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ON STRONGER FORMS OF FIRST-ORDER NECESSARY CONDITIONS OF OPTIMALITY FOR STATE-CONSTRAINED CONTROL PROBLEMS

Sofia O. Lopes[1], Fernando A.C.C. Fontes[2 §]

[1,2]Department of Mathematics for Science and Technology
University of Minho
Campus de Azurém, Guimarães, 4800-058, PORTUGAL
[1]e-mail: sofialopes@mct.uminho.pt
[2]e-mail: ffontes@mct.uminho.pt

Abstract: Recent research in necessary conditions for optimization problems has pursued two main directions. On the one hand, there is a quest to address problems with more and more generality. On the other, there is the investigation of tighter and tighter conditions which are able to identify a reduced set of candidates to minimizers. This work describes a line of research that has been pursued in the latter direction. We discuss the degeneracy phenomenon in optimization problems with inequality constraints. We start by describing this phenomenon in the context of mathematical programming problems. Later, we address the degeneracy phenomenon in the context of optimal control problems. We report some forms of nondegenerate and normal forms of necessary conditions of optimality for optimal control problems with state constraints. A brief overview of some significant literature in this area is made.

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[§]Correspondence author

1. Introduction

The main purpose of Necessary Conditions of Optimality (NCO) is to identify a small set of candidates to local minimizers among the overall set of admissible solutions. However, for optimization problems with constraints, it may happen that the NCO are useless to give useful information about the minimizers. When this happens, we say that the NCO degenerate.

We start motivating by considering a problem in the context of Mathematical Programming (we assume that all functions involved are are continuously differentiable):

$$(MP) \quad \begin{array}{lll} \text{Minimize} & f(x) & \\ \text{subject to} & h^i(x) \leq 0, & i = 1, 2, ..., n. \end{array}$$

If $\bar{x}$ is a solution to the problem (MP), then the NCO in the form of Fritz-John conditions [12] guarantee the existence of nonnegative multipliers λ and μ_i, with $i = 0, 1, 2, ..., n$ such that

$$(\lambda, \mu_1, ..., \mu_n) \neq 0$$
$$\lambda f_x(\bar{x}) + \textstyle\sum_{i=1}^{n} \mu_i h_x^i(\bar{x}) = 0$$
$$\mu_i h^i(\bar{x}) = 0, \quad \text{for } i = 1, ..., n.$$

If these conditions are satisfied with $\lambda = 0$, the cost function is not involved in the choice of candidates to minimizers. Naturally, it would be desirable to force the cost function to be involved in the NCO by imposing that $\lambda = 1$. However, we have to guarantee that the NCO are still satisfied at local minimum. If that is not the case, the NCO would not be necessary.

So, additional hypotheses, known as Constraint Qualification (CQ), have to be considered to identify the problems under which λ can be chosen to be positive. There are several forms of CQ for this problem (see e.g. [1, 14, 4]). As an example, we can impose that the gradients of the active constraints at the minimizer are linearly independent. Alternatively, we have the Mangasarian-Fromovitz form of CQ:

For every local minimizer $\bar{x}$ there exists a vector $v \in \mathbb{R}^n$ such that

$$h_x^i(\bar{x}) \cdot v < 0 \quad \textit{if} \quad h^i(\bar{x}) = 0, \quad i = 1, 2, ..., n.$$

The Kuhn-Tucker conditions are precisely a normal version of the Fritz John conditions valid under a suitable CQ. They state that we can choose $\lambda = 1$ for all problems complying with the CQ. The work of Kuhn and Tucker, probably one of the most cited results in optimization, is in fact a stronger, and nondegenerate form, of a previous result. This fact justifies the importance of studying nondegenerate versions of NCO for constrained optimization problems.

This phenomenon is well-studied in the context of mathematical programming for a long time. However, results in the optimal control context have witnessed many important advances in the recent years. In addition to the importance *per se* of having stronger NCO, normal forms of the maximum principle have important applications in deducing regularity properties of the minimizers (fundamental for certain optimization algorithms) and also properties of Hamilton-Jacobi equations.

2. Degeneracy in Optimal Control

Let consider an Optimal Control Problem (OCP) with inequality (state) constraints and fixed left-endpoint:

$$
\begin{array}{lll}
(P) & \text{Minimize} & g(x(1)) \\
 & \text{subject to} & \dot{x}(t) = f\left(t, x(t), u(t)\right) \quad \text{a.e. } t \in [0,1] \\
 & & x(0) = x_0, \quad x(1) \in C \\
 & & u(t) \in \Omega(t) \qquad\qquad\qquad \text{a.e. } t \in [0,1] \\
 & & h\left(x(t)\right) \leq 0 \qquad\qquad\qquad \text{for all } t \in [0,1].
\end{array}
$$

The data comprise functions $g : \mathbb{R}^n \mapsto \mathbb{R}$, $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$, $h : \mathbb{R}^n \mapsto \mathbb{R}$, $x_0 \in \mathbb{R}^n$, a set $C \subset \mathbb{R}^n$ and a multifunction $\Omega : [0,1] \rightrightarrows \mathbb{R}^m$.

It is well known that, the NCO for such problems appears in the form of Maximum Principle (MP) (see e.g. [15, 7, 18]). They typically assert (under smoothness hypotheses) existence of an absolutely continuous function p, a non-negative measure $\mu \in C^*([0,1] : \mathbb{R})$ and $\lambda \geq 0$ such that

$$
\begin{gathered}
\mu\{[0,1]\} + \|p\|_{L^\infty} + \lambda \neq 0, \\
-\dot{p}(t) = q(t) \cdot f_x(t, \bar{x}(t), \bar{u}(t)) \text{ a.e.} \\
-q(1) \in N_C(\bar{x}(1)) + \lambda g_x(\bar{x}(1)), \\
\text{supp}\,\{\mu\} \subset \{t \in [0,1] : h(\bar{x}(t)) = 0\}, \quad \text{and} \\
\bar{u}(t) \text{ maximizes over } \Omega(t) \quad u \mapsto q(t) \cdot f(t, \bar{x}(t), u), \quad \text{a.e. } t \in [0,1],
\end{gathered}
$$

where

$$
q(t) = \begin{cases} p(t) + \int_{[0,t)} h_x(\bar{x}(s))\mu(ds) & t \in [0,1) \\ \\ p(1) + \int_{[0,1]} h_x(\bar{x}(s))\mu(ds) & t = 1. \end{cases}
$$

Assuming that the pathwise state constraint is active in the initial instant

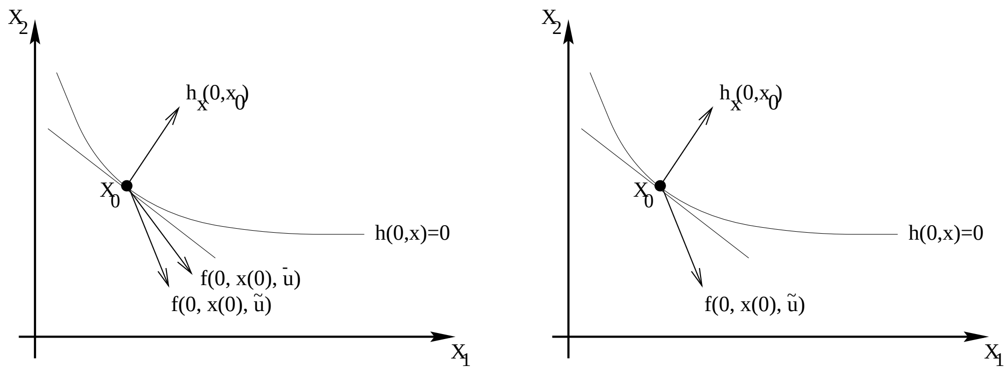

Figure 1: Constraint qualification $\mathbf{CQ1_d}$ and $\mathbf{CQ2_d}$, respectively (adapted from [10])

of time $h(x_0) = 0$, the set of multipliers (degenerate multipliers)[1]

$$\lambda = 0,\ \mu = \delta_{t=0}, p = -h_x(x_0) \tag{1}$$

satisfies the MP for all admissible process (x, u). This can be easily seen by noting that the quantity $p(t) + \int_{[0,t)} h_x(\bar{x}(s))\mu(ds)$ vanishes almost everywhere and all conditions of the MP are satisfied independently of the value of the minimizer process $(\bar{x}, \bar{u})$. In this case, the NCO are said to degenerate.

3. Avoiding the Degeneracy Phenomenon

In order to avoid the degeneracy phenomenon, the NCO can be strengthened with additional conditions, typically a stronger form of the nontriviality condition such as:

$$\lambda + \int_{(0,1]} \mu(ds) + \|q\|_{L_\infty} > 0,$$

which is not satisfied by the degenerate multipliers.

The term *normality* is used when the NCO for OCP can be written with the multiplier associated with the objective function λ not zero.

The CQ existent in the literature to avoid degeneracy in OCP are, typically, of two types (see Figure 1):

[1]Here $\delta_{\{0\}}$ denotes the unit measure concentrated at $\{0\}$.

$\mathbf{CQ1_d}$: $\exists\delta, \epsilon > 0$ and $\exists\tilde{u}(t) \in \Omega(t)$:

$$h_x(x_0) \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t)] < -\delta \quad \text{a.e. } t \in [0, \epsilon).$$

Loosely speaking, this is the requirement that there exist a control function pulling the state away from the boundary of the state constraint set faster than the optimal control on a neighborhood of the initial time.

$\mathbf{CQ2_d}$: $\exists\delta, \epsilon > 0$ and $\exists\tilde{u}(t) \in \Omega(t)$:

$$h_x(x_0) \cdot f(t, x_0, \tilde{u}(t)) < -\delta \quad \text{a.e. } t \in [0, \epsilon).$$

That means, this $\mathbf{CQ2_d}$ requires the existence of a control functions pulling the state away from the state constraint boundary on a neighborhood of the initial time.

Extending $\mathbf{CQ1_d}$ and $\mathbf{CQ2_d}$ in such way that they are verifiable not only on a neighborhood of the initial time, but also on neighborhood of each instant in which the minimizer trajectory touches the boundary, allows us to write the MP with $\lambda = 1$. Here, we denote by $\mathbf{CQ1_n}$ and $\mathbf{CQ2_n}$ (respectively), the CQ that ensure the normality of MP.

3.1. Nondegenerate Results Involving $CQ1_d$ / $CQ1_n$

In Ferreira and Vinter [9], nondegeneracy necessary conditions for a free final state are developed, where it is required $f(t, x, \Omega(t))$ to be convex and data are merely required to be measurable in time.

The result in Ferreira, Fontes and Vinter [8] generalizes the result in [9] by allowing the final state to belong a given set C, the data to be nonsmooth and by not requiring the velocity set $f(t, x, \Omega(t))$ to be convex.

Based on theses previous nondegenerate result, Fontes (in [11]), ensures the normality for free final state problems.

3.2. Nondegenerate Results Involving $CQ2_d$ / $CQ2_n$

In Arutyunov and Aseev [3], a nondegenerate Maximum Principle is developed where the velocity has to be locally Lipschitz, convex and compact.

In Rampazzo and Vinter paper [16], the Maximum Principle can be written with $\lambda = 1$, if the dynamics are Lipschitz continuous with respect to time, the final state is free, and the initial state belongs to a given set. In [17] nondegenerate NCO were derived for problems with general endpoint constraints and which allow measurable time-dependence and nonconvex velocity sets.

In the paper of Cernea and Frankowska [6], to guarantee that the Maximum Principle is valid with $\lambda = 1$, it was necessary to establish the existence of a "linearization" of F along $(\bar{x}, \dot{\bar{x}})$ by closed convex processes, which are Lipschitz with respect to the state and the existence of a convex "linearizations" along the optimal trajectories.

For OCP with Lipschitz continuous trajectories, where the initial state belongs to a given set and the final state is free, the normality is ensured in Bettiol and Frankowska [5]. One of the advantages of this result is that it allows nonsmooth and nonconvex state constraints.

The book [2] provides several references to a vast earlier Russian literature on the degeneracy/abnormality phenomenon.

3.3. Comments

Clearly, a normal form of MP implies a nondegenerate form of MP. However most of these results require more regularity on data. See for example [9], [16], [11], [6] and [5].

Comparing theses results, we conclude that the results that use constraint qualification of the type **CQ1** $= (CQ1_d, CQ1_n)$, as in [9], [8], [11], typically require less regularity. However, **CQ1** involves the minimizing $\bar{u}$ which we do not know in advance, and consequently the condition is, in general not easily verifiable, except in special cases, such as problems in the calculus of variations (see [13] and also [9]).

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