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FAMILIES OF ALGEBRAIC LATTICES IN EVEN DIMENSIONS UP TO 8

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Abstract: In this paper, starting from suitable ideals in the ring of algebraic integers of cyclotomic fields $\mathbb{Q}(\zeta_m)$, where $m = 3, 6, 8, 9, 20$, we present new rotated versions of lattices in dimensions 2, 4, 6 and 8, by use of the canonical homomorphism. These rotated versions reproduce densest algebraic lattices in dimensions 2, 4, 6 and 8.

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1. Introduction

The theory of lattices via algebraic numbers theory has shown to be extremely useful in information theory. The problem of finding densest algebraic lattices has been studied in last years. In [2], Boutros et al constructed rotated versions of lattices $D_4 = \Lambda_4$, K_{12} and $E_6 = \Lambda_6$ via ideals of the ring of algebraic integers of the cyclotomic fields $\mathbb{Q}(\zeta_m)$, for $m = 8, 21$ and 40, respectively. In [4],

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Nóbrega Neto et al presented rotated versions of lattices $E_8 = \Lambda_8$, K_{12} and Λ_{24} via the canonical homomorphism applied in ideals of the ring of algebraic integers of subfields of cyclotomic fields $\mathbb{Q}(\zeta_{pq})$, where p and q are distinct prime numbers. In [1], Andrade et al presented rotated versions of lattices $A_2 = \Lambda_2$ and $D_4 = \Lambda_4$ via the canonical homomorphism applied in ideals of the ring of algebraic integers of cyclotomic fields $\mathbb{Q}(\zeta_m)$, for $m = 3, 6$ and 8. Thus, having the construction of algebraic lattices as the main motivation, in this paper, starting from suitable ideals of the ring of algebraic integers of $\mathbb{Q}(\zeta_m)$, where $m = 3, 6, 8, 9, 20$, we construct new rotated versions of dense lattices, for example, $A_2 = \Lambda_2$, $A_4 = \Lambda_4$, $E_6 = \Lambda_6$ and $E_8 = \Lambda_8$. These rotated versions reproduce densest lattices in dimensions 2, 4, 6 and 8.

This paper is organized as follows. In Section 2, we give basic results from numbers field and quadratics form over the cyclotomic fields $\mathbb{Q}(\zeta_m)$, where m is a positive integer. In Section 3, we present the concept of algebraic lattices via the canonical homomorphism (or Minkowski). In Section 4, we present new examples of densest algebraic lattices obtained via the canonical homomorphism applied in ideals of the ring of algebraic integers of cyclotomic fields $\mathbb{Q}(\zeta_m)$, where $m = 3, 6, 8, 9, 20$, in dimensions 2, 4, 6 and 8.

2. Basic Results from Numbers Field

In this section, we give a briefly review key concepts from the theory of algebraic number fields [6]-[7] which are necessary for the development of the subsequent sections.

Let $\mathbb{L}$ be an algebraic number field of degree n, i.e., $\mathbb{L} = \mathbb{Q}(\alpha)$, with $\alpha \in \mathbb{C}$ a root of a monic irreducible polynomial $p(x) \in \mathbb{Z}[x]$. The n distinct roots of $p(x)$, namely, $\alpha_1, \alpha_2, \cdots, \alpha_n$, are the conjugates of α. If $\sigma : \mathbb{L} \to \mathbb{C}$ is a $\mathbb{Q}$-homomorphism then $\sigma(\alpha) = \alpha_i$ for some $i = 1, 2, \cdots, n$. Furthermore, there are exactly n $\mathbb{Q}$-homomorphism σ_i, for $i = 1, 2, \cdots, n$, of $\mathbb{L}$ in $\mathbb{C}$.

Definition 1. Let $\mathbb{L}$ be a number field. The *trace* of any element $\alpha \in \mathbb{L}$ is defined as the rational number $\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha)$.

An element $\alpha \in \mathbb{L}$ is called an *algebraic integer* if there is a monic polynomial $f(x)$ with integer coefficients such that $f(\alpha) = 0$. The set $\mathcal{O}_{\mathbb{L}} = \{\alpha \in \mathbb{L} : \alpha \text{ is an algebraic integer}\}$ is called *ring of algebraic integers* of $\mathbb{L}$. It can be shown that $\mathcal{O}_{\mathbb{L}}$, as a $\mathbb{Z}$-module, has a basis $\{\alpha_1, \cdots, \alpha_n\}$ over $\mathbb{Z}$, where n is the degree of $\mathbb{L}$ [6]. In other words, every element $\alpha \in \mathcal{O}_{\mathbb{L}}$ can be uniquely written as $\alpha = \sum_{i=1}^{n} a_i\alpha_i$, where $a_i \in \mathbb{Z}$ for all $i = 1, 2, \cdots, n$. Furthermore, if $\alpha \in \mathcal{O}_{\mathbb{L}}$

then $Tr_{\mathbb{L}/\mathbb{Q}}(\alpha)$ is a number integer.

Definition 2. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{L}}$ and let $\mathcal{A}$ be an ideal of $\mathcal{O}_{\mathbb{L}}$. The *discriminant* of $\mathbb{L}$ over $\mathbb{Q}$ is defined by $\mathcal{D}_{\mathbb{L}} = \det(\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha_i\alpha_j))_{i,j=1}^n$ and the *norm* of $\mathcal{A}$ is defined by $\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A}) = card(\mathcal{O}_{\mathbb{L}}/\mathcal{A})$.

Definition 3. Let n be a positive integer. We say that ζ_m is a m-th *root of unity* if $\zeta_m^m = 1$. We say that ζ_m is a *primitive* m-th *root of unity* if $\zeta_m^m = 1$ and $\zeta_m^r \neq 1$ for all $1 \leq r \leq m-1$. The field $\mathbb{Q}(\zeta_m)$ is called a *cyclotomic field*.

Let m be a positive integer, ζ_m a primitive m-th root of unity and $\mathbb{L} = \mathbb{Q}(\zeta_m)$ the corresponding cyclotomic field. It can be shown that $[\mathbb{L} : \mathbb{Q}] = \phi(m)$, where ϕ is the Euler function, $\mathcal{O}_{\mathbb{L}} = \mathbb{Z}[\zeta_m]$ is the ring of algebraic integers of $\mathbb{Z}[\zeta_m]$ and $\{1, \zeta_m, \zeta_m^2, \cdots, \zeta_m^{\phi(m)-1}\}$ is an integral basis of $\mathbb{L}$ [8].

Theorem 4. (see [5]) *If* $m = \prod_{i=1}^{r} p_i^{\alpha_i}$, $P = \prod_{i=1}^{r} p_i$, $\mathbb{L} = \mathbb{Q}(\zeta_m)$, $\mathcal{O}_{\mathbb{L}}$ *is the ring of algebraic integers of* $\mathbb{L}$ *and* $\alpha = \sum_{i=0}^{\phi(m)-1} a_i\zeta_m^i \in \mathcal{O}_{\mathbb{L}}$ *then*

$$\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) = \frac{2m}{P}[\frac{\phi(P)}{2}\sum_{i=0}^{\phi(m)-1} a_i^2 + \mu(P)\sum_{i=1}^{\phi(P)-1} A_{\frac{m}{P}i}\phi(gcd(i,P))\mu(gcd(i,P))],$$

where μ *is a Moebius function,* ϕ *is the Euler function and* $A_i = \sum_{j=0}^{\phi(m)-(i+1)} a_j a_{i+j}$, *for all* $i = 0, 1, \cdots, \phi(m)-1$.

3. Algebraic Lattice

In this section, we give the concept of algebraic lattice via the canonical homomorphism of a number field [6], [7].

Let $\mathbb{L}$ be a number field of degree n, and $\sigma_1, \sigma_2, \cdots, \sigma_n$ be the $\mathbb{Q}$-homomorphisms of $\mathbb{L}$ into $\mathbb{C}$, ordered in such a way that σ_i is real for $i = 1, 2, \cdots, r_1$ and σ_{j+r_2} is the complex conjugate of σ_j for $j = r_1+1, r_1+2, \cdots, r_1+r_2$. Let $\mathcal{R}(z)$ and $\mathcal{I}(z)$ be the real and imaginary part of the complex number z, respectively.

Definition 5. The group homomorphism $\sigma_{\mathbb{L}} : \mathbb{L} \to \mathbb{R}^n$ defined by

$$\sigma_{\mathbb{L}}(\alpha) = (\sigma_1(\alpha), \cdots, \sigma_{r_1}(\alpha), \mathcal{R}(\sigma_{r_1+1}(\alpha)), \mathcal{I}(\sigma_{r_1+1}(\alpha)), \cdots,$$

$$\mathcal{R}(\sigma_{r_1+r_2}(\alpha)), \mathcal{I}(\sigma_{r_1+r_2}(\alpha))),$$

where $\alpha \in \mathbb{L}$, is called *canonical (or Minkowski's) homomorphism.*

Theorem 6. (see [6]) *If $\mathcal{A}$ is a non-zero ideal of $\mathcal{O}_{\mathbb{L}}$, then $\sigma_{\mathbb{L}}(\mathcal{A})$ is an algebraic lattice in $\mathbb{R}^n$ and its volume is given by $v(\sigma_{\mathbb{L}}(\mathcal{A})) = 2^{-r_2}\sqrt{\mathcal{D}_{\mathbb{L}}}\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A})$.*

Proposition 7. (see [3]) *If $\mathbb{L} = \mathbb{Q}(\zeta_m)$ and $\alpha \in \mathbb{L}$ then $\mid \sigma_{\mathbb{L}}(\alpha) \mid^2 = \frac{1}{2}\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha})$.*

The parameter

$$\rho = \frac{1}{2}\min\{\mid \sigma_{\mathbb{L}}(\alpha) \mid : \alpha \in \mathcal{A}, \alpha \neq 0\}$$

is called of *packing radius* of $\sigma_{\mathbb{L}}(\mathcal{A})$. If $t = \min\{\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) : \alpha \in \mathcal{A}, \alpha \neq 0\}$ then the center density of $\sigma_{\mathbb{L}}(\mathcal{A})$ is given by

$$\delta(\sigma_{\mathbb{L}}(\mathcal{A})) = \frac{1}{2^n \mid \mathcal{D}_{\mathbb{L}} \mid^{1/2}} \frac{t^{n/2}}{\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A})},$$

where $n = \phi(m)$.

4. Families of Algebraic Lattices in Even Dimension

In this section, we give examples of densest algebraic lattices obtained via the canonical homomorphism in dimensions 2, 4, 6 and 8. These lattices reproduce rotated versions of $\Lambda_2 = A_2$, $\Lambda_4 = D_4$, $\Lambda_6 = E_6$ and $\Lambda_8 = E_8$, respectively.

4.1. Algebraic Lattices in Dimension 2

In this section, we present examples of algebraic lattices in dimension 2 that have the same center density that the lattice $A_2 = \Lambda_2$.

Example 8. Let $\mathbb{L} = \mathbb{Q}(\zeta_6)$ and $\mathcal{A} = (\zeta_6)\mathcal{O}_{\mathbb{L}}$ an ideal of $\mathcal{O}_{\mathbb{L}}$. We have that $[\mathbb{L} : \mathbb{Q}] = 2$, $\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A}) = 1$ and $\mathcal{D}_{\mathbb{L}} = 3$. If $\alpha = (a_0 + a_1\zeta_6)\zeta_6 \in \mathcal{A}$, where $a_0, a_1 \in \mathbb{Z}$, then $\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) = 2(a_0^2 + a_1^2 + a_0a_1)$. Thus $t = \min\{\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) : \alpha \in \mathcal{O}_{\mathbb{L}}, \alpha \neq 0\} = 2$ (with $a_0 = 1$ and $a_1 = 0$) and therefore

$$\delta(\sigma_{\mathbb{L}}(\mathcal{A})) = \frac{1}{2^2 \mid D_{\mathbb{L}} \mid^{1/2}} \frac{t^{2/2}}{\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A})} = \frac{1}{2\sqrt{3}} \simeq 0,28868.$$

Thus the lattice $\sigma_{\mathbb{L}}(\mathcal{A})$ has the same density that the hexagonal lattice $A_2 = \Lambda_2$ (best density in $\mathbb{R}^2$).

Similarly, in the next table, we have that the lattices $\sigma_{\mathbb{L}}(\mathcal{A})$, where $\mathcal{A}$ is an ideal of $\mathcal{O}_{\mathbb{L}} = \mathbb{Z}[\zeta_6]$, have the same density that the hexagonal lattice $A_2 = \Lambda_2$.

$\mathcal{A}$	$\mathcal{N}(\mathcal{A})$	t	$\mathcal{A}$	$\mathcal{N}(\mathcal{A})$	t
$(1+\zeta_6)\mathcal{O}_{\mathbb{L}}$	3	6	$(-4\zeta_6)\mathcal{O}_{\mathbb{L}}$	16	32
$2\mathcal{O}_{\mathbb{L}}$	4	8	$(3-5\zeta_6)\mathcal{O}_{\mathbb{L}}$	19	38
$(1+2\zeta_6)\mathcal{O}_{\mathbb{L}}$	7	14	$(5-\zeta_6)\mathcal{O}_{\mathbb{L}}$	21	42
$(3\zeta_6)\mathcal{O}_{\mathbb{L}}$	9	18	$(5-5\zeta_6)\mathcal{O}_{\mathbb{L}}$	25	50
$(2-4\zeta_6)\mathcal{O}_{\mathbb{L}}$	12	24	$6\mathcal{O}_{\mathbb{L}}$	36	72
$(-4+\zeta_6)\mathcal{O}_{\mathbb{L}}$	13	26			

The next theorem shows that there exists several algebraic lattices in dimension 2.

Theorem 9. *If $\mathbb{L} = \mathbb{Q}(\zeta_3)$ and $\mathcal{A}$ is a non-zero principal ideal of $\mathcal{O}_{\mathbb{L}}$ then the algebraic lattice $\sigma_{\mathbb{L}}(\mathcal{A})$ has the same center density of the lattice $\Lambda_2 = A_2$.*

Proof. If $\mathcal{A}$ is a non-zero principal ideal of $\mathcal{O}_{\mathbb{L}}$ then $\mathcal{A} = (a + b\zeta_3)\mathcal{O}_{\mathbb{L}}$, where $a, b \in \mathbb{Z}$ and $(a, b) \neq (0, 0)$. Thus

$$\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A}) = \mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\langle a + b\zeta_3\rangle) = |\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(a + b\zeta_3)| = |a^2 + b^2 - ab| = a^2 + b^2 - ab,$$

since $a^2 + b^2 - ab > 0$. If $\alpha \in \mathcal{A}$ then

$$\alpha = (a + b\zeta_3)(a_0 + a_1\zeta_3) = (aa_0 - ba_1) + (aa_1 + ba_0 - ba_1)\zeta_3,$$

where $a_0, a_1 \in \mathbb{Z}$. Thus

$$\begin{aligned}\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) &= 2[(aa_0 - ba_1)^2 + (aa_1 + ba_0 - ba_1)^2 - (aa_0 - ba_1)(aa_1 + ba_0 - ba_1)] \\ &= 2a^2a_0^2 - 2a^2a_0a_1 + 2a^2a_1^2 - 2aba_0^2 + 2aba_0a_1 - 2aba_1^2 + 2b^2a_0^2 - 2b^2a_0a_1 + 2b^2a_1^2 \\ &= 2(a^2 + b^2 - ab)(a_0^2 + a_1^2 - a_0a_1),\end{aligned}$$

and $t = \min\{\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) : \alpha \in \mathcal{A}, \alpha \neq 0\} = 2(a^2 + b^2 - ab)$ (with $a_0 = 1$ and $a_1 = 0$). Therefore

$$\delta(\sigma_{\mathbb{L}}(\mathcal{A})) = \frac{1}{2^2\sqrt{|\mathcal{D}_{\mathbb{L}}|}} \frac{t^{2/2}}{\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A})} = \frac{1}{4\sqrt{3}} \frac{2(a^2 + b^2 - ab)}{(a^2 + b^2 - ab)} \simeq 0,28868. \qquad \square$$

4.2. Algebraic Lattices in Dimension 4

In this section, we present examples of algebraic lattices in dimension 4. These lattices have the same center density that the lattice $D_4 = \Lambda_4$.

Example 10. Let $\mathbb{L} = \mathbb{Q}(\zeta_8)$ and $\mathcal{A} = (1 + \zeta_8^3)\mathcal{O}_{\mathbb{L}}$ an ideal of $\mathcal{O}_{\mathbb{L}} =$

$\mathbb{Z}[\zeta_8]$. We have that $[\mathbb{L}:\mathbb{Q}] = 4$, $\mathcal{D}_{\mathbb{L}} = 256$ and $\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A}) = 2$. If $\alpha \in \mathcal{A}$ then $\alpha = (1+\zeta_8^3)(a_0 + a_1\zeta_8 + a_2\zeta_8^2 + a_3\zeta_8^3)$, where $a_0, a_1, a_2, a_3 \in \mathbb{Z}$. Thus $\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) = 8(a_0^2 + a_1^2 + a_2^2 + a_3^2 - a_0a_1 - a_1a_2 + a_0a_3 - a_2a_3)$. Therefore $t = \min\{\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) : \alpha \in \mathcal{A}, \alpha \neq 0\} = 8$ (with $a_0 = a_1 = 1$ and $a_2 = a_3 = 0$) and

$$\delta(\sigma_{\mathbb{L}}(\mathcal{A})) = \frac{1}{2^4|\mathcal{D}_{\mathbb{L}}|^{1/2}}\ \frac{t^{4/2}}{\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A})} = 0,125.$$

Thus the lattice $\sigma_{\mathbb{L}}(\mathcal{A})$ has the same density that the lattice $A_4 = \Lambda_4$ (best density in $\mathbb{R}^4$).

Similarly, in the next table, we have that the lattices $\sigma_{\mathbb{L}}(\mathcal{A})$, where $\mathcal{A}$ is an ideal of $\mathcal{O}_{\mathbb{L}} = \mathbb{Z}[\zeta_8]$, have the same density that the lattice $A_4 = \Lambda_4$.

$\mathcal{A}$	$\mathcal{N}(\mathcal{A})$	t	$\mathcal{A}$	$\mathcal{N}(\mathcal{A})$	t
$(1+\zeta_8+\zeta_8^2+\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	8	16	$(1+3\zeta_8-2\zeta_8^2+2\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	162	72
$(1-\zeta_8+2\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	18	24	$(3+\zeta_8+\zeta_8^2-3\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	200	80
$(2\zeta_8+2\zeta_8^2)\mathcal{O}_{\mathbb{L}}$	32	32	$(1+4\zeta_8+2\zeta_8^2-\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	242	88
$(2+2\zeta_8+\zeta_8^2+\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	50	40	$(4\zeta_8+2\zeta_8^2+2\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	288	96
$(-3+5\zeta_8-5\zeta_8^2+\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	72	48	$(2+3\zeta_8-3\zeta_8^2+2\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	338	104
$(2+6\zeta_8+6\zeta_8^2+2\zeta_8^3)\mathcal{O}_{\mathbb{L}}$	128	64			

4.3. Algebraic Lattices in Dimension 6

In this section, we present examples of algebraic lattices in dimension 6. These lattices have the same center density that lattice $E_6 = \Lambda_6$.

Example 11. Let $\mathbb{L} = \mathbb{Q}(\zeta_9)$ and $\mathcal{A} = (1+\zeta_9+\zeta_9^2+\zeta_9^3+\zeta_9^4+\zeta_9^5)\mathcal{O}_{\mathbb{L}}$ an ideal of $\mathcal{O}_{\mathbb{L}} = \mathbb{Z}[\zeta_9]$. We have that $[\mathbb{L}:\mathbb{Q}] = 6$, $\mathcal{D}_{\mathbb{L}} = 3^9$ and $\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A}) = 9$. If $\alpha \in \mathcal{A}$ then $\alpha = (1+\zeta_9+\zeta_9^2+\zeta_9^3+\zeta_9^4+\zeta_9^5)(a_0+a_1\zeta_9+a_2\zeta_9^2+a_3\zeta_9^3+a_4\zeta_9^4+a_5\zeta_9^5)$, where $a_i \in \mathbb{Z}$, for all $i = 0,1,2,3,4,5$. Thus $\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) = 18(\sum_{i=0}^{5} a_i^2 + a_0a_1 + a_1a_2 - a_0a_3 + a_2a_3 - a_0a_4 - a_1a_4 + a_3a_4 - a_0a_5 - a_1a_5 - a_2a_5 + a_4a_5)$ and $t = \min\{\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) : \alpha \in \mathcal{A}, \alpha \neq 0\} = 18$ (with $a_0 = 1$ and $a_1 = a_2 = a_3 = a_4 = a_5 = 0$). Therefore

$$\delta(\sigma_{\mathbb{L}}(\mathcal{A})) = \frac{1}{2^6|\mathcal{D}_{\mathbb{L}}|^{1/2}}\ \frac{t^{6/2}}{\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A})} \simeq 0,07217.$$

Thus the lattice $\sigma_{\mathbb{L}}(\mathcal{A})$ has the same density that the lattice E_6 (best density in $\mathbb{R}^6$).

Similarly, in the next table, we have that the lattices $\sigma_{\mathbb{L}}(\mathcal{A})$, where $\mathcal{A}$ is an ideal of $\mathcal{O}_{\mathbb{L}} = \mathbb{Z}[\zeta_9]$, have the same density that the lattice E_6.

$\mathcal{A}$	$\mathcal{N}(\mathcal{A})$	t
$(2 - \zeta_9^4 - \zeta_9^5)\mathcal{O}_{\mathbb{L}}$	9	18
$(1 - 2\zeta_9^2 + \zeta_9^3 + 2\zeta_9^4 - 2\zeta_9^5)\mathcal{O}_{\mathbb{L}}$	9	18
$(1 + 2\zeta_9 + 2\zeta_9^4 + \zeta_9^5)\mathcal{O}_{\mathbb{L}}$	9	18
$(1 + \zeta_9 + \zeta_9^2 - \zeta_9^3 - \zeta_9^4 - \zeta_9^5)\mathcal{O}_{\mathbb{L}}$	243	54
$(-2 + \zeta_9 + \zeta_9^2 - \zeta_9^3 - \zeta_9^4 + 2\zeta_9^5)\mathcal{O}_{\mathbb{L}}$	243	54
$(2 + 2\zeta_9 + 2\zeta_9^2 + \zeta_9^3 - 2\zeta_9^4 - 2\zeta_9^5)\mathcal{O}_{\mathbb{L}}$	243	54
$(2 + 2\zeta_9 + 2\zeta_9^2)\mathcal{O}_{\mathbb{L}}$	576	72
$(-2\zeta_9 - 2\zeta_9^3 - 2\zeta_9^5)\mathcal{O}_{\mathbb{L}}$	576	72
$(-2\zeta_9 + 2\zeta_9^2 + 2\zeta_9^3 - 2\zeta_9^4)\mathcal{O}_{\mathbb{L}}$	576	72
$(1 + \zeta_9 + \zeta_9^2 - 2\zeta_9^3 - 2\zeta_9^4 - 2\zeta_9^5)\mathcal{O}_{\mathbb{L}}$	3087	126
$(2 + 2\zeta_9 + 2\zeta_9^2 - \zeta_9^3 - \zeta_9^4 - \zeta_9^5)\mathcal{O}_{\mathbb{L}}$	3087	126

4.4. Algebraic Lattices in Dimension 8

In this section we present examples of algebraic lattices in dimension 8. These lattices have the same center density of lattice $E_8 = \Lambda_8$.

Example 12. Let $\mathbb{L} = \mathbb{Q}(\zeta_{20})$ and $\mathcal{A} = (-1 - \zeta_{20} + \zeta_{20}^2 + \zeta_{20}^3 + \zeta_{20}^4)\mathcal{O}_{\mathbb{L}}$ an ideal of $\mathcal{O}_{\mathbb{L}} = \mathbb{Z}[\zeta_{20}]$. We have that $[\mathbb{L} : \mathbb{Q}] = 8$, $\mathcal{D}_{\mathbb{L}} = 2^8 5^6$ and $\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A}) = 80$. If $\alpha \in \mathcal{A}$ then $\alpha = (-1 - \zeta_{20} + \zeta_{20}^2 + \zeta_{20}^3 + \zeta_{20}^4)(a_0 + a_1\zeta_{20} + a_2\zeta_{20}^2 + a_3\zeta_{20}^3 + a_4\zeta_{20}^4 + a_5\zeta_{20}^5 + a_6\zeta_{20}^6 + a_7\zeta_{20}^7)$, where $a_i \in \mathbb{Z}$, for all $i = 0, 1, \cdots, 7$. Thus $\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) = 40(\sum_{i=0}^{7} a_i^2 + a_0a_1 + a_1a_2 - a_0a_3 + a_2a_3 - a_0a_4 - a_1a_4 + a_3a_4 - a_1a_5 - a_2a_5 + a_4a_5 + a_0a_6 - a_2a_6 - a_3a_6 + a_5a_6 + a_0a_7 + a_1a_7 - a_3a_7 - a_4a_7 + a_6a_7)$ and $t = \min\{\mathcal{T}_{\mathbb{L}/\mathbb{Q}}(\alpha\overline{\alpha}) : \alpha \in \mathcal{A}, \alpha \neq 0\} = 40$ (with $a_0 = a_1 = a_4 = -1$, $a_2 = a_6 = 1$ and $a_3 = a_5 = a_7 = 0$). Therefore

$$\delta(\sigma_{\mathbb{L}}(\mathcal{A})) = \frac{1}{2^8|\mathcal{D}_{\mathbb{L}}|^{1/2}} \; \frac{t^{8/2}}{\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(\mathcal{A})} \simeq 0,06250.$$

Thus the lattice $\sigma_{\mathbb{L}}(\mathcal{A})$ has the same density that the lattice E_8 (best density in $\mathbb{R}^8$).

Similarly, in the next table, we have that the lattices $\sigma_{\mathbb{L}}(\mathcal{A})$, where $\mathcal{A}$ is an ideal of $\mathcal{O}_{\mathbb{L}} = \mathbb{Z}[\zeta_{20}]$, have the same density that the lattice E_8.

$\mathcal{A}$	$\mathcal{N}(\mathcal{A})$	t
$(\zeta_{20} - \zeta_{20}^2 - \zeta_{20}^3 + \zeta_{20}^4 - \zeta_{20}^5)\mathcal{O}_{\mathbb{L}}$	80	40
$(-1 - \zeta_{20} + \zeta_{20}^2 - \zeta_{20}^5 + \zeta_{20}^6 - \zeta_{20}^7)\mathcal{O}_{\mathbb{L}}$	80	40
$(\zeta_{20}^3 + \zeta_{20}^4 - \zeta_{20}^5 - \zeta_{20}^6 - \zeta_{20}^7)\mathcal{O}_{\mathbb{L}}$	80	40
$(-1 + \zeta_{20}^2 - \zeta_{20}^3 - \zeta_{20}^4 - \zeta_{20}^6)\mathcal{O}_{\mathbb{L}}$	80	40
$(-\zeta_{20}^2 + \zeta_{20}^3 + \zeta_{20}^4 - \zeta_{20}^5 + \zeta_{20}^6)\mathcal{O}_{\mathbb{L}}$	80	40
$(-1 - \zeta_{20} - \zeta_{20}^2 - \zeta_{20}^3 + \zeta_{20}^4 + \zeta_{20}^5 - \zeta_{20}^7)\mathcal{O}_{\mathbb{L}}$	405	60
$(1 - \zeta_{20}^2 + \zeta_{20}^3 + \zeta_{20}^4 - \zeta_{20}^5 + \zeta_{20}^6 - \zeta_{20}^7)\mathcal{O}_{\mathbb{L}}$	405	60
$(1 + \zeta_{20} + \zeta_{20}^2 + \zeta_{20}^4 - \zeta_{20}^5 - \zeta_{20}^6 - \zeta_{20}^7)\mathcal{O}_{\mathbb{L}}$	405	60
$(1 + \zeta_{20} + \zeta_{20}^2 + \zeta_{20}^3 - \zeta_{20}^4 - \zeta_{20}^5 + \zeta_{20}^7)\mathcal{O}_{\mathbb{L}}$	405	60
$(1 + \zeta_{20} + \zeta_{20}^3 + \zeta_{20}^4 + \zeta_{20}^5 + \zeta_{20}^6 - \zeta_{20}^7)\mathcal{O}_{\mathbb{L}}$	405	60
$(1 - \zeta_{20} + \zeta_{20}^2 + \zeta_{20}^3 + \zeta_{20}^5 - \zeta_{20}^6 + \zeta_{20}^7)\mathcal{O}_{\mathbb{L}}$	405	60

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