta\gamma\delta$ is nonsingular, a contradiction.

Thus $\varepsilon \parallel \alpha$. Then $y \in \varepsilon \cap \gamma$ and $\gamma \parallel \alpha$ imply that $\varepsilon = \gamma$. We conclude that $u \in \gamma$, so δ intersects γ in u. □

8. Proof of Theorem 1

Since we demonstrated in [4] that $\mathfrak{B}1^2$ characterizes ovoidal Laguerre planes, it suffices to show that $\mathfrak{B}1^1$ and $\mathfrak{B}0$ each implies $\mathfrak{B}1^2$.

Combining Lemmas 1, 2, and 3, we obtain that $\mathfrak{B}1^1$ implies $\mathfrak{B}1^2$. The fact that $\mathfrak{B}0$ implies $\mathfrak{B}1^2$ follows directly from Lemmas 4 and 5.

9. Further Observations

The bundle forms $\mathfrak{B}0$, $\mathfrak{B}1^1$, $\mathfrak{B}1^2$, and $\mathfrak{B}2$ are the only nonsingular bundle forms consisting exclusively of positive pencils, or, in the traditional terminology, for which no two spears "degenerate" to one spear. We have shown that the first three characterize ovoidal Laguerre planes. What about $\mathfrak{B}2$? We note that $\mathfrak{B}2$ is, in fact, identical to $\mathfrak{M}2$, the degenerate Miquelian form that characterizes elation Laguerre planes (Knarr [3]). Since there are elation Laguerre planes that are not ovoidal (see, e.g., [6]), we conclude that $\mathfrak{B}2$ cannot characterize ovoidal Laguerre planes.

To see that $\mathfrak{B}2 = \mathfrak{M}2$, let us examine the statement of $\mathfrak{M}2$ by Knarr.

$\mathfrak{M}2$. If eight distinct points $x_i, y_i \in \mathcal{P}$, $i = 1, \dots, 4$, with $v_i \parallel y_i$, $i = 1, \dots, 4$ are given such that each of the sets $\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_2, y_3, y_4\}$ and $\{x_3, x_4, y_1, y_2\}$ is contained in a circle, then the set $\{y_1, y_2, y_3, y_4\}$ is also contained in a circle.

Here "points" refers to spears, $\mathcal{P}$ is the set of all spears, and a set "contained in a circle" means the spears in the set each touch a single cycle. Set $\alpha = x_3x_4$, $\beta = x_1x_2$, $\gamma = y_3y_4$, and $\delta = y_1y_2$. Then the antecedent of $\mathfrak{M}2$ says that $\alpha \parallel \gamma$, $\beta \parallel \delta$, $\alpha \cap \beta$, $\beta \cap \gamma$, and $\alpha \cap \delta$. The consequent is that $\delta \cap \gamma$. This is exactly the statement of $\mathfrak{B}2$.

Notice that this observation, the characterization of elation Laguerre planes by Knarr, and the characterizations of ovoidal Laguerre planes in [2] and [4] provide an alternative understanding of the result of Schroth [7] that ovoidal Laguerre planes are elation Laguerre planes.

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References

[1] L.M. Batten, *Combinatorics of Finite Geometries*, Second Edition, Cambridge University Press, Cambridge (1997).

[2] J. Kahn, Locally projective-planar lattices which satisfy the bundle theorem, *Math. Z.*, **175** (1980), 219-247.

[3] N. Knarr, A geometric characterization of elation Laguerre planes, *Arch. Math.*, **78** (2002), 162-165.

[4] R.D. Knight, The bundle form $\mathfrak{B}1^2$ characterizes ovoidal Laguerre planes, *J. Geom.*, To Appear.

[5] E. Laguerre, Sur la Géométrie de Direction, In: *Œuvres De Laguerre*, Chelsea, New York (1972), 592-603.

[6] R. Löwen, U. Pfüller, Two-dimensional Laguerre planes over convex functions, *Geom. Dedicata*, **23** (1987), 73-85.

[7] A.E. Schroth, Ovoidal Laguerre planes are weakly miquelian, *Arch. Math.*, **72** (1999), 77-80.

[8] B.L. van der Waerden, L.J. Smid, Eine Axiomatik der Kreisgeometrie und der Laguerregeometrie, *Math. Ann.*, **118** (1935), 753-776.