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A NOTE ON (m, n)-QUASI-IDEALS IN SEMIRINGS

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Abstract: The notion of quasi-ideals for semirings was introduced by K. Iseki. In this paper, we study (m, n)-quasi-ideals of semirings. We generalize some facts of left ideals, right ideals and quasi-ideals of semirings.

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1. Introduction

The notions of quasi-ideals for rings and semigroups were first introduced by O. Steinfeld [3] and [4], respectively. Quasi-ideals for rings and semigroups have been widely studied. Many papers cited in [5], a book written by O. Steinfeld in 1978. K. Iseki [1] introduced the notion of quasi-ideals for semirings without zero and proved some results. We can see some properties of quasi-ideals of semirings in [1] and [2].

In this paper, we study (m, n)-quasi-ideals of semirings. We generalize some facts of left ideals, right ideals and quasi-ideals of semirings.

2. Preliminaries

A semiring S in this paper is a nonempty set S together with two binary operations additive $+$ and multiplication $\cdot$ such that $(S, +)$ is a commutative semigroup, $(S, \cdot)$ is a semigroup and are connected by distributive laws;

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$$a(b+c) = ab+ac \text{ and } (b+c)a = ba+ca \text{ for all } a,b,c \in S.$$

In this paper, we shall assume that a semiring S has an absorbing zero 0, that is, $a+0 = a = 0+a$ and $0a = 0 = a0$ for all $a \in S$.

Let $\mathbb{N}$ denote the set of all natural numbers. For nonempty subsets A, B of a semiring S, let $(\mathbb{N} \cup \{0\})A$ and AB denote respectively the set of all finite sums of the form $\sum k_i a_i$ where $k_i \in \mathbb{N} \cup \{0\}$ and $a_i \in A$ and the set of all finite sums of the form $\sum a_i b_i$ where $a_i \in A$ and $b_i \in B$.

A nonempty subset T of a semiring S is called a *subsemiring* of S if $a+b, ab \in T$ for all $a, b \in T$. A subsemiring L of S is called a *left ideal* of S if $SL \subseteq L$. A *right ideal* of S is defined analogously. A subsemiring Q of S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. The class of quasi-ideals in semirings is a generalization of the class of one-sided ideals in semirings. Moreover, the intersection of a left ideal and a right ideal of a semiring S is a quasi-ideal of S. However, a quasi-ideal of a semiring S need not be obtianed in this way.

3. Main Results

A subsemiring Q of a semiring S is called an (m,n)-*quasi-ideal* of S if $S^m Q \cap QS^n \subseteq Q$ where m and n are positive integers. We have that a quasi-ideal Q of a semiring S is a $(1,1)$-quasi-ideal of S. Note that if S is a semiring having an identity, then all (m,n)-quasi-ideals of S are quasi-ideals of S for all $m, n \in \mathbb{N}$. Moreover, we have an (m,n)-quasi-ideal of S is a (k,l)-quasi-ideal of S for all $k \geq m$ and $l \geq n$. The following example shows that any (m,n)-quasi-ideal of a semiring S need not be a quasi-ideal of S.

Example 3.1. Let $SU_4(\mathbb{N} \cup \{0\})$ be the semiring of all strictly upper triangular 4×4 matrices over $\mathbb{N} \cup \{0\}$ under the usual addition and multiplication of matrices and

$$Q = \left\{ \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid x \in \mathbb{N} \cup \{0\} \right\}.$$

It is easy to prove that Q is a subsemiring of $SU_4(\mathbb{N} \cup \{0\})$. We have that

$$SU_4(\mathbb{N} \cup \{0\})Q \cap QSU_4(\mathbb{N} \cup \{0\}) = \left\{ \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid x \in \mathbb{N} \cup \{0\} \right\} \not\subseteq Q$$

but

$$SU_4(\mathbb{N}\cup\{0\})^2Q\cap QSU_4(\mathbb{N}\cup\{0\})^5=\left\{\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right\}\subseteq Q.$$

This implies that Q is a (2,5)-quasi-ideal but not a quasi-ideal of $SU_4(\mathbb{N}\cup\{0\})$.

The following lemma is well-known.

Lemma 3.1. *Let S be a semiring and T_i be a subsemiring of S for all $i\in I$. If $\bigcap_{i\in I}T_i\neq\emptyset$, then $\bigcap_{i\in I}T_i$ is a subsemiring of S.*

Proof. Assume $\bigcap_{i\in I}T_i\neq\emptyset$. Let $a,b\in\bigcap_{i\in I}T_i$. Thus $a,b\in T_i$ for all $i\in I$. Since T_i is a subsemiring of S for all $i\in I$, $a+b,ab\in T_i$ for all $i\in I$. Therefore $a+b,ab\in\bigcap_{i\in I}T_i$. Hence $\bigcap_{i\in I}T_i$ is a subsemiring of S. □

Note that the condition $\bigcap_{i\in I}T_i\neq\emptyset$ is necessary. For example, let $\mathbb{N}\cup\{0\}$ be a semiring under the usual addition and multiplication. For each $i\in\mathbb{N}$, let $T_i=\{k\in\mathbb{N}\mid k\geq i\}$. It is easy to see that T_i is a subsemiring of a semiring $\mathbb{N}$ for all $i\in\mathbb{N}$ but $\bigcap_{i\in\mathbb{N}}T_i=\emptyset$.

The following theorem shows that the intersection of (m,n)-quasi-ideals of a semiring S is an (m,n)-quasi-ideal of S.

Theorem 3.2. *Let S be a semiring and Q_i be an (m,n)-quasi-ideal of S for all $i\in I$. Then $\bigcap_{i\in I}Q_i$ is an (m,n)-quasi-ideal of S.*

Proof. Since $0\in\bigcap_{i\in I}Q_i$, $\bigcap_{i\in I}Q_i$ is nonempty. By Lemma 3.1, we have $\bigcap_{i\in I}Q_i$ is a subsemiring of S. Next, let $c\in S^m(\bigcap_{i\in I}Q_i)\cap(\bigcap_{i\in I}Q_i)S^n$. Then $c=\sum x_kp_k=\sum q_ly_l$ for some $x_k\in S^m,y_l\in S^n$ and $p_k,q_l\in\bigcap_{i\in I}Q_i$. Then for each k and l, we have $p_k,q_l\in Q_i$ for all $i\in I$. Thus $c\in S^mQ_i\cap Q_iS^n$ for all $i\in I$. Since Q_i is an (m,n)-quasi-ideal of S for all $i\in I,c\in Q_i$ for all $i\in I$. Then $c\in\bigcap_{i\in I}Q_i$. Hence $\bigcap_{i\in I}Q_i$ is an (m,n)-quasi-ideal of S. □

Let A be a subset of a semiring S and $\mathcal{I} = \{Q \mid Q$ is an (m,n)-quasi-ideal of S containing $A\}$. Therefore $\mathcal{I}$ is nonempty because $S \in \mathcal{I}$. Let $(A)_{q(m,n)} = \bigcap_{Q \in \mathcal{I}} Q$. It is clearly see that $(A)_{q(m,n)}$ is nonempty because $0 \in (A)_{q(m,n)}$. By Theorem 3.2, $(A)_{q(m,n)}$ is an (m,n)-quasi-ideal of S. Moreover, $(A)_{q(m,n)}$ is the smallest (m,n)-quasi-ideal of S containing A. The (m,n)-quasi-ideal $(A)_{q(m,n)}$ is called the *(m,n)-quasi-ideal of S generated by A*. It is clear that $(\emptyset)_{q(m,n)} = (\{0\})_{q(m,n)} = \{0\}$. The following theorem is true.

Theorem 3.3. *Let A be a nonempty subset of a semiring S. Then*

$$(A)_{q(m,n)} = (\sum_{i=1}^{\max\{m,n\}} (\mathbb{N} \cup \{0\})A^i) + (S^m A \cap AS^n).$$

Proof. Let $k = \max\{m,n\}$ and $Q = (\sum_{i=1}^{k}(\mathbb{N} \cup \{0\})A^i) + (S^m A \cap AS^n)$. Since S has an absorbing zero, $A \subseteq Q$. Let $a, b \in Q$. Therefore $a = x_1 + y_1$ and $b = x_2 + y_2$ where $x_1, x_2 \in \sum_{i=1}^{k}(\mathbb{N} \cup \{0\})A^i$ and $y_1, y_2 \in S^m A \cap AS^n$. So $x_1 + x_2 \in \sum_{i=1}^{k}(\mathbb{N} \cup \{0\})A^i, x_1x_2 \in \sum_{i=1}^{k}(\mathbb{N} \cup \{0\})A^i + (S^m A \cap AS^n)$ and $y_1 + y_2, x_1y_2, y_1x_2, y_1y_2 \in S^m A \cap AS^n$. Thus

$$a + b = (x_1 + x_2) + (y_1 + y_2) \in (\sum_{i=1}^{k}(\mathbb{N} \cup \{0\})A^i) + (S^m A \cap AS^n)$$

and

$$ab = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2 \in \sum_{i=1}^{k}(\mathbb{N} \cup \{0\})A^i) + (S^m A \cap AS^n).$$

Then Q is a subsemiring of S containing A. Next, we have

$$\begin{aligned} S^m Q \cap QS^n = S^m((&\sum_{i=1}^{k}(\mathbb{N} \cup \{0\})A^i) + (S^m A \cap AS^n)) \\ &\cap ((\sum_{i=1}^{k}(\mathbb{N} \cup \{0\})A^i) + (S^m A \cap AS^n))S^n \end{aligned}$$

$$\subseteq S^m((\sum_{i=1}^{k}(\mathbb{N}\cup\{0\})A^i)+S^m A)\cap((\sum_{i=1}^{k}(\mathbb{N}\cup\{0\})A^i)+AS^n))S^n \subseteq S^m A \cap AS^n \subseteq Q.$$

Now, we have Q is an (m,n)-quasi-ideal of S containing A.

To show Q is smallest, let Q' be any (m,n)-quasi-ideal of S containing A. Then $(\mathbb{N}\cup\{0\})A^i \subseteq Q'$ for all $i\in\mathbb{N}$ and $S^mA\cap AS^n \subseteq S^mQ'\cap Q'S^n \subseteq Q'$. Therefore $Q = (\sum_{i=1}^{k}(\mathbb{N}\cup\{0\})A^i) + (S^mA\cap AS^n) \subseteq Q'$.

Hence, Q is the smallest (m,n)-quasi-ideal of S containing A. Therefore $(A)_{q(m,n)} = (\sum_{i=1}^{\max\{m,n\}}(\mathbb{N}\cup\{0\})A^i) + (S^mA\cap AS^n)$, as required. □

Let S be a semiring. A subsemiring L of S is called an *m-left ideal* of S if $S^mL\subseteq L$ where m is a positive integer. An *n-right ideal* of S is defined analogously where n is a positive integer. The proof of the following theorem is similar to the proof of Theorem 3.2.

Theorem 3.4. *Let S be a semiring.*

(i) Let L_i be an m-left ideal of S for all $i\in I$. Then $\bigcap_{i\in I} L_i$ is an m-left ideal of S.

(ii) Let R_i be an n-right ideal of S for all $i\in I$. Then $\bigcap_{i\in I} R_i$ is an n-right ideal of S.

Let A be a subset of a semiring S and $\mathcal{I} = \{L \mid L$ is an m-left ideal of S containing $A\}$. Thus $\mathcal{I}$ is not empty because $S\in\mathcal{I}$. Let $(A)_{l(m)} = \bigcap_{L\in\mathcal{I}} L$. It is clearly see that $(A)_{l(m)}$ is nonempty because $0\in(A)_{l(m)}$. By Theorem 3.4(i), $(A)_{l(m)}$ is an m-left ideal of S. Moreover, $(A)_{l(m)}$ is the smallest m-left ideal of S containing A. The m-left ideal $(A)_{l(m)}$ is called the *m-left ideal of S generated by A*. The *n-right ideal $(A)_{r(n)}$ of S generated by A* is defined analogously. The proof of the following theorem is similar to the proof of Theorem 3.3.

Theorem 3.5. *Let A be a nonempty subset of a semiring S. The following statements hold.*

(i) $(A)_{l(m)} = (\sum_{i=1}^{m}(\mathbb{N}\cup\{0\})A^i) + S^mA.$

(ii) $(A)_{r(n)} = (\sum_{i=1}^{n}(\mathbb{N}\cup\{0\})A^i) + AS^n.$

The following theorem shows that the intersection of an m-left ideal and an

n-right ideal of a semiring S is an (m,n)-quasi-ideal of S.

Theorem 3.6. *Let L and R be an m-left ideal and an n-right ideal of a semiring S. Then $L \cap R$ is an (m,n)-quasi-ideal of S.*

Proof. Since $0 \in L \cap R$, by Lemma 3.1, we have $L \cap R$ is a subsemiring of S. Next, we have

$$(S^m(L \cap R)) \cap ((L \cap R)S^n) \subseteq S^m L \cap RS^n \subseteq L \cap R.$$

Hence $L \cap R$ is an (m,n)-quasi-ideal of S. □

We have known that a quasi-ideal of a semiring S need not be the intersection of a left ideal and a right ideal of S. Since a quasi-ideal of a semiring S is a $(1,1)$-quasi-ideal of S, an (m,n)-quasi-ideal need not be the intersection of an m-left ideal and an n-right ideal of S.

Lemma 3.7. *Let Q be an (m,n)-quasi-ideal of semiring S. Then $Q+S^mQ$ and $Q+QS^n$ are an m-left ideal and an n-right ideal of S, respectively.*

Proof. Since Q is a subsemiring of S, $\sum_{i=1}^{m}(\mathbb{N} \cup \{0\})Q^i = Q$. By Theorem 3.5(i), $Q + S^mQ$ is an m-left ideal (generated by Q) of S. Similarly, $Q + QS^n$ is an n-right ideal of S. □

An m-left ideal L of S is called an *m-left $*$-ideal* of S provided that $a, a+x \in L$ implies $x \in L$. An *n-right $*$-ideal* of S is defined analogously. Below we examine semirings in which each (m,n)-quasi-ideal is an intersection of an m-left ideal and an n-right ideal.

Theorem 3.8. *Let Q be an (m,n)-quasi-ideal of semiring S.*

(i) If $Q \subseteq S^mQ$ and S^mQ is an m-left $*$-ideal of S, then Q is the intersection of the m-left ideal $Q + S^mQ$ and the right ideal $Q + QS^n$.

(ii) If $Q \subseteq QS^n$ and QS^n is an n-right $*$-ideal of S, then Q is the intersection of the m-left ideal $Q + S^mQ$ and the right ideal $Q + QS^n$.

Proof. (i) Assume that $Q \subseteq S^mQ$ and S^mQ is an m-left $*$-ideal of S. By Lemma 3.7, $Q + S^mQ$ and $Q + QS^n$ are an m-left ideal and an n-right ideal of S, respectively. Since S has an absorbing zero, $Q \subseteq (Q + S^mQ) \cap (Q + QS^n)$.

Conversely, let $x \in (Q + S^mQ) \cap (Q + QS^n)$. Since $Q \subseteq S^mQ, Q + S^mQ = S^mQ$. Thus $(Q + S^mQ) \cap (Q + QS^n) = S^mQ \cap (Q + QS^n)$. Then $x = a = q + b$ where $a \in S^mQ, b \in QS^n$ and $q \in Q$. Now, we have $q \in Q \subseteq S^mQ$ and $q + b \in S^mQ$, $b \in S^mQ$ because S^mQ is an m-left $*$-ideal of S. Therefore $b \in S^mQ \cap QS^n \subseteq Q$. Hence $x = b+q \in Q$. This implies $Q = (Q+S^mQ) \cap (Q+QS^n)$.

(ii) It is similar to (i). □

Corollary 3.9. *Let S be a semiring.*

(i) If S has a left identity and every m-left ideal of S is an m-left $*$-ideal of S, then every (m,n)-quasi-ideal of S is the intersection of an m-left ideal and an n-right ideal of S.

(ii) If S has a right identity and every n-right ideal of S is an n-right $*$-ideal of S, then every (m,n)-quasi-ideal of S is the intersection of an m-left ideal and an n-right ideal of S.

Proof. (i) Let Q be an (m,n)-quasi-ideal of S. Since S has a left identity, $Q \subseteq S^mQ$. It is easy to prove that S^mQ is an m-left ideal of S. By assumption, S^mQ is an m-left $*$-ideal of S. By Theorem 3.8 (i), Q is an intersection of an m-left ideal and an n-right ideal of S.

(ii) It is similar to (i). □

An element a of a semiring S is called *regular* if there exists $x \in S$ such that $a = axa$. A semiring S is called *regular* if every element in S is regular.

Theorem 3.10. *Every (m,n)-quasi-ideal in a regular semiring is the intersection of an m-left ideal and an n-right ideal of S.*

Proof. Let S be a regular semiring and Q be an (m,n)-quasi-ideal of S. By Lemma 3.7, $Q + S^mQ$ and $Q + Q^nS$ is an m-left ideal and an n-right ideal of S, respectively. Let $L = Q + S^mQ$ and $R = Q + QS^n$. So $Q \subseteq L \cap R$. Since S is regular, $Q \subseteq S^mQ$ and $Q \subseteq QS^n$. Thus $L = S^mQ$ and $R = QS^n$. Then $R \cap L = S^mQ \cap QS^n \subseteq Q$. Therefore $Q = R \cap L$. □

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